Journal of Mathematical Research with Applications
Jul., 2024, Vol. 44, No. 4, pp. 470–474
DOI:10.3770/j.issn:2095-2651.2024.04.005
Http://jmre.dlut.edu.cn

On the Zeros of a Special Class of Polynomials

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Abstract Let P be a complex polynomial of the form $P(z)=(\lambda z-a)^m\prod_{j=1}^{n-m}(z-z_j)$, where $|z_j|\geq 1,\ 1\leq j\leq n-m$. The aim of this paper is to obtain generalisation of a result due to Zargar and Manzoor and a result due to Mir, Nazir and Wani. We shall also obtain an interesting bound which contains the zeros of the second derivative of $P(z)$.

Keywords bound; coefficient; polynomial; zeros

MR(2020) Subject Classification 12D10; 12D05

1. Introduction

The relative position of real zeros and critical points of a real differentiable function f is described in a well-known theorem of Rolle. However, this classical theorem is not true in general for analytic functions of a complex variable. As an analogous result of Rolle's theorem for at least a restricted class of analytic functions such as polynomials in complex variables, Gauss and Lucas [1] established a result which pertains to the regional location of the critical points of a polynomial $P(z)$ when the zeros of $P(z)$ are known. Concerning the location of critical points of polynomial $P(z)$ relative to each individual zero of $P(z)$, Blagovest Sendov established a conjecture, which entered into the literature as 'Ileif's Conjecture'. In connection with this conjecture, Brown [2] posed a problem that for what best constant C_n, $P'(z)$ does not vanish in $|z|<C_n$. Brown [2] himself conjectured that $C_n=\frac{1}{n}$. Aziz and Zargar [3] settled this problem by proving the following theorem.

Theorem 1.1 *If* $P(z)=z\prod_{j=1}^{n-1}(z-z_j)$ *is a polynomial of degree* n *with* $|z_j|\geq 1$, $1\leq j\leq n-1$, *then* $P'(z)$ *does not vanish in* $|z|<\frac{1}{n}$.

Rather and Ahmad [4] proved the following generalizations of Theorem 1.1.

Theorem 1.2 *If* $P(z)=(z-a)\prod_{j=1}^{n-1}(z-z_j)$ *is a polynomial of degree* n *with* $|a|\leq 1$ *and* $|z_j|\geq 1$ *for* $1\leq j\leq n-1$, *then* $P'(z)$ *does not vanish in the region*

$$|z-(\frac{n-1}{n})a|<\frac{1}{n}.$$

Received February 11, 2024; Accepted March 12, 2024
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Theorem 1.3 *If* $P(z) = (z-a)^m \prod_{j=1}^{n-m}(z-z_j)$ *is a polynomial of degree* n *with* $|a| \leq 1$ *and* $|z_j| \geq 1$ *for* $1 \leq j \leq n-m$*, then* $P'(z)$ *has* $(m-1)$ *fold zero at* $z = a$ *and remaining* $n-m$ *zeros of* $P'(z)$ *lie in the region*

$$|z - (\frac{n-m}{n})a| \geq \frac{m}{n}.$$

Zargar and Manzoor [5] extended Theorem 1.1 to second derivative of $P(z)$ by establishing the following result.

Theorem 1.4 *If* $P(z) = z^m \prod_{j=1}^{n-m}(z-z_j)$ *is a polynomial of degree* n *with* $|z_j| \geq 1$, $1 \leq j \leq n-1$*, then* $P''(z)$ *does not vanish in*

$$0 < |z| < \frac{m(m-1)}{n(n-1)}.$$

To obtain zero-free region for critical points of a polynomial, Zargar and Manzoor [6] established the following result for a special class of polynomials.

Theorem 1.5 *If* $P(z)$ *is a polynomial of degree* n *such that* $P(z)$ *does not vanish in* $|z| < 1$*, then the polynomial* $zP'(z) + 2P(z)$ *does not vanish in*

$$|z| < \frac{2}{n+2}.$$

Nazir, Mir and Wani [7] generalized Theorem 1.5 by proving the following result.

Theorem 1.6 *If the zeros of a polynomial* $P(z)$ *of degree* n *lie in* $|z| \geq 1$*, then for every* $\lambda > 0$*, the polynomial* $\lambda P(z) + (z-a)P'(z)$, $|a| \leq 1$*, has no zero in*

$$|z - \frac{na}{n+\lambda}| < \frac{\lambda}{n+\lambda}.$$

There exist many results [5–9] on the location of critical points of a polynomial and that of its derivatives. In this paper, we obtain a result which is a generalisation of Theorems 1.5 and 1.6. We also obtain an interesting bound concerning the zeros of the derivative of a special class of polynomial. For the proof of our results, we shall use the following theorem which is known in the literature as Walsh's Coincidence Theorem [10].

