Journal of Mathematical Research with Applications
Jul., 2024, Vol. 44, No. 4, pp. 483–494
DOI:10.3770/j.issn:2095-2651.2024.04.007
Http://jmre.dlut.edu.cn

The Complementarity of Normalized Solutions for Kirchhoff Equations with Mixed Nonlinearity

Lin XU, Qilin XIE*
School of Mathematics and Statistics, Guangdong University of Technology, Guangdong 510520, P. R. China

Abstract In this paper, we study the existence of solutions for Kirchhoff equation

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big)\Delta u=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u,\quad x\in\mathbb{R}^3$$

with mass constraint condition

$$S_c:=\Big\{u\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}|u|^2\mathrm{d}x=c\Big\},$$

where a, b, $c>0$, $\mu\in\mathbb{R}$, $2<q<p<6$, and $\lambda\in\mathbb{R}$ appears as a Lagrange multiplier. For the range of p and q, the Sobolev critical exponent 6 and mass critical exponent $\frac{14}{3}$ are involved where corresponding energy functional is unbounded from below on S_c. We consider the focusing case, i.e., $\mu>0$ when (p,q) belongs to a certain domain in $\mathbb{R}^2$. We prove the existence of normalized solutions by using constraint minimization, concentration compactness principle and Minimax methods. We partially extend the results which have been studied.

Keywords normalized solutions; Kirchhoff type equation; mixed nonlinearty

MR(2020) Subject Classification 35J20; 35J60; 35J15

1. Introduction and main results

In this paper, we study the existence of solutions with prescribed mass to the following Kirchhoff equation

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big)\Delta u=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u\ \text{ in }\ \mathbb{R}^3,\tag{1.1}$$

where a, $b>0$ and $2<q<p<6$. The Eq. (1.1) is closely related to the equation

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big)\Delta u=f(x,u),\tag{1.2}$$

which is the stationary analog of the equation

$$u_{tt}-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big)\Delta u=f(x,u),\tag{1.3}$$

Received October 16, 2023; Accepted March 8, 2024
Supported by the Basic and Applied Basic Research Foundation of Guangdong Province (Grant No. 2022A1515 010644).
* Corresponding author
E-mail address: xl1244176808@163.com (Lin XU); xieql@gdut.edu.cn (Qilin XIE)

where $f(x,u)$ is of general nonlinearity. The Eq. (1.3) was initially proposed by Kirchhoff in 1883 as an extension of the classical D'Alembert's wave equations. In Eq. (1.3), the variable u denotes the displacement, f is the external force and a represents the initial tension. Additionally, b is associated with the intrinsic properties of the string (such as Young's modulus). It is worth noting that the nonlocal problems also appear in other fields as biological systems, where u describes a process which depends on the average of itself (for example, population density). Mathematically, the Eq. (1.3) is often referred to be nonlocal as the appearance of the term $b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x$ implies that Eq. (1.3) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, which make the study of Eq. (1.3) particularly interesting.

After the pioneering work of Lions [1], the Eq. (1.3) began to receive much attention and many researchers studied its steadystate model, see [2–6] for more important research progress.

At present, there are two substantially different view points in terms of the frequency λ in Eq. (1.1). One is to regard the frequency λ as a given constant. In this situation, solutions of Eq. (1.1) are critical points of the corresponding action functional on the working space and has been extensively studied in [7–10] and the references therein. We point out that the existence, multiplicity and concentration of solutions for Eq. (1.1) involving Sobolev subcritical, critical and supercritical exponents have been extensively studied under different assumptions about the nonlinearity term, see [4] and their references therein.

The other one is to regard λ as an unknown quantity to the Eq. (1.1). Nowadays, some physicists are very interested in the solutions satisfying the normalized condition $\int_{\mathbb{R}^3}|u|^2\mathrm{d}x=c$, for a priori given c, since the mass admits a clear physical meaning. In this situation, it is natural to prescribe the value of the mass so that λ can be interpreted as a Lagrange multiplier. For example, from a physical point of view, the normalized condition may represent the number of particles of each component in Bose-Einstein condensates or the power supply in the nonlinear optics framework. In addition, such solutions can give a better insight of the dynamical properties, like orbital stability or instability and can describe attractive Bose-Einstein condensates. This type of solutions is usually called prescribed mass solutions or normalized solutions in mathematics. In order to study the solution of Eq. (1.1) satisfying the normalized condition, it suffices to consider the critical point of the functional

$$E_\mu(u):=\frac{a}{2}\|\nabla u\|_2^2+\frac{b}{4}\|\nabla u\|_2^4-\frac{\mu}{q}\|u\|_q^q-\frac{1}{p}\|u\|_p^p,$$

on

$$S_c=\{u\in H^1(\mathbb{R}^3):\|u\|_2^2=c\},$$

where $\|\cdot\|_r$ denotes the standard norm in $L^r(\mathbb{R}^3)$ for $r\in[1,+\infty)$. We consider the following minimization problem

$$m(c):=\inf_{S_c}E_\mu(u). \tag{1.4}$$

If $u\in S_c$ is a minimizer of problem (1.4), then there exists $\lambda_c\in\mathbb{R}$ as a Lagrange multiplier such that $E'_\mu(u)=\lambda_c u$, namely, $u\in S_c$ is a solution of Eq. (1.1) for some λ_c.

Taking $a=1$ and $b=0$, then Eq. (1.1) reduces to the classical Schrödinger equation.

Cazenave and Lions [11] and Soave [12, 13], Jeanjean et al. [14, 15] considered the following equation,

$$-\Delta u = \lambda u + \mu|u|^{q-2}u + |u|^{p-2}u \ \text{ in } \ \mathbb{R}^N, \tag{1.5}$$

where $N \geq 1$, $\mu \in \mathbb{R}$, p, $q \in (2, 2^*]$, and $2^* := \frac{2N}{(N-2)^+}$ when $N \geq 3$, $2^* := +\infty$ when $N = 1, 2$. Soave [13] raised a question whether there is a second normalized solution to Eq. (1.5) for $p = 2^*$ and $2 < q < 2 + \frac{4}{N}$. It is worth pointing out that Jeanjean and Le [15] solved this question if $N \geq 4$. When dimensions $N = 3$, Wei and Wu used the approach different from [15] and solved the above problem in [16]. Chen and Tang [17] constructed alternative testing functions and found an unified scheme to treat two cases, working for all dimensions $N \geq 3$. In addition, the generalizations and improvements for [13] was carried out in the recent paper [18, 19]. To be more precise, Li [19] removed the restriction on μ and got the ground state normalized solution for Eq. (1.5) when $2 + \frac{4}{N} < q < p = 2^*$. Alves et al. [18] considered the existence of normalized solution with exponential critical growth for $N = 2$.

