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FI-t-Lifting Modules and T-Quasi-Dual Baer Modules

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Abstract The notions of FI-t-lifting modules and t-quasi-dual Baer modules are introduced and the relations between them are studied in this paper. It is shown that an amply supplemented module M is an FI-t-lifting module if and only if every fully invariant t-coclosed submodule of M is a direct summand of M if and only if $\bar{Z}^2(M)$ is a direct summand of M and $\bar{Z}^2(M)$ is an FI-lifting module if and only if M is t-quasi-dual Baer and an FI-t-$\mathcal{K}$-module.

Keywords t-small module; FI-t-lifting module; t-quasi-dual Baer module

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1. Introduction and preliminaries

Throughout this paper all rings are associative with unity and all modules will be unital right R-modules, where R is a ring. We use $N \leq M$ ($N \ll M$, $S =$End(M)) to indicate that N is a submodule (a small submodule, the ring of all R-endomorphisms) of M.

Lifting modules play important roles in rings and categories of modules. Lifting modules and their generalizations have been studied extensively by many authors recently. The notion of FI-lifting modules, as a proper generalization of lifting modules, and strongly FI-lifting modules were introduced by Talebi and Amouzegar [1]. A module M is an FI-lifting module if every fully invariant submodule A of M contains a direct summand B of M such that $A/B \ll M/B$. A module M is a strongly FI-lifting module if every fully invariant submodule A of M contains a fully invariant direct summand B of M such that $A/B \ll M/B$. T-lifting modules and t-dual Baer modules [2] and quasi-dual Baer modules [3] were posed by Talebi and Amouzegar. A module M is said to be t-lifting if for every submodule A of M there is a direct summand K of M such that $A/K \ll_t M/K$. M is said to be a quasi-dual Baer module if for any fully invariant submodule N of M, there exists an idempotent e in S such that $\{\phi \in S \mid \mathrm{Im}\phi \leq N\} = eS$, equivalently, $\sum_{\phi\in I}\mathrm{Im}\phi$ is a direct summand of M for every ideal I of S. Inspired by these, we introduce the notions of FI-t-lifting modules and t-quasi-dual Baer modules and the relations between them are studied in this paper. It is shown that an amply supplemented module M is an FI-t-lifting module if and only if every fully invariant t-coclosed submodule of M is a direct

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summand of M if and only if $\bar{Z}^2(M)$ is a direct summand of M and $\bar{Z}^2(M)$ is an FI-lifting module if and only if M is t-quasi-dual Baer and an FI-t-$\mathcal{K}$-module.

Let M be a module and $V \leq M$. V is said to be small in M if $M \neq V + T$ for any proper submodule T of M. Let N and L be submodules of M, N is called a supplement of L in M if $N + L = M$ and N is minimal with respect to this property. Equivalently, $M = N + L$ and $N \cap L \ll N$. M is called supplemented if every submodule of M has a supplement in M. M is called amply supplemented if for any submodules A, B of M with $M = A + B$ there exists a supplement P of A such that $P \leq B$. M is called a lifting module if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$. Let $A \leq B \leq M$. We say that A is a coessential submodule of B in M (denoted by $A \overset{ce}{\hookrightarrow} B$ in M) (see [4]) if $B/A \ll M/A$ and A is a corational submodule of B in M (denoted by $A \overset{cr}{\hookrightarrow} B$ in M) (see [4]) if $\mathrm{Hom}(M/A, B/X) = 0$ for any submodule X such that $A \leq X \leq B$. A submodule of M is coclosed in M if it has no proper coessential submodule in M. A coclosure of a submodule B of M is a coessential submodule of B in M which is also a coclosed submodule of M. M is called a UCC module [5] if every submodule has a unique coclosure in M. M is called a small module if M is small in some modules. It is well known that that a module M is a small module if and only if it is small in its injective hull. Following [6], for a module M, let

$$\bar{Z}(M) = \mathrm{Rej}(M, \mathscr{S}) = \cap\{\mathrm{Ker} f | f : M \to V, V \in \mathscr{S}\} = \cap\{U \leq M | M/U \in \mathscr{S}\},$$