Theorem 1.7 *Let* $G(z_1, z_2, \ldots, z_n)$ *be a symmetric* n*-linear form of total degree* n *in* $(z_1, z_2, \ldots, z_n)$ *and let* C *be a circular region containing the* n *points* $\alpha_1, \alpha_2, \ldots, \alpha_n$*, then there exist at least one point* $\alpha \in C$ *such that*

$$G(\alpha_1, \alpha_2, \ldots, \alpha_n) = G(\alpha, \alpha, \ldots, \alpha).$$

2. Main results

In this section, we have established a result for a polynomial of the form

$$P(z) = (\lambda z - a)^m \prod_{j=1}^{n-m}(z - z_j),$$

where $|z_j| \geq 1$, $1 \leq j \leq n-m$, which is a generalisation of a result due to Zargar and Manzoor [6] and Nazir, Mir and Wani [7]. We have also established another result concerning the zeros of second derivative of the same polynomial.

Theorem 2.1 *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \geq 1$, then for every real $\lambda \geq 1$ and for any positive integer m with $m < n$, the polynomial $mP(z) + (\lambda z - a)P'(z)$, $|a| \leq 1$, has no zero in*

$$|z - \frac{na}{m+n\lambda}| < \frac{m}{m+n\lambda}.$$

Remark 2.2 By taking $m = 2$, $\lambda = 1$ and $a = 0$ in Theorem 2.1, we obtain Theorem 1.5.

Remark 2.3 By taking $m = \lambda$ and $\lambda = 1$ in Theorem 2.1, we obtain Theorem 1.6.

Theorem 2.4 *Let $P(z) = (\lambda z - a)^m \prod_{j=1}^{n-m}(z - z_j)$ be a polynomial of degree n with $|a| \leq 1$ and $|z_j| \geq 1$, $1 \leq j \leq n-m$. Then the second derivative of the polynomial $P(z)$ has $(m-2)$ fold zero at $z = a$ and the remaining $n-m$ zeros lie in*

$$|z - \frac{n(n-1)a}{(m+n\lambda)[(m-1)+(n-1)\lambda]}| \geq \frac{m(m-1)}{(m+n\lambda)[(m-1)+(n-1)\lambda]}.$$

The result is best possible as shown by the polynomial

$$P(z) = (z-a)^m (z - e^{\iota\alpha})^{n-m}, \quad 0 \leq \alpha < 2\pi.$$

The following result immediately follows by taking $m = 1$ and $\lambda = 1$ in Theorem 2.1.

Corollary 2.5 *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \geq 1$, then the polynomial $P(z) + (z-a)P'(z)$, $|a| \leq 1$, does not vanish in $|z - \frac{na}{n+1}| < \frac{1}{n+1}$.*

Putting $\lambda = 1$ in Theorem 2.4, we get the following result.

Corollary 2.6 *If $P(z) = (z-a)^m \prod_{j=1}^{n-m}(z - z_j)$ is a polynomial of degree n with $|a| \leq 1$ and $|z_j| \geq 1$, $1 \leq j \leq n-m$, then $P''(z)$ has $(m-2)$ fold zero at $z = a$ and the remaining $n-m$ zeros lie in*

$$|z - \frac{n(n-1)a}{(m+n)(m+n-2)}| \geq \frac{m(m-1)}{(m+n)(m+n-2)}.$$

Taking $a = 0$ in Corollary 2.6, the following result follows.

Corollary 2.7 *If $P(z) = z^m \prod_{j=1}^{n-m}(z - z_j)$ is a polynomial of degree n with $|a| \leq 1$ and $|z_j| \geq 1$, $1 \leq j \leq n-m$, then the second derivative of the polynomial $P(z)$ has $(m-2)$ fold zero at $z = a$ and the remaining $n-m$ zeros lie in $|z| \geq \frac{m(m-1)}{(m+n)(m+n-2)}$.*

3. Proof of the Theorems

In this section, we have established the proofs of the above results. We have applied Walsh's Coincidence Theorem [10] to prove these results.