In [20], when $\mu = 0$, problem (1.4) admits a minimizer if and only if mass is within the appropriate range for $2 < p < \frac{14}{3}$ and no minimizer for $\frac{14}{3} < p < 6$. Moreover, Ye [21] proved the existence of normalized solutions for Kirchhoff equation with general mass subcritical term.

Next, we briefly review the history of the potential case. For Eq. (1.1) with a potential and $\mu = 0$,

$$-\Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\Big)\Delta u + V(x)u + \lambda u = |u|^{p-2}u, \ \text{in } \mathbb{R}^N, \tag{1.6}$$

when the positive potential V satisfies some suitable assumptions and $p \in (2 + \frac{4}{N}, 2^*)$, Mo and Ma stablished the existence of bound state normalized solutions of Eq. (1.6) in [22]. When $N = 3$ and $p \in (\frac{14}{3}, 6)$, Cai and Zhang [23] obtained the existence of mountain pass solutions of Eq. (1.6) with positive energy and the nonexistence of solutions with negative energy for the negative bounded potential $V(x)$. We refer the readers to [24,25] for more general nonlinearities.

Recently, Xie and Chen [26] proved the existence of infinitely many solution for Kirchhoff equation with general mass supercritical term. When $\frac{14}{3} < q < p = 6$, Zhang and Han [27] considered the existence of normalized solutions for problem (1.4) by calculating the threshold of the mountain pass level. Li et al. [28] considered the existence and asymptotic properties of normalized solutions to Eq. (1.1). In addition to studying the existence of solutions to Eq. (1.1) in Sobolev critical cases, they obtained the existence of multiple solutions in the case $2 < q < \frac{10}{3}$, $\frac{14}{3} < p < 6$ by constructing a special Palais-Smale sequence and Ekeland's variational principle. Meanwhile, they also obtained the existence of ground state solution in the case where $\frac{14}{3} < q < p < 6$ by using minimax methods in [12]. Hu and Mao [29] considered the existence of normalized solutions to Eq. (1.1) for $2 < q < \frac{10}{3}$, $2 < p \leq \frac{14}{3}$ and $\frac{14}{3} < q < p < 6$. They showed the existence of normalized solutions for above two cases by minimizing method. In addition, they also discussed how μ, p and q affect the existence of normalized solutions to Eq. (1.1). Li et al. [30] obtained the multiplicity of the normalized solutions for $4 < q < \frac{14}{3} < p = 6$ and $q > 4$ only ensured the corresponding Lagrange multiplier is negative. For $p = 6$, and q in four cases, i.e., $2 < q < \frac{10}{3}$, $q = \frac{10}{3}$, $\frac{10}{3} < q < \frac{14}{3}$, $\frac{14}{3} \le q < 6$, Feng et al. [31] considered the existence and multiplicity of normalized solutions under suitable assumptions on μ and c.

Some of their results on normalized solutions to Eq. (1.1) are summarized in the following table and we present the following pictures for the sake of clarity in the above discussion. Through the above discussion, we know the existence of normalized solutions to Eq. (1.1) with $\frac{10}{3} < q < p \le \frac{14}{3}$ and $\frac{10}{3} < q < \frac{14}{3} < p < 6$ are still unknown, i.e., (I) and (II) two areas in the Figure 1.

p and q	types of solutions	references
$2 < q < \frac{10}{3}, 2 < p \le \frac{14}{3}$, $q < p$	a global minimizer	[21, 29]
$2 < q < 6 = p$	Mountain Pass solution; ground state solution	[27, 28, 31]
$2 < q < \frac{10}{3}, \frac{14}{3} < p < 6$	Mountain Pass solution; ground state solution	[28]
$\frac{14}{3} < q < p < 6$	ground state solution; infinity many solutions	[26, 28]

Table 1 The above summary

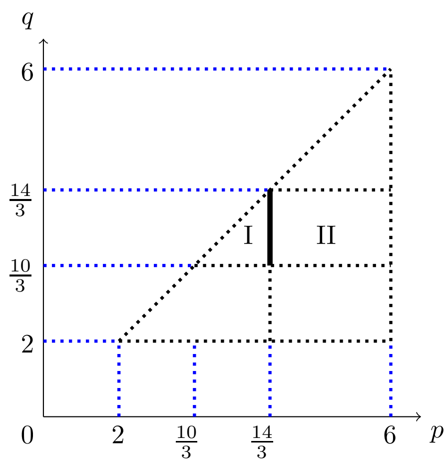

Figure 1 $\mu > 0$

Before presenting our main results, let us first recall the sharp Gagliardo-Nirenberg inequality [32, 33]

$$\|u\|_t^t \le C_t^t \|\nabla u\|_2^{\gamma_t t} \|u\|_2^{(1-\gamma_t)t}, \text{ for all } u \in H^1(\mathbb{R}^3),$$

where $C_t^t := \frac{t}{2\|Q\|_2^{t-2}}$ with $2 < t \le 6$ and $\gamma_t := \frac{3(t-2)}{2t}$. By combining the Pohozaev identity, Q satisfies

$$\|\nabla Q\|_2^2 = \|Q\|_2^2 = \frac{2}{p}\|Q\|_p^p. \tag{1.7}$$

Thanks to the Gagliardo-Nirenberg inequality, it is known that $p = \frac{14}{3}$ is a mass critical exponent for Kirchhoff equation. To be more precise, the functional $E_0(u)$ is bounded from below on manifold S_c for $2 < p < \frac{14}{3}$ and unbounded for $\frac{14}{3} < p < 6$.