where $\mathscr{S}$ denotes the class of all small modules. M is called cosingular (non-cosingular) if $\bar{Z}(M) = 0$ ($\bar{Z}(M) = M$). Note that $\bar{Z}^2(M)$ is defined as $\bar{Z}(\bar{Z}(M))$. Let M be a module and $A \leq M$. A is called t-small in M (see [2]) (written $A \ll_t M$) if for every submodule B of M, $\bar{Z}^2(M) \leq A + B$ implies that $\bar{Z}^2(M) \leq B$. A submodule C of M is called t-coclosed if $C/C' \ll_t M/C'$ implies that $C = C'$. Recall that a submodule K of M is called fully invariant in M if $f(K) \leq K$ for all $f \in \mathrm{End}(M)$. For other standard definitions we refer to [7,8].

We list a few lemmas for later use.

Lemma 1.1 ([3, Proposition 2.2]) *Let M be an amply supplemented module and $A \leq M$. The following conditions are equivalent:*

(1) A is t-small in M.

(2) $A \cap \bar{Z}^2(M) \ll \bar{Z}^2(M)$.

(3) $A \cap \bar{Z}^2(M) \ll M$.

(4) $\bar{Z}^2(A) = 0$, namely, $\bar{Z}(A)$ is cosingular.

Lemma 1.2 ([9, Lemma 1.1], [10, Proposition 1.3]) *Let M be a module.*

(1) Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M.

(2) If $X \leq Y \leq M$ such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y, then X is a fully invariant submodule of M.

(3) Let $M = M_1 \oplus M_2$. If N is a fully invariant submodule of M, then $N = N_1 \oplus N_2$, where $N_i = N \cap M_i$ is fully invariant in M_i for $i = 1, 2$.

(4) Let $K \leq N$ be submodules of M. If K is fully invariant in M and N/K is fully invariant in M/K, then N is fully invariant in M.

Lemma 1.3 ([8, Proposition 3.7]) *Let M be a module and $K \leq L \leq M$. If K is coclosed in M, then K is coclosed in L and the converse is true if L is coclosed in M.*

2. FI-t-lifting modules

The concept of FI-t-lifting modules is introduced and some characterizations are given in this section. It is proven that an amply supplemented module is an FI-t-lifting module if and only if every fully invariant t-coclosed submodule of M is a direct summand of M if and only if $\bar{Z}^2(M)$ is a direct summand of M and $\bar{Z}^2(M)$ is an FI-lifting module. We start with the following.

Definition 2.1 *A module M is called an FI-t-lifting module if for every fully invariant submodule N of M, there is a direct summand K of M such that $K \leq N$ and $N/K \ll_t M/K$.*

Clearly, every t-lifting module is an FI-t-lifting module, but the converse is not true in general.

Example 2.2 $\mathbb{Q}_{\mathbb{Z}}$ is an FI-t-lifting module but not a t-lifting module.

Lemma 2.3 *Let M be a non-cosingular amply supplemented module and N be a fully invariant submodule of M. Then every coclosure of N in M is a fully invariant submodule of M.*

Proof Let X be a coclosure of N. Then $X \overset{ce}{\hookrightarrow} N$ and X is coclosed in M. Because M is non-cosingular, $X \overset{cr}{\hookrightarrow} N$ in M by [4, Proposition 1.10]. Let $f \in \mathrm{End}(M)$, then $f(X) \overset{cr}{\hookrightarrow} f(N)$ in $f(M)$ by [4, Proposition 1.1], and hence $f(X) \overset{ce}{\hookrightarrow} f(N)$ in $f(M)$ by [4, Proposition 1.1]. Thus $f(X) \overset{ce}{\hookrightarrow} f(N)$ in M. By [6, Lemma 2.3, Propostion 2.4], $f(X)$ is coclosed in M. So $f(X)$ is a coclosure of $f(N)$. Since M is a non-cosingular amply supplemented module, M is UCC by [5, Proposition 4.2]. Because $f(N) \leq N$, $f(X) \leq X$ by [5, Theorem 3.7]. □

Corollary 2.4 *Let M be a non-cosingular amply supplemented module. Then the following are equivalent:*

(1) M is an FI-lifting module.