Proof of Theorem 2.1 Let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$ so that $|z_j| \geq 1$, $1 \leq j \leq n$. Let

w be any zero of $mP(z) + (\lambda z - a)P'(z)$. Then

$$mP(w) + (\lambda w - a)P'(w) = 0. \tag{3.1}$$

This equation is linear and symmetric in $z_1, z_2, \ldots, z_n$. Hence by using Walsh's Coincidence Theorem for the circular region $C = \{z : |z| \geq 1\}$, there exists $\alpha \in C$ such that $P(z) = (z - \alpha)^n$. Thus Eq. (3.1) reduces to

$$m(w - \alpha)^n + (\lambda w - a)n(w - \alpha)^{n-1} = 0.$$

Equivalently,

$$(w - \alpha)^{n-1}\{w(m + n\lambda) - (na + m\alpha)\} = 0.$$

This gives, $w = \alpha$ and $w = \frac{na}{m+n\lambda} + \frac{m\alpha}{m+n\lambda}$. Now if $w = \alpha$, then using the fact that $|a| \leq 1$, we have

$$\begin{aligned}|w - \frac{na}{m + n\lambda}| &= |\alpha - \frac{na}{m + n\lambda}| \geq |\alpha| - \frac{n}{m + n\lambda}|a| \\ &\geq 1 - \frac{n}{m + n\lambda}|a| \geq 1 - \frac{n}{m + n\lambda} \\ &= \frac{n(\lambda - 1)}{m + n\lambda} + \frac{m}{m + n\lambda} \geq \frac{m}{m + n\lambda}.\end{aligned}$$

Again, if $w = \frac{na}{m+n\lambda} + \frac{m\alpha}{m+n\lambda}$, then

$$|w - \frac{na}{m + n\lambda}| = |\frac{m\alpha}{m + n\lambda}| = \frac{m}{m + n\lambda}|\alpha| \geq \frac{m}{m + n\lambda}.$$

Since w is any zero of $mP(z)+(\lambda z-a)P'(z)$, it follows that every zero of $mP(z)+(\lambda z-a)P'(z)$ lies in the region $|z - \frac{na}{m+n\lambda}| \geq \frac{m}{m+n\lambda}$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.4 Consider the polynomial $P(z) = (\lambda z - a)^m Q(z)$, therefore,

$$\begin{aligned}P'(z) &= m(\lambda z - a)^{m-1}Q(z) + (\lambda z - a)^m Q'(z) = (\lambda z - a)^{m-1}\{mQ(z) + (\lambda z - a)Q'(z)\} \\ &= (\lambda z - a)^{m-1}R(z),\end{aligned}$$

where $R(z) = mQ(z) + (\lambda z - a)Q'(z)$ has $m - 1$ fold zero at $z = a$ and remaining $n - m$ lie in the region $|z - \frac{na}{m+n\lambda}| \geq \frac{m}{m+n\lambda}$. Now, consider the polynomial

$$\begin{aligned}S(z) =& P'(\frac{mz}{m + n\lambda} + \frac{na}{m + n\lambda}), \\ S(z) =& [\lambda(\frac{mz}{m + n\lambda} + \frac{na}{m + n\lambda}) - a]^{m-1}R(\frac{mz}{m + n\lambda} + \frac{na}{m + n\lambda}) \\ =& (\frac{m}{m + n\lambda})^{m-1}(\lambda z - a)^{m-1}R(\frac{mz}{m + n\lambda} + \frac{na}{m + n\lambda}),\end{aligned} \tag{3.2}$$

then $S(z)$ is a polynomial of degree $n - 1$ with $m - 1$ fold zero at $z = a$ and remaining $n - m$ zeros lie in $|z| \geq 1$. Applying Walsh's Coincidence theorem to the polynomial $S(z)$, the derivative $S'(z)$ has $m - 2$ fold zero at $z = a$ and the remaining $n - m$ zeros lie in the region

$$|z - \frac{(n - 1)a}{(m - 1) + (n - 1)\lambda}| \geq \frac{(n - 1)a}{(m - 1) + (n - 1)\lambda}.$$

Replacing z by $\frac{m+n\lambda}{m}z - \frac{na}{m+n\lambda}$ in Eq. (3.2) and differentiating

$$P''(z) = S'(z) = (\lambda z - a)^{m-2}T(z),$$

where $T(z) = (m-1)R(z) + (\lambda z - a)R'(z)$. Applying above, we see $P''(z)$ has $(m-2)$ fold zero at $z = a$ and remaining $n - m$ zeros lie in

$$|z - \frac{n(n-1)a}{(m+n\lambda)[(m-1)+(n-1)\lambda)]}| \geq \frac{m(m-1)}{(m+n\lambda)[(m-1)+(n-1)\lambda]}.$$

This completes the proof of Theorem 2.4. □

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