Inspired by [29], Hu and Mao considered the existence of normalized solutions to Eq. (1.1) for $2 < q < \frac{10}{3}$, $2 < p \le \frac{14}{3}$. Our first result demonstrates the existence of a normalized solution to Eq. (1.1) in the case that $\frac{10}{3} < q < p \le \frac{14}{3}$, i.e., (I) area in Figure 1.

Theorem 1.1 *If $\frac{10}{3} < q < p \leq \frac{14}{3}$, there exists $\mu_* > 0$ such that the Eq. (1.1) has a normalized solution when $\mu > \mu_*$ and $c \in (0, c_p^*)$, where*

$$c^* := (\frac{b\|Q\|_2^{\frac{8}{3}}}{2})^3, \quad c_p^* = \begin{cases} +\infty, & \text{if } 2 < p < \frac{14}{3}, \\ c^*, & \text{if } p = \frac{14}{3}. \end{cases}$$

For $\frac{10}{3} < q < \frac{14}{3} < p < 6$, i.e., (II) area in Figure 1, $E_\mu(u)$ is unbounded from below on S_c. Hence, it is unlikely to obtain a solution to Eq. (1.1) by minimizing method. However, we adopt some ideas from [18, 28, 31, 34] to overcome the difficulty and obtain the existence of Mountain Pass solution to Eq. (1.1). Exactly, we show the mountain pass geometry of $E_\mu(u)$ on S_c and obtain a compactness lemma from [28, Proposition 3.1].

Theorem 1.2 *Assume that $\frac{10}{3} < q < \frac{14}{3} < p < 6$. Then the Eq. (1.1) has a normalized solution of Mountain Pass type for $\mu > 0$.*

The remaining part of this paper is organized as follows. We present the proof of Theorem 1.1 in Section 2. In Section 3, we shall show the proof of Theorem 1.2.

Notation. From now on in this paper, otherwise mentioned, we use the following notations:

- $\|\cdot\|$ is standard norm in $H^1(\mathbb{R}^3)$.
- $H_r^1(\mathbb{R}^3)$ stands for the space of radially symmetric functions in $H^1(\mathbb{R}^3)$.
- $B_r(u)$ is an open ball centered at u with radius $r > 0$, $B_r = B_r(0)$.
- C, C_1, C_2, ... denote any positive constant, whose value is not relevant.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- $:=$ and $=:$ denote definitions.
- $\rightharpoonup$ and $\to$ denote the weak and strong convergence in the related function space, respectively.

2. The area I

In this section, we consider $\frac{10}{3} < q < p \leq \frac{14}{3}$. In addition, we see that $2 < q\gamma_q < p\gamma_p \leq 4$.

Lemma 2.1 *Let a_i, b_i, c_i, $d_1 > 0$ for $i = 1, 2$ and define f, $g : (0, \infty) \to \mathbb{R}$ by*

$$f(t) = a_1 t + b_1 t^2 - c_1 t^{\tilde{q}} - d_1 t^{\tilde{p}}, \quad g(t) = a_2 t + b_2 t^2 - c_2 t^{\tilde{q}},$$

where $1 < \tilde{q} < \tilde{p} < 2$. Then, f and g have their strict minimum at negative level, when following conditions hold,

$$\frac{1}{a_1}(c_1(\frac{d_1}{b_1})^{\frac{\tilde{q}-1}{2-\tilde{p}}} A_1 + \frac{d_1^{\frac{1}{2-\tilde{p}}}}{b_1^{\frac{\tilde{p}-1}{2-\tilde{p}}}}(A_2 - A_3)) > 1, \quad \frac{1}{a_2}\frac{c_2^{\frac{1}{2-\tilde{q}}}}{b_2^{\frac{\tilde{q}-1}{2-\tilde{q}}}}(B_1 - B_2) > 1$$

and

$$A_1 := [\frac{\tilde{p}(\tilde{p}-1)(\tilde{p}-\tilde{q})}{2(2-\tilde{q})}]^{\frac{\tilde{q}-1}{2-\tilde{p}}}, \quad A_2 := [\frac{\tilde{p}(\tilde{p}-1)(\tilde{p}-\tilde{q})}{2(2-\tilde{q})}]^{\frac{\tilde{p}-1}{2-\tilde{p}}}, \quad A_3 := [\frac{\tilde{p}(\tilde{p}-1)(\tilde{p}-\tilde{q})}{2(2-\tilde{q})}]^{\frac{1}{2-\tilde{p}}},$$

$$B_1 := [\frac{\tilde{q}(\tilde{q}-1)}{2}]^{\frac{\tilde{q}-1}{2-\tilde{q}}}, \quad B_2 := [\frac{\tilde{q}(\tilde{q}-1)}{2}]^{\frac{1}{2-\tilde{q}}}.$$

Proof It is elementary, similar proofs can be found in [28] and we omit the details. □

In order to search minimizers for problem (1.4), we first discuss the coerciveness of E_μ on S_c.

Lemma 2.2 *E_μ is coercive on S_c if one of the following conditions holds.*

(i) $\frac{10}{3} < q < p < \frac{14}{3}$.

(ii) $\frac{10}{3} < q < \frac{14}{3} = p$ *and* $0 < c < c^*$.

Proof (i) If $\frac{10}{3} < q < p < \frac{14}{3}$, by the Gagliardo-Nirenberg inequality, for every $u \in S_c$,

$$\begin{aligned} E_\mu(u) =& \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p}\|u\|_p^p \\ \geq& \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu c^{\frac{q(1-\gamma_q)}{2}}}{q} C_q^q \|\nabla u\|_2^{q\gamma_q} - \frac{c^{\frac{p(1-\gamma_p)}{2}}}{p} C_p^p \|\nabla u\|_2^{p\gamma_p} \to \infty, \end{aligned}$$

as $\|\nabla u\|_2 \to \infty$.