(2) M is a strongly FI-lifting module.

(3) Every fully invariant coclosed submodule of M is a direct summand of M.

Proof (1)⇔(2). By [11, Corollary 3.2].

(1)⇒(3). It is clear.

(3)⇒(1). Let C be a fully invariant submodule of M. As M is amply supplemented, there exists a coclosure C' of C in M. By Lemma 2.3, C' is a fully invariant submodule of M, and hence C' is a direct summand of M, as desired. □

Theorem 2.5 *Let M be an amply supplemented module. Then the following statements are* *equivalent:*

(1) M is an FI-t-lifting module.

(2) For every fully invariant submodule A of M, there is a decomposition $A = N \oplus N'$ such that N is a direct summand of M and $N' \ll_t M$.

(3) Every fully invariant t-coclosed submodule of M is a direct summand of M.

(4) For every fully invariant submodule A of M, $\bar{Z}^2(A)$ is a direct summand of M.

(5) For every fully invariant coclosed submodule A of M, $\bar{Z}^2(A)$ is a direct summand of M.

(6) $\bar{Z}^2(M)$ is a direct summand of M and $\bar{Z}^2(M)$ is an FI-lifting module.

(7) Every fully invariant submodule A of M which is contained in $\bar{Z}^2(M)$, contains a direct summand N of M such that $A/N \ll M/N$.

Proof (1)⇒(2). Let A be a fully invariant submodule of M. Then there is a decomposition $M = N \oplus L$ such that $A/N \ll_t M/N$, and so $A = N \oplus (A \cap L)$. By Lemma 1.1, $\bar{Z}^2(A/N) = 0$, and hence $\bar{Z}^2(A \cap L) = 0$. Therefore, $A \cap L \ll_t M$.

(2)⇒(3). Let A be a fully invariant t-coclosed submodule of M. By assumption, $A = N \oplus N'$ such that N is a direct summand of M and $N' \ll_t M$. By Lemma 1.1, $\bar{Z}^2(N') = 0$, and so $\bar{Z}^2(A/N) = 0$. Again by Lemma 1.1, $A/N \ll_t M/N$. Because A is a fully invariant t-coclosed submodule of M, $A = N$ is a direct summand of M.

(3)⇒(4). Let A be a fully invariant submodule of M. Since $\bar{Z}^2(A)$ is non-cosingular, $\bar{Z}^2(A)$ is t-coclosed by [2, Proposition 2.5]. Because $\bar{Z}^2(A)$ is a fully invariant submodule of A and A is a fully invariant submodule of M, $\bar{Z}^2(A)$ is a fully invariant submodule of M by Lemma 1.2. Thus $\bar{Z}^2(A)$ is a direct summand of M by assumption.

(4)⇒(5). It is clear.

(5)⇒(6). As $\bar{Z}^2(M)$ is a fully invariant coclosed submodule of M, $\bar{Z}^2(\bar{Z}^2(M)) = \bar{Z}^2(M)$ is a direct summand of M. Next we shall show that $\bar{Z}^2(M)$ is an FI-lifting module. Let C be a fully invariant submodule of $\bar{Z}^2(M)$. As $\bar{Z}^2(M)$ is amply supplemented and non-cosingular, there is a coclosure C' of C in $\bar{Z}^2(M)$. By Lemma 2.3, C' is a fully invariant coclosed submodule of $\bar{Z}^2(M)$. Since $\bar{Z}^2(M)$ is a fully invariant submodule of M, C' is a fully invariant coclosed submodule of M by Lemmas 1.2 and 1.3. Thus C' is non-cosingular by [6, Lemma 2.3], and hence $\bar{Z}^2(C') = C'$ is a direct summand of M by assumption. So C' is a direct summand of $\bar{Z}^2(M)$. Let $\bar{Z}^2(M) = C' \oplus C''$. Then $\bar{Z}^2(M) = C + C''$ and $C = C \cap \bar{Z}^2(M) = C' \oplus (C \cap C'')$. Because $\bar{Z}^2(M)/C' \cong C''$, $C/C' \cong (C \cap C'')$ and $C/C' \ll \bar{Z}^2(M)/C'$, it follows that $C \cap C'' \ll C''$. Thus C'' is a direct summand supplement of C in $\bar{Z}^2(M)$, and so $\bar{Z}^2(M)$ is an FI-lifting module by [11, Proposition 3.7].

(6)⇒(7). Let A be a fully invariant submodule of M which is contained in $\bar{Z}^2(M)$. Since $\bar{Z}^2(M)$ is a direct summand of M, A is a fully invariant submodule of $\bar{Z}^2(M)$ by Lemma 1.2. As $\bar{Z}^2(M)$ is an FI-lifting module, there is a direct summand N of $\bar{Z}^2(M)$ such that $N \leq A$ and $A/N \ll \bar{Z}^2(M)/N$, and so $A/N \ll M/N$, as required.

(7)⇒(1). Let A be a fully invariant submodule of M. Because $\bar{Z}^2(M)$ is a fully invariant submodule of M, $A \cap \bar{Z}^2(M)$ is a fully invariant submodule of M by Lemma 1.2. By assumption, there is a direct summand N of M such that $(A\cap \bar{Z}^2(M))/N \ll M/N$. Since $(A\cap \bar{Z}^2(M))/N = A/N \cap \bar{Z}^2(M/N)$, $A/N \ll_t M/N$ by Lemma 1.1, and so M is an FI-t-lifting module. □

Corollary 2.6 *Let $M = M_1 \oplus M_2$ be an amply supplemented module. If M_1 and M_2 are FI-t-lifting modules, then M is an FI-t-lifting module.*

Proof Because every direct summand of amply supplemented modules is amply supplemented, M_1 and M_2 are amply supplemented modules. By assumption and Theorem 2.5, $\bar{Z}^2(M_i)$ is a direct summand of M_i and $\bar{Z}^2(M_i)$ is an FI-lifting module for $i = 1, 2$. Since $\bar{Z}^2(M) = \bar{Z}^2(M_1) \oplus \bar{Z}^2(M_2)$ and any finite direct sum of FI-lifting modules is FI-lifting [1, Theorem 2.6], $\bar{Z}^2(M)$ is a direct summand of M and $\bar{Z}^2(M)$ is FI-lifting, and so M is an FI-t-lifting module by Theorem 2.5. □

Corollary 2.7 *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be an amply supplemented module. If each M_i is an FI-t-lifting module, then M is an FI-t-lifting module.*

Corollary 2.8 *Let M be an amply supplemented FI-t-lifting module. Then*

(1) If K is a fully invariant submodule of M and K is amply supplemented, then K is an FI-t-lifting module;

(2) If L is a fully invariant submodule of M, then M/L is an FI-t-lifting module;

(3) Every direct summand of M is an FI-t-lifting module.

Proof (1) Let L be a fully invariant submodule of K. As K is a fully invariant submodule of M, L is a fully invariant submodule of M by Lemma 1.2. As M is an amply supplemented FI-t-lifting module, there is a direct summand N of M such that $L/N \ll_t M/N$, and hence $\bar{Z}^2(L/N) = 0$ by Lemma 1.1. Because K is amply supplemented and N is a direct summand of K, $L/N \ll_t K/N$ by Lemma 1.1. So K is an FI-t-lifting module.