(ii) If $\frac{10}{3} < q < \frac{14}{3} = p$ and $0 < c < c^*$, then for every $u \in S_c$ by the Gagliardo-Nirenberg inequality,

$$E_\mu(u) \geq \frac{a}{2}\|\nabla u\|_2^2 + \Big(\frac{b}{4} - \frac{c^{\frac{1}{3}}}{2\|Q\|_2^{\frac{8}{3}}}\Big)\|\nabla u\|_2^4 - \frac{\mu c^{\frac{q(1-\gamma_q)}{2}}}{q} C_q^q \|\nabla u\|_2^{q\gamma_q} \to \infty,$$

as $\|\nabla u\|_2 \to \infty$. The proof of lemma is completed. □

Lemma 2.3 *Assume that* $\frac{10}{3} < q < p \leq \frac{14}{3}$, *there exists* $\mu_* := \max\{\mu_0, \mu_1\}$, $m(c) < 0$ *for every* $c \in (0, c_p^*)$ *when* $\mu > \mu_*$.

Proof Let

$$u_s(x) = c^{\frac{1}{2}} \frac{s^{\frac{3}{4}} Q(s^{\frac{1}{2}} x)}{\|Q\|_2}, \quad s > 0. \tag{2.1}$$

Then $u_s \in S_c$ and $\|\nabla u_s\|_2^2 = cs$, $\|u_s\|_p^p = \frac{pc^{\frac{p}{2}}}{2\|Q\|_2^{p-2}} s^{\frac{p\gamma_p}{2}}$, $\|u_s\|_q^q = \frac{qc^{\frac{q}{2}}}{2\|Q\|_2^{q-2}} s^{\frac{q\gamma_q}{2}}$. We have

$$\begin{aligned} e_\mu(s) :=& \frac{a}{2}\|\nabla u_s\|_2^2 + \frac{b}{4}\|\nabla u_s\|_2^4 - \frac{\mu}{q}\|u_s\|_q^q - \frac{1}{p}\|u_s\|_p^p \\ =& \frac{a}{2}cs + \frac{b}{4}c^2 s^2 - \frac{\mu c^{\frac{q}{2}}}{2\|Q\|_2^{q-2}} s^{\frac{q\gamma_q}{2}} - \frac{c^{\frac{p}{2}}}{2\|Q\|_2^{p-2}} s^{\frac{p\gamma_p}{2}}. \end{aligned}$$

It suffices to show that there exists $s_0 > 0$ such that $e_\mu(s_0) < 0$, which means that

$$e_\mu(s) = \frac{a}{2}cs + \frac{b}{4}c^2 s^2 - \frac{\mu c^{\frac{q}{2}}}{2\|Q\|_2^{q-2}} s^{\frac{q\gamma_q}{2}} - \frac{c^{\frac{p}{2}}}{2\|Q\|_2^{p-2}} s^{\frac{p\gamma_p}{2}}.$$

(i) If $\frac{10}{3} < q < p < \frac{14}{3}$, by Lemma 2.1, it is clear that there exists μ_0 such that $e_\mu(s)$ attains its negative global minimum when $\mu > \mu_0$. Then, there exists $s_0 > 0$ such that $E_\mu(u_{s_0}) < 0$.

(ii) If $p = \frac{14}{3}$ and $0 < c < c^*$, then

$$e_\mu(s) = \frac{a}{2}cs + \Big(\frac{b}{4} - \frac{c^{\frac{1}{3}}}{2\|Q\|_2^{\frac{8}{3}}}\Big)c^2 s^2 - \frac{\mu c^{\frac{q}{2}}}{2\|Q\|_2^{q-2}} s^{\frac{q\gamma_q}{2}}.$$

We know that there exists μ_1 by Lemma 2.1, and there exists $s_0 > 0$ such that $E_\mu(u_{s_0}) < 0$ when $\mu > \mu_1$. □

Lemma 2.4 *Assume that $\frac{10}{3} < q < p \leq \frac{14}{3}$ and $\mu > \mu_*$. Then $m(c)$ is continuous on $(0, c_p^*)$.*

Proof We need to prove that $m(c_n) \to m(c)$ if $c_n \to c$. By Lemma 2.3, we have $m(c) < 0$ for every $c \in (0, c_p^*)$ and $\mu > \mu_*$. For every $n \in \mathbb{N}$, let $u_n \in S_{c_n}$ such that

$$E_\mu(u_n) < m(c_n) + \frac{1}{n} < \frac{1}{n}.$$

We deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ by Lemma 2.2, and $\{u_n\}$ is bounded in $L^r(\mathbb{R}^3)$ for $r \in [2, 6]$ by Sobolev embedding. Moreover, the fact $\{\frac{c}{c_n} u_n\} \subset S_c$ implies that

$$\begin{aligned} m(c) &\leq E_\mu(\frac{c}{c_n} u_n) \\ &= \frac{a}{2}(\frac{c}{c_n})^2 \|\nabla u_n\|_2^2 + \frac{b}{4}(\frac{c}{c_n})^4 \|\nabla u_n\|_2^4 - \frac{\mu}{q}(\frac{c}{c_n})^q \|u_n\|_q^q - \frac{1}{p}(\frac{c}{c_n})^p \|u_n\|_p^p \\ &= E_\mu(u_n) + o_n(1) < m(c_n) + \frac{1}{n} + o_n(1). \end{aligned}$$

On the other hand, let $\{v_n\} \subset S_c$ be a minimizing sequence for $m(c)$. Then

$$m(c_n) \leq E_\mu(\frac{c_n}{c} v_n) = E_\mu(v_n) + o_n(1) = m(c) + o_n(1).$$

Thus $\lim_{n\to\infty} m(c_n) = m(c)$. □

Lemma 2.5 *Assume that $\frac{10}{3} < q < p \leq \frac{14}{3}$ and $\mu > \mu_*$. Then for every $c \in (0, c_p^*)$, $m(c) < m(\alpha) + m(c - \alpha)$ when $0 < \alpha < c$.*

Proof Let $\{u_n\} \subset S_c$ be a minimizing sequence for $m(c)$. Then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ since E_μ is coercive on S_c. We first claim that there exists a constant $k_1 > 0$ such that $\|\nabla u_n\|_2^2 > k_1$. In fact, if $\|\nabla u_n\|_2 \to 0$, then

$$\begin{aligned} E_\mu(u_n) &= \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{\mu}{q}\|u_n\|_q^q - \frac{1}{p}\|u_n\|_p^p \\ &\geq \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{\mu c^{\frac{q(1-\gamma_q)}{2}}}{q} C_q^q \|\nabla u_n\|_2^{\gamma_q q} - \frac{c^{\frac{p(1-\gamma_p)}{2}}}{p} C_p^p \|\nabla u_n\|_2^{\gamma_p p} \to 0. \end{aligned}$$

Since

$$E_\mu(u_n) \leq \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 \to 0,$$

then $E_\mu(u_n) \to 0$ as $\|\nabla u_n\|_2 \to 0$, which contradicts $m(c) < 0$.