(2) Let K/L be a fully invariant submodule of M/L. Then K is a fully invariant submodule of M by Lemma 1.2. Because M is an FI-t-lifting module, there is a decomposition $M = N \oplus N'$ such that $N \le K$ and $K/N \ll_t M/N$. As L is a fully invariant submodule of M, $L = (N\cap L)\oplus(N'\cap L) = (N + L)\cap(N' + L)$, and hence $M/L = (N + L)/L \oplus (N' + L)/L$. It is easy to see that $(N + L)/L \le K/L$. Because $K/N \ll_t M/N$, $\bar{Z}^2(K/N) = (\bar{Z}^2(K) + N)/N = 0$. Thus $\bar{Z}^2(K) \le N$, and so $\bar{Z}^2(K/(N + L)) = 0$. Since $M/(N + L)$ is amply supplemented, $K/(N + L) \ll_t M/(N + L)$ by Lemma 1.1. Therefore, M/L is an FI-t-lifting module.

(3) As M is an amply supplemented FI-t-lifting module, $M = \bar{Z}^2(M) \oplus L$ and $\bar{Z}^2(M)$ is an FI-lifting module by Theorem 2.5. Let N be a direct summand of M. Case one: Let N be non-cosingular. Then $N \le \bar{Z}^2(M)$, and hence N is a direct summand of $\bar{Z}^2(M)$. As $\bar{Z}^2(M)$ is non-cosingular, it is a strongly FI-lifting module by [11, Corollary 3.2]. So N is a strongly FI-lifting module by [1, Theorem 3.4]. Thus N is an FI-t-lifting module. Case two: Assume that N is not non-cosingular and $M = N \oplus N'$, then $\bar{Z}^2(M) = \bar{Z}^2(N) \oplus \bar{Z}^2(N')$. Thus $N = N \cap M = N \cap (\bar{Z}^2(M) \oplus L) = \bar{Z}^2(N) \oplus L', L' = N \cap (\bar{Z}^2(N') \oplus L)$, and hence $\bar{Z}^2(N)$ is a non-cosingular direct summand of M. By Case one, $\bar{Z}^2(N)$ is an FI-lifting module. Because N

is amply supplemented, it is an FI-t-lifting module by Theorem 2.5. □

3. T-quasi-dual Baer modules

As a proper generalization of quasi-dual Baer modules, the notion of t-quasi-dual Baer modules is introduced in this section. It is shown that an amply supplemented module M is an FI-t-lifting module if and only if M is t-quasi-dual Baer and an FI-t-$\mathcal{K}$-module.

Definition 3.1 *A module M is called a t-quasi-dual Baer module if for every ideal I of S, $I(\bar{Z}^2(M))$ is a direct summand of M.*

Example 3.2 $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}_{p^3}$ are t-quasi-dual Baer modules, but not quasi-dual Baer modules.

Recall that a module M has SSSP if the sum of every number of direct summands of M is a direct summand of M.

Theorem 3.3 *The following statements are equivalent for a module M:*

(1) M is a t-quasi-dual Baer module.

(2) $\bar{Z}^2(M)$ is a direct summand of M and $\bar{Z}^2(M)$ is a quasi-dual Baer module.

(3) M has SSSP on fully invariant direct summands which are contained in $\bar{Z}^2(M)$ and $I(\bar{Z}^2(M))$ is a direct summand of M for every principal ideal I of S.

(4) For any subset A of S, $\sum_{\varphi\in A} S\varphi S(\bar{Z}^2(M))$ is a direct summand of M.

Proof (1)⇒(2). As M is a t-quasi-dual Baer module, $\bar{Z}^2(M)=S(\bar{Z}^2(M))$ is a direct summand of M. Let I be an ideal of $\bar{S}$ =End$(\bar{Z}^2(M))$, $A=\{i\phi\pi \mid \phi\in I\}$, where π is the canonical projection onto $\bar{Z}^2(M)$, i is the inclusion map from $\bar{Z}^2(M)$ to M and $I'=SAS$. As $I(\bar{Z}^2(M))$ is a fully invariant submodule of $\bar{Z}^2(M)$ and $\bar{Z}^2(M)$ is a fully invariant submodule of M, $I(\bar{Z}^2(M))$ is a fully invariant submodule of M by Lemma 1.2, and hence $SI(\bar{Z}^2(M))\le I(\bar{Z}^2(M))$. Again $I'(\bar{Z}^2(M))=SAS(\bar{Z}^2(M))=SI(\bar{Z}^2(M))\ge I(\bar{Z}^2(M))$, so $I'(\bar{Z}^2(M))=I(\bar{Z}^2(M))$. Because M is a t-quasi-dual Baer module, $I'(\bar{Z}^2(M))=I(\bar{Z}^2(M))$ is a direct summand of M. Thus $I(\bar{Z}^2(M))$ is a direct summand of $\bar{Z}^2(M)$. Therefore, $\bar{Z}^2(M)$ is a quasi-dual Baer module.

(2)⇒(1). Let I be an ideal of S, $A'=\{\pi'\phi\mid_{\bar{Z}^2(M)}\colon \phi\in I\}$ where π' is the canonical projection onto $\bar{Z}^2(M)$, $\bar{S}$ =End$(\bar{Z}^2(M))$ and $I'=\bar{S}A'\bar{S}$. Since $\bar{Z}^2(M)$ is a quasi-dual Baer module, $I'(\bar{Z}^2(M))$ is a direct summand of $\bar{Z}^2(M)$. It is easy to see that $I'(\bar{Z}^2(M))=I(\bar{Z}^2(M))$. Because $\bar{Z}^2(M)$ is a direct summand of M, $I(\bar{Z}^2(M))$ is a direct summand of M.

(1)⇒(3). Let $e_i^2=e_i\in S, i\in\Lambda$, $e_i(M)\le\bar{Z}^2(M)$ and $e_i(M)$ is a fully invariant submodule of M. Take $I=\sum_{i\in\Lambda} Se_iS$, then $I(\bar{Z}^2(M))=\sum_{i\in\Lambda} Se_iS(\bar{Z}^2(M))\le\sum_{i\in\Lambda} Se_i(M)\le\sum_{i\in\Lambda} e_i(M)$. It is clear that $e_i(M)\le\sum_{i\in\Lambda} Se_iS(\bar{Z}^2(M))$, and hence $\sum_{i\in\Lambda} e_i(M)=I(\bar{Z}^2(M))$ is a direct summand of M as M is a t-quasi-dual Baer module. The rest is obvious.

(3)⇒(4). By assumption, $S\varphi S(\bar{Z}^2(M))$ is a direct summand of M for every $\varphi\in A$. It is easy to see that $S\varphi S(\bar{Z}^2(M))\le\bar{Z}^2(M)$ and $S\varphi S(\bar{Z}^2(M))$ is a fully invariant submodule of M. Again by assumption, $\sum_{\varphi\in A} S\varphi S(\bar{Z}^2(M))$ is a direct summand of M.

(4)⇒(1). Let I be an ideal of S. Then $I(\bar{Z}^2(M)) = \sum_{\varphi\in I} S\varphi S(\bar{Z}^2(M))$. By assumption, $I(\bar{Z}^2(M))$ is a direct summand of M, and so M is a t-quasi-dual Baer module. □

Proposition 3.4 *Every direct summand of a t-quasi-dual Baer module is a t-quasi-dual Baer module.*

Proof Let M be a t-quasi-dual Baer module and N a direct summand of M. Write $M = N\oplus N'$, then $\bar{Z}^2(M) = \bar{Z}^2(N)\oplus\bar{Z}^2(N')$. By Theorem 3.3, $\bar{Z}^2(M)$ is a direct summand of M and $\bar{Z}^2(M)$ is a quasi-dual Baer module, and hence $\bar{Z}^2(N)$ is a direct summand of N and $\bar{Z}^2(N)$ is a quasi-dual Baer module by [3, Theorem 2.1]. Again by Theorem 3.3, N is a t-quasi-dual Baer module.