For every $u \in S_c$, let

$$u^\theta(x) = u(\theta^{-\frac{1}{3}} x), \theta > 0.$$

Then $u^\theta, u_n^\theta \in S_{\theta c}$. Let $\theta > 1$ such that $\theta c < c_p^*$. Then

$$\begin{aligned} m(\theta c) \leq E_\mu(u_n^\theta) &= \frac{a}{2}\theta^{\frac{1}{3}} \|\nabla u_n\|_2^2 + \frac{b}{4}\theta^{\frac{2}{3}} \|\nabla u_n\|_2^4 - \frac{\mu}{q}\theta \|u_n\|_q^q - \frac{1}{p}\theta \|u_n\|_p^p \\ &= \theta E_\mu(u_n) + \theta[\frac{a}{2}(\theta^{-\frac{2}{3}} - 1)\|\nabla u_n\|_2^2 + \frac{b}{4}(\theta^{-\frac{1}{3}} - 1)\|\nabla u_n\|_2^4] \\ &\leq \theta E_\mu(u_n) + \theta[\frac{a}{2}(\theta^{-\frac{2}{3}} - 1)k_1 + \frac{b}{4}(\theta^{-\frac{1}{3}} - 1)k_1^2] \\ &< \theta E_\mu(u_n). \end{aligned}$$

Let $n \to +\infty$. We have

$$m(\theta c) < \theta m(c). \tag{2.2}$$

And from (2.2) the conclusion follows as [35, Lemma II.1]. □

Proof of Theorem 1.1 Let $\{u_n\} \subset S_c$ be a minimizing sequence for $m(c)$. Since E_μ is coercive on S_c, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Hence,

$$\delta := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 \mathrm{d}x > 0.$$

Otherwise, by Vanishing Lemma of Lions, $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for $2 < r < 2^*$, and then

$$0 \leq \lim_{n\to\infty} (\frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4) = \lim_{n\to\infty} E_\mu(u_n) = m(c) < 0,$$

a contradiction. Therefore, there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2} > 0.$$

Let $v_n = u_n(x + y_n)$. Then

$$\int_{B_1} |v_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{2.3}$$

Moreover, $\{v_n\} \subset S_c$ is also a bounded minimizing sequence for $m(c)$, and then we may assume that

$$\begin{cases} v_n \rightharpoonup v_0, & \text{in } H^1(\mathbb{R}^3), \\ v_n \to v_0, & \text{in } L^t_{\text{loc}}(\mathbb{R}^3), t \in [1,6), \\ v_n(x) \to v_0(x), & \text{a.e. } x \in \mathbb{R}^3. \end{cases} \tag{2.4}$$

Then (2.3) implies that $v_0 \neq 0$. Hence, $\alpha := \|v_0\|_2^2 \in (0, c]$. We now prove that $\alpha = c$. Suppose that $\alpha < c$, by (2.4), we have

$$c = \|v_n\|_2^2 = \|v_0\|_2^2 + \|v_n - v_0\|_2^2 + o_n(1) \tag{2.5}$$

By Brezis-Lieb Lemma, (2.5) and Lemma 2.4, we see that

$$m(c) = \lim_{n\to\infty} E_\mu(v_n) \geq E_\mu(v_0) + \lim_{n\to\infty} E_\mu(v_n - v_0) \geq m(\alpha) + m(c - \alpha),$$

which contradicts Lemma 2.5. So, $\alpha = c$, namely, $v_0 \in S_c$. Since

$$m(c) \leq E_\mu(v_0) \leq \underline{\lim}_{n\to\infty} E_\mu(v_n) = m(c),$$

$v_0 \in S_c$ is a minimizer of $m(c)$.

3. The area II

Throughout this subsection, we always assume that $\frac{10}{3} < q < \frac{14}{3} < p < 6$ and denote $S_{c,r} := S_c \cap H_r^1(\mathbb{R}^3)$. In addition, we see that $2 < q\gamma_q < 4 < p\gamma_p$.

Lemma 3.1 ([28, Lemma 2.3]) *Let $a \geq 0$, $b > 0$, p, $q \in (2, 6]$ and μ, $\lambda \in \mathbb{R}$. If $u \in H^1(\mathbb{R}^3)$ is a* *weak solution of*

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big)\Delta u=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u,\quad \textit{in } \mathbb{R}^3,$$

then the following Pohozaev identity holds

$$P_\mu(u)=a\|\nabla u\|_2^2+b\|\nabla u\|_2^4-\mu\gamma_q\|u\|_q^q-\gamma_p\|u\|_p^p=0.$$

We introduce the Pohozaev set:

$$\mathcal{P}_{c,\mu}:=\{u\in S_{c,r}:0=P_\mu(u)=a\|\nabla u\|_2^2+b\|\nabla u\|_2^4-\mu\gamma_q\|u\|_q^q-\gamma_p\|u\|_p^p\}.$$

Lemma 3.1 and the elliptic regularity theory imply that $u\in C^2(\mathbb{R}^3)$. Similarly to [36, Appendix B and C], one can show that the critical points of the functional $E_\mu|_{S_{c,r}}$ lie in $\mathcal{P}_{c,\mu}$.