Let M be a module. Recall that M is a t-$\mathcal{K}$-module [2] if $T_S(N) = T_S(0)$ implies that $N \ll_t M$ for any submodule N of M, where $T_S(N) = \{\phi \in S \mid \phi(\bar{Z}^2(M)) \leq N\}$; M is a $\mathcal{K}$-module [2] if $D_S(N) = 0$ implies that $N \ll M$ for any submodule N of M, where $D_S(N) = \{\phi\in S \mid \phi(M) \leq N\}$; M is an FI-$\mathcal{K}$-module [3] if $D_S(N) = 0$ implies that $N \ll M$ for any fully invariant submodule N of M. □

Definition 3.5 *A module M is called an FI-t-$\mathcal{K}$-module if $T_S(N) = T_S(0)$ implies that $N \ll_t M$ for any fully invariant submodule N of M.*

Proposition 3.6 *Let M be an amply supplemented module. Then:*

(1) M is an FI-t-$\mathcal{K}$-module if and only if $T_S(N) = T_S(0)$ implies that $N \ll M$ for any fully invariant submodule N which is contained in $\bar{Z}^2(M)$;

(2) If M is an FI-t-$\mathcal{K}$-module, then $\bar{Z}^2(M)$ is an FI-$\mathcal{K}$-module.

Proof (1) "⇒". It is clear by Lemma 1.1.

"⇐". Let N be a fully invariant submodule of M with $T_S(N) = T_S(0)$. As $\bar{Z}^2(M)$ is a fully invariant submodule of M, $T_S(N\cap\bar{Z}^2(M)) = T_S(N) = T_S(0)$. By Lemma 1.2, $N\cap\bar{Z}^2(M)$ is a fully invariant submodule of M, so $N\cap\bar{Z}^2(M) \ll M$ by assumption, and hence $N \ll_t M$ by Lemma 1.1.

(2) Let $\bar{S}$ =End$(\bar{Z}^2(M))$ and N a fully invariant submodule of $\bar{Z}^2(M)$ with $D_{\bar{S}}(N) = 0$. We shall show that $T_S(N) = T_S(0)$. Let $\phi \in T_S(N)$. Then $\bar{\phi} = \phi\mid_{\bar{Z}^2(M)}$ is an endomorphism of $\bar{Z}^2(M)$ with $\bar{\phi}(\bar{Z}^2(M)) \leq N$, and so $\bar{\phi} \in D_{\bar{S}}(N) = 0$. Thus $T_S(N) = T_S(0)$. By assumption $N \ll_t M$, so $N \ll \bar{Z}^2(M)$ by Lemma 1.1. □

Theorem 3.7 *Let M be an amply supplemented module. Then the following statements are equivalent:*

(1) M is an FI-t-lifting module.

(2) M is t-quasi-dual Baer and an FI-t-$\mathcal{K}$-module.

(3) M is a t-quasi-dual Baer module and $C = T_S(C)(\bar{Z}^2(M))$ for any fully invariant t-coclosed submodule C of M.

Proof (1)⇒(2). Because M is an FI-t-lifting module, $\bar{Z}^2(M)$ is a direct summand of M and $\bar{Z}^2(M)$ is an FI-lifting module. Since $\bar{Z}^2(M)$ is non-cosingular, $\bar{Z}^2(M)$ is a quasi-dual

Baer module by [3, Theorem 2.4] and hence M is a t-quasi-dual Baer module by Theorem 3.3. Next we shall show that M is an FI-t-$\mathcal{K}$-module. Let N be a fully invariant submodule of M, $N \leq \bar{Z}^2(M)$ and $T_S(N) = T_S(0)$. Because M is an FI-t-lifting module, there is a direct summand K of M such that $N/K \ll_t M/K$. By Lemma 1.1, $N/K \cap \bar{Z}^2(M/K) \ll M/K$. Because $N/K \leq (\bar{Z}^2(M) + K)/K = \bar{Z}^2(M/K)$, $N/K \ll M/K$. Let $M = K \oplus K'$ and $K \neq 0$, then $\bar{Z}^2(K) \neq 0$ (since if $\bar{Z}^2(K) = 0$, then $0 \neq K \leq N \leq \bar{Z}^2(M) = \bar{Z}^2(K') \leq K'$, this is a contradiction). Now consider the canonical projection $\pi_K : M \to K$, then $\pi_K \in T_S(N)$ and $\pi_K \bar{\in} T_S(0)$. This contradicts $T_S(N) = T_S(0)$, hence $K = 0$. Thus $N \ll M$. By Proposition 3.6, M is an FI-t-$\mathcal{K}$-module.