For $u\in S_{c,r}$ and $s\in\mathbb{R}$, we define

$$(s\star u)(x):=e^{\frac{3}{2}s}u(e^sx).$$

Then, $s\star u\in S_c$ and the map $(s,u)\in\mathbb{R}\times H^1(\mathbb{R}^3)\mapsto s\star u\in H^1(\mathbb{R}^3)$ is continuous by [37, Lemma 3.5]. Let $S_{c,r}$ and $\mu\in\mathbb{R}^+$ be fixed. We define the fiber map, for all $s\in\mathbb{R}$,

$$\Psi_u^\mu(s):=E_\mu(s\star u)=\frac{a}{2}e^{2s}\|\nabla u\|_2^2+\frac{b}{4}e^{4s}\|\nabla u\|_2^4-\mu\frac{e^{q\gamma_q s}}{q}\|u\|_q^q-\frac{e^{p\gamma_p s}}{p}\|u\|_p^p.$$

Direct calculation gives

$$(\Psi_u^\mu)'(s)=ae^{2s}\|\nabla u\|_2^2+be^{4s}\|\nabla u\|_2^4-\mu\gamma_qe^{q\gamma_qs}\|u\|_q^q-\gamma_pe^{p\gamma_ps}\|u\|_p^p=P_\mu(s\star u). \tag{3.1}$$

Therefore, $(\Psi_u^\mu)'(s)=0$ if and only if $s\star u\in\mathcal{P}_{c,\mu}$. From (3.1), we see immediately that: $s\in\mathbb{R}$ is a critical point for Ψ_u^μ if and only if $s\star u\in\mathcal{P}_{c,\mu}$.

We shall investigate the mountain pass geometry of E_μ on $S_{c,r}$. Denote $A_k:=A_k(c)=\{u\in S_{c,r}:\|\nabla u\|_2<k\}$, for $k>0$.

Lemma 3.2 *There exist two positive numbers $k_1<k_2$ such that*

(i) $0<\sup_{u\in\overline{A_{k_1}}}E_\mu(u)<\inf_{u\in\partial A_{k_2}}E_\mu(u)$ *and* $E_\mu(u)>0$ *and* $P_\mu(u)>0$*, for* $u\in A_{k_2}$.

(ii) *There exists $u_0\in S_{c,r}\backslash A_{k_2}$ such that $E_\mu(u_0)<0$.*

Proof (i) It follows from Gagliardo-Nirenberg inequality that, for $u\in S_{c,r}$,

$$\begin{aligned}E_\mu(u)&=\frac{a}{2}\|\nabla u\|_2^2+\frac{b}{4}\|\nabla u\|_2^4-\frac{\mu}{q}\|u\|_q^q-\frac{1}{p}\|u\|_p^p\\&\geq\frac{a}{2}\|\nabla u\|_2^2+\frac{b}{4}\|\nabla u\|_2^4-\frac{\mu}{q}c^{\frac{q(1-\gamma_q)}{2}}C_q^q\|\nabla u\|_2^{q\gamma_q}-\frac{c^{\frac{p(1-\gamma_p)}{2}}C_p^p}{p}\|\nabla u\|_2^{p\gamma_p}\end{aligned}$$

and

$$\begin{aligned}P_\mu(u)&=a\|\nabla u\|_2^2+b\|\nabla u\|_2^4-\mu\gamma_q\|u\|_q^q-\gamma_p\|u\|_p^p\\&\geq a\|\nabla u\|_2^2+b\|\nabla u\|_2^4-\mu\gamma_qc^{\frac{q(1-\gamma_q)}{2}}C_q^q\|\nabla u\|_2^{q\gamma_q}-\gamma_pc^{\frac{p(1-\gamma_p)}{2}}C_p^p\|\nabla u\|_2^{p\gamma_p}.\end{aligned}$$

It is also clear that

$$E_\mu(u)\leq\frac{a}{2}\|\nabla u\|_2^2+\frac{b}{4}\|\nabla u\|_2^4.$$

Taking two small positive numbers $k_1 < k_2$, we arrive at the desired result.

(ii) Let $u \in S_{c,r}$. Then $\lim_{s\to+\infty} E_\mu(s \star u) = -\infty$. Choosing $u_0 = s \star u$ with $s > 0$ large enough, we have $u_0 \in S_{c,r} \backslash A_{k_2}$ and $E_\mu(u_0) < 0$. □

By Lemma 3.2, we define the mountain pass level of the functional E_μ on $S_{c,r}$ by

$$m^* := m^*(c,\mu) := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} E_\mu(\gamma(t)),$$

where

$$\Gamma := \Gamma(c,\mu) = \{\gamma \in C([0,1], S_{c,r}) : \gamma(0) \in \overline{A_{k_1}}, E_\mu(\gamma(1)) \le 0\}.$$

Clearly, $m^* \ge \inf_{u\in\partial A_{k_2}} E_\mu(u) > 0$. We also need the following lemma.

Lemma 3.3 *There exists a Palais-Smale sequence $\{u_n\} \subset S_{c,r}$ for $E_\mu|_{S_{c,r}}$ at the level m^* with the following properties $P_\mu(u_n) \to 0$, as $n \to \infty$.*

Proof Inspired by [38], we introduce a stretched functional $\widetilde{E_\mu} : S_{c,r} \times \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{E_\mu}(u,s) := E_\mu(s \star u) = \frac{ae^{2s}}{2}\|\nabla u\|_2^2 + \frac{be^{4s}}{4}\|\nabla u\|_2^4 - \mu\frac{e^{q\gamma_q s}}{q}\|u\|_q^q - \frac{e^{p\gamma_p s}}{p}\|u\|_p^p.$$

Set

$$\widetilde{\Gamma} = \{\tilde\gamma \in C([0,1], S_{c,r} \times \mathbb{R}) : \tilde\gamma(0) \in \overline{A_{k_1}} \times \{0\}, \tilde\gamma(1) \in E_\mu^0 \times \{0\}\},$$

where $E_\mu^0 := \{u \in S_{c,r} : E_\mu(u) \le 0\}$. It is easy to see that if $\gamma \in \Gamma$, then $\tilde\gamma := (\gamma, 0) \in \widetilde{\Gamma}$ and $\widetilde{E_\mu}(\tilde\gamma(s)) = E_\mu(\gamma(s))$ for $s \in [0,1]$; If $\tilde\gamma = (\tilde\gamma_1, s) \in \widetilde{\Gamma}$, then $\gamma(\cdot) := s \star \tilde\gamma_1(\cdot) \in \Gamma$ and $E_\mu(\gamma(s)) = \widetilde{E_\mu}(\tilde\gamma(s))$ for $s \in [0,1]$. Therefore, we have

$$m^* = \inf_{\tilde\gamma\in\tilde\Gamma} \sup_{s\in[0,1]} \widetilde{E_\mu}(\tilde\gamma(s)).$$

By the definition of m^*, for $\varepsilon_n = \frac{1}{n^2}$, there exists $\gamma_n \in \Gamma$ such that

$$\sup_{s\in[0,1]} E_\mu(\gamma_n(s)) \le m^* + \frac{1}{n^2}.$$

Setting $\tilde\gamma_n = (\gamma_n, 0) \in \widetilde{\Gamma}$, we have

$$\sup_{s\in[0,1]} \widetilde{E_\mu}(\tilde\gamma_n(s)) \le m^* + \frac{1}{n^2}.$$

Using Ekeland's variational principle as [38, Lemma 2.3] yields the existence of a sequence $\{(v_n, s_n)\} \subset S_{c,r} \times \mathbb{R}$ such that, as $n \to +\infty$,

$$|s_n| + \mathrm{dist}(v_n, \gamma_n([0,1])) \to 0, \tag{3.2}$$

$$\widetilde{E_\mu}(v_n, s_n) \to m^*, \tag{3.3}$$

$$(\widetilde{E_\mu}|_{S_{c,r}\times\mathbb{R}})'(v_n, s_n) \to 0. \tag{3.4}$$

Note that $\widetilde{E_\mu}(v_n, s_n) = \widetilde{E_\mu}(s_n \star v_n, 0)$ and

$$\langle(\widetilde{E_\mu}|_{S_{c,r}\times\mathbb{R}})'(v_n, s_n), (\varphi, t)\rangle = \langle(\widetilde{E_\mu}|_{S_{c,r}\times\mathbb{R}})'(s_n \star v_n, 0), (s_n \star \varphi, t)\rangle, \tag{3.5}$$

for $(\varphi, t) \in H_r^1(\mathbb{R}^3) \times \mathbb{R}$ with $\int_{\mathbb{R}^3} v_n \varphi = 0$. Setting $u_n = s_n \star v_n \in S_{c,r}$, we see from (3.3) that

$$E_\mu(u_n) = \widetilde{E_\mu}(s_n \star v_n, 0) = \widetilde{E_\mu}(v_n, s_n) \to m^*, \quad \text{as} \quad n \to \infty.$$

Taking $(0,1)$ as a test function in (3.5), we deduce from (3.4) that

$$P_\mu(u_n) = \partial_s \widetilde{E_\mu}(u_n, 0) \to 0, \quad \text{as} \quad n \to \infty.$$

For $w \in H_r^1(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} (s_n \star v_n) w = 0$, we take $((-s_n) \star w, 0)$ as a test function in (3.5) and then deduce from (3.2) and (3.4) that $(E_\mu|_{S_{c,r}})'(u_n) \to 0$ as $n \to \infty$. The proof is completed. □

Lemma 3.4 ([28, Proposition 3.1]) *Let $\{u_n\} \subset S_{c,r}$ be a Palais-Smale sequence for E_μ at energy level $m \neq 0$ with $P_\mu(u_n) \to 0$ as $n \to \infty$. Then up to a subsequence $u_n \to u$ strongly in $H^1(\mathbb{R}^3)$ for some $u \in H^1(\mathbb{R}^3)$. Moreover, $u \in S_c$ and u is a radial solution to Eq.* (1.1) *for some $\lambda < 0$.*

Proof of Theorem 1.2 In view of Lemma 3.3, there exists a sequence $\{u_n\} \subset S_{c,r}$ with the following properties

$$E_\mu(u_n) \to m^*, \ (E_\mu|_{S_{c,r}})'(u_n) \to 0, \ P_\mu(u_n) \to 0, \quad \text{as} \quad n \to \infty.$$

Then, $u_n \to u$ strongly in $H^1(\mathbb{R}^3)$ by Lemma 3.4. The proof is completed. □

Acknowledgements We thank the referees for their time and comments.

References

[1] J. L. LIONS. *On Some Questions in Boundary Value Problems of Mathematical Physics*. North-Holland, Amsterdam-New York, 1978.

[2] C. O. ALVES, F. CORRÊA, T. F. MA. *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*. Comput. Math. Appl., 2005, **49**(1): 85–93.

[3] Xiaoming HE, Wenming ZOU. *Infinitely many positive solutions for Kirchhoff-type problems*. Nonlinear Anal., 2009, **70**(3): 1407–1414.

[4] Xiaoming HE, Wenming ZOU. *Existence and concentration behavior of positive solutions for a Kirchhoff equation in R^3*. J. Differential Equations, 2012, **252**(2): 1813–1834.

[5] Jiu LIU, Jiafeng LIAO, Chunlei TANG. *Positive solutions for Kirchhoff-type equations with critical exponent in R^N*. J. Math. Anal. Appl., 2015, **429**(2): 1153–1172.

[6] Chunyu LEI, Jiafeng LIAO, Chunlei TANG. *Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents*. J. Math. Anal. Appl., 2015, **421**(1): 521–538.

[7] Yinbin DENG, Shuangjie PENG, Wei SHUAI. *Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in R^3*. J. Funct. Anal., 2015, **269**(11): 3500–3527.

[8] Yi HE, Gongbao LI. *Standing waves for a class of Kirchhoff type problems in R^3 involving critical sobolev exponents*. Calc. Var. Partial Differential Equations, 2015, **54**(3): 3067–3106.

[9] Gongbao LI, Hongyu YE. *On the concentration phenomenon of L^2-subcritical constrained minimizers for a class of Kirchhoff equations with potentials*. J. Differential Equations, 2019, **266**(11): 7101–7123.