(2)⇒(1). As M is a t-quasi-dual Baer module, $\bar{Z}^2(M)$ is a direct summand of M and $\bar{Z}^2(M)$ is a quasi-dual Baer module. As M is an FI-t-$\mathcal{K}$-module, $\bar{Z}^2(M)$ is an FI-$\mathcal{K}$-module. By [3, Proposition 2.3], $\bar{Z}^2(M)$ is an FI-lifting module, and so M is an FI-t-lifting module by Theorem 2.5.

(1)⇒(3). By the proof of (1) ⇒ (2), M is t-quasi-dual Baer. Let C be a fully invariant t-coclosed submodule of M. Obviously, $T_S(C)(\bar{Z}^2(M)) \leq C$. By hypothesis, C is a direct summand of M, say $M = C \oplus C'$. Consider the canonical projection π onto C. It is clear that $\pi \in T_S(C)$. By [2, Proposition 2.5], $C \leq \bar{Z}^2(M)$, thus $C = \pi(C) \leq \pi(\bar{Z}^2(M)) \leq T_S(C)(\bar{Z}^2(M))$. So $C = T_S(C)(\bar{Z}^2(M))$.

(3)⇒(1). Let C be a fully invariant t-coclosed submodule of M. By assumption, $C = T_S(C)(\bar{Z}^2(M))$. It is easy to see that $T_S(C) \leq ST_S(C)S$. Because $ST_S(C)S(\bar{Z}^2(M)) \leq SC \leq C$, $T_S(C) = ST_S(C)S$. Because M is a t-quasi-dual Baer module, $C = ST_S(C)S(\bar{Z}^2(M))$ is a direct summand of M, it follows that M is an FI-t-lifting module by Theorem 2.5. □

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References

[1] Y. TALEBI, T. AMOUZEGAR. *Strongly FI-lifting modules*. Int. Electron. J. Algebra, 2008, **3**: 75–82.

[2] T. AMOUZEGAR, D. K. TÜTÜNCÜ, Y. TALEBI. *t-dual Baer modules and t-lifting modules*. Vietnam J. Math., 2014, **42**(2): 159–169.

[3] T. AMOUZEGAR, Y. TALEBI. *On quasi-dual Baer modules*. TWMS J. Pure Appl. Math., 2013, **4**(1): 78–86.

[4] Y. TALEBI, N. VANAJA. *Copolyform modules*. Comm. Algebra, 2002, **30**(3): 1461–1473.

[5] L. GANESAN, N. VANAJA. *Modules for which every submodule has a unique coclosure*. Comm. Algebra, 2002, **30**(5): 2355–2377.

[6] Y. TALEBI, N. VANAJA. *The torsion theory cogenerated by M-small modules*. Comm. Algebra, 2002, **30**(3): 1449–1460.

[7] F. W. ANDERSON, K. R. FULLER. *Rings and Categories of Modules*. Second edition. Graduate Texts in Mathematics, 13. Springer-Verlag, New York, 1992.

[8] J. CLARK, C. LOMP, N. VANAJA, et al. *Lifting Modules. Supplements and Projectivity in Module Theory*. Birkhäuser Verlag, Basel, 2006.

[9] G. F. BIRKENMEIER, B. J. MÜLLER, S. T. RIZVI. *Modules in which every fully invariant submodule is essential in a direct summand*. Comm. Algebra, 2002, **30**(3): 1395–1415.

[10] S. ASGARI, A. HAGHANY. *T-extending modules and t-Baer modules*. Comm. Algebra, 2011, **39**(5): 1605–1623.

[11] D. K. TÜTÜNCÜ, R. TRIBAK. *On τ-noncosingular modules*. Bull. Aust. Math. Soc., 2009, **80**(3): 462–471.