[10] K. PERERA, Zhitao ZHANG. *Nontrivial solutions of Kirchhoff-type problems via the Yang index*. J. Differential Equations, 2006, **221**(1): 246–255.

[11] T. CAZENAVE, P. L. LIONS. *Orbital stability of standing waves for some nonlinear Schrödinger equations*. Commun. Math. Phys., 1982, **85**(4): 549–561.

[12] N. SOAVE. *Normalized ground states for the NLS equation with combined nonlinearities*. J. Differential Equations, 2020, **269**(9): 6941–6987.

[13] N. SOAVE. *Normalized ground states for the NLS equation with combined nonlinearities: the sobolev critical case*. J. Funct. Anal., 2020, **279**(6): 108610, 43 pp.

[14] L. JEANJEAN, J. JENDREJ, T. T. LE, et al. *Orbital stability of ground states for a sobolev critical Schrödinger equation*. J. Math. Pures Appl. (9), 2022, **164**: 158–179.

[15] L. JEANJEAN, T. T. LE. *Multiple normalized solutions for a sobolev critical Schrödinger equation*. Math. Ann., 2022, **384**(1-2): 101–134.

[16] Juncheng WEI, Yuanze WU. *Normalized solutions for Schrödinger equations with critical sobolev exponent and mixed nonlinearities*. J. Funct. Anal., 2022, **283**(6): 109574, 46 pp.

[17] Sitong CHEN, Xianhua TANG. *New approaches for Schrödinger equations with prescribed mass: the sobolev subcritical case and the sobolev critical case with mixed dispersion*. arXiv:2210.14503, 2022.

[18] C. O. ALVES, Chao JI, O. H. MIYAGAKI. *Normalized solutions for a Schrödinger equation with critical growth in R^N*. Calc. Var. Partial Differential Equations, 2022, **61**(1): Paper No. 18, 24 pp.

[19] Xinfu LI. *Existence of normalized ground states for the sobolev critical Schrödinger equation with combined nonlinearities*. Calc. Var. Partial Differential Equations, 2021, **60**(5): Paper No. 169, 14 pp.

[20] Hongyu YE. *The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations*. Math. Methods Appl. Sci., 2015, **38**(13): 2663–2679.

[21] Hongyu YE, Lina ZHANG. *The existence and nonexistence of global L^2-constrained minimizers for Kirchhoff equations with L^2-subcritical general nonlinearity*. Math. Methods Appl. Sci., 2023, **46**(5): 5234–5244.

[22] Shuai MO, Shiwang MA. *Normalized solutions to Kirchhoff equation with nonnegative potential*. arXiv:2301.07926, 2023.

[23] Li CAI, Fubao ZHANG. *Normalized solutions of mass supercritical Kirchhoff equation with potential*. J. Geom. Anal., 2023, **33**(3): Paper No. 107, 33 pp.

[24] Qun WANG, Aixia QIAN. *Normalized solutions to the Kirchhoff equation with potential term: mass supercritical case*. Bull. Malays. Math. Sci. Soc., 2023, **46**(2): Paper No. 77, 25 pp.

[25] Helin GUO, Yimin ZHANG, Huansong ZHOU. *Blow-up solutions for a Kirchhoff type elliptic equation with trapping potential*. Commun. Pure Appl. Anal., 2018, **17**(5): 1875–1897.

[26] Weihong XIE, Haibo CHEN. *Existence and multiplicity of normalized solutions for the nonlinear Kirchhoff type problems*. Comput. Math. Appl., 2018, **76**(3): 579–591.

[27] Penghui ZHANG, Zhiqing HAN. *Normalized ground states for Kirchhoff equations in R^3 with a critical nonlinearity*. J. Math. Phys., 2022, **63**(2): Paper No. 021505, 15 pp.

[28] Gongbao LI, Xiao LUO, Tao YANG. *Normalized solutions to a class of Kirchhoff equations with sobolev critical exponent*. arXiv:2103.08106, 2021.

[29] Jiaqing HU, Anming MAO. *Normalized solutions to the Kirchhoff equation with a perturbation term*. J. Differential Equations, 2023, **36**(3/4): 289–312.

[30] Quanqing LI, V. RăDULESCU, Jian ZHANG, et al. *Normalized solutions of the autonomous Kirchhoff equation with sobolev critical exponent: sub-and super-critical cases*. Proc. Amer. Math. Soc., 2023, **151**(2): 663–678.

[31] Xiaojun FENG, Haidong LIU, Zhitao ZHANG. *Normalized solutions for Kirchhoff type equations with combined nonlinearities: the sobolev critical case*. Discrete Contin. Dyn. Syst., 2023, **43**(8): 2935–2972.

[32] C. A. STUART. *Bifurcation from the continuous spectrum in the L^2 theory of elliptic equations on R^N*. Recent Methods Nonlinear Anal. Appl., Liguori, Napoli, 1981, 231–300.

[33] M. I. WEINSTEIN. *Nonlinear Schrödinger equations and sharp interpolation estimates*. Commun. Math. Phys., 1982, **87**(4): 567–576.

[34] C. O. ALVES, Chao JI, O. H. MIYAGAKI. *Multiplicity of normalized solutions for a Schrödinger equation with critical growth in R^N*. arXiv:2103.07940, 2021.

[35] P. L. LIONS. *The concentration-compactness principle in the calculus of variations. The locally compact case. I.*. Ann. Inst. H. Poincaré Anal. Non Linéaire., 1984, **1**(2): 109–145.

[36] M. WILLEM. *Minimax Theorems. Progress Nonlinear Differential Equations and Their Applications*. Birkhäuser Boston, 1996.

[37] T. BARTSCH, N. SOAVE. *Multiple normalized solutions for a competing system of Schrödinger equations*. Calc. Var. Partial Differential Equations, 2019, **58**(1): Paper No. 22, 24 pp.

[38] L. JEANJEAN. *Existence of solutions with prescribed norm for semilinear elliptic equations*. Nonlinear Anal., 1997, **28**(10): 1633–1659.