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Gradient Estimate for an Exponentially Harmonic Type Heat Equation on Riemannian Manifolds

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Abstract In this paper, we derive a Hamilton-Souplet-Zhang type gradient estimate for exponentially harmonic type heat equation on Riemannian manifolds. As its application, we obtain a Liouville type theorem.

Keywords Gradient estimate; exponentially harmonic type heat equation; Riemannian manifold; Liouville theorem

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1. Introduction

Let (M^n, g) be a complete Riemannian manifold, and $u \in C^\infty(M)$. We consider the following functional,

$$\mathrm{E}(u) = \int_M \exp\big(\frac{1}{2}|\nabla u|^2\big)\mathrm{d}v, \tag{1.1}$$

where $|\nabla u|^2 = g^{ij}\nabla_i u \nabla_j u$, $(g^{ij}) = (g_{ij})^{-1}$, and $\mathrm{d}v$ is volume form of M.

The Euler-Lagrange equation of (1.1) is

$$\mathrm{div}\{\exp\big(\frac{1}{2}|\nabla u|^2\big)\nabla u\} = 0, \tag{1.2}$$

where div is divergence operator of (M^n, g).

If $u \in C^\infty(M)$ and u is the solution to (1.2), we call u an exponentially harmonic function on M.

After calculation, u is the solution to the equation (1.2) if and only if u satisfies the following equation

$$\triangle u + \frac{1}{2}\langle \nabla|\nabla u|^2, \nabla u\rangle = 0. \tag{1.3}$$

So if u is the solution to the equation (1.3), we also call u an exponentially harmonic function.

In [1], Yau first proved the Liouville type theorem for a positive harmonic function on complete Riemannian manifolds with nonnegative Ricci curvature by gradient estimate of harmonic function. It is a natural question if a similar Liouville type theorem for exponentially harmonic functions on Riemannian manifolds is also true.

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In [2], Hong partly answered the question. He got the following theorem by gradient estimate.

Theorem 1.1 ([2]) *Let (M^n, g) be a Riemannian manifold with nonnegative sectional curvature. Then any bounded exponentially harmonic function on M has to be constant.*

In [3], using gradient estimate, Wu, Ruan and Yang improved Theorem 1.1.

Theorem 1.2 ([3]) *Let (M^n, g) be a Riemannian manifold with nonnegative Ricci curvature and sectional curvature $K_M \geq -k^2$ for some constant $k \geq 0$. Then any bounded exponentially harmonic function on M has to be constant.*

In this paper, we consider smooth solutions to the parabolic analog of (1.3), that is,

$$\frac{\partial u}{\partial t} = \triangle u + \frac{1}{2}\langle \nabla |\nabla u|^2, \nabla u\rangle, \tag{1.4}$$

where $\triangle$ is the Laplace-Beltrami operator of g and ∇ is the gradient operator of g.

We will derive a Hamilton-Souplet-Zhang type gradient estimate for solutions of (1.4). As its application, we obtain a Liouville type theorem.

Now let $f = u^{\frac{5}{6}}$, using (1.4), we have

$$\frac{\partial f}{\partial t} = \triangle f + \frac{18}{25} f^{\frac{2}{5}} \langle \nabla |\nabla f|^2, \nabla f\rangle + \frac{1}{5}\frac{|\nabla f|^2}{f} + \frac{36}{125} f^{-\frac{3}{5}} |\nabla f|^4. \tag{1.5}$$

Theorem 1.3 *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$ with* $\mathrm{Ric}(M) \geq -(n-1)k_1$ *for some constant $k_1 \geq 0$ and sectional curvature $K_M \geq -k_2^2$ for some constant $k_2 \geq 0$. Suppose that $f(x,t)$ is any positive solution to the equation (1.5) in*

$$Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty).$$

Suppose also that $0 < f \leq \mathcal{D}$ in $Q_{R,T}$, and $\mathcal{D} \geq 1$. Then there exists a constant $c = c(n)$ such that

$$|\nabla f| \leq c\mathcal{D}\Big(\frac{1}{R} + \frac{1}{\sqrt{R}} + \frac{1}{\sqrt[3]{R}} + \frac{\sqrt{k_2}}{\sqrt{R}} + \sqrt[4]{k_1} + \frac{1}{\sqrt[4]{T}}\Big) \tag{1.6}$$

in $Q_{\frac{R}{2},\frac{T}{2}}$.

Using maximum principle, we prove the above result. Our method draws from the papers [4–8].

By the theorem, we get the following Liouville type theorem.

Corollary 1.4 *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$ with* $\mathrm{Ric}(M) \geq 0$ *and sectional curvature $K_M \geq -k_2^2$ for some constant $k_2 \geq 0$. Suppose that u is a positive ancient solution to the equation (1.4) and $u = o([\sqrt[3]{d(x)} + \sqrt[4]{|t|}]^{\frac{6}{5}})$ near infinity, where $d(x)$ is the distance function from a fixed point. Then u is a constant.*

Remark 1.5 This result is a parabolic generalization of Theorem 1.2. In fact, if u is a bounded solution of the equation (1.3), we have $u_t = 0$, then we can apply this corollary and obtain that u is a constant.

Remark 1.6 From Remark 2.2 and the proof of Theorem 1.3, the growth rate of u depends on the ε.

Exponentially harmonic function is a special exponentially harmonic map. In 1990, exponentially harmonic maps were first explored by Eells and Lemaire [9]. In the past 30 years, exponentially harmonic maps have been thoroughly studied (see [10, 11] and their references). We believe our method is useful to study exponentially harmonic map.

2. Proof of Theorem 1.3

In order to prove Theorem 1.3, we need a lemma.

Lemma 2.1 *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$ with $\mathrm{Ric}(M) \geq -(n-1)k_1$ for some constant $k_1 \geq 0$. Then we have*

$$A_{ij}\omega_{ij} - \omega_t \geq -2(n-1)k_1\omega - \frac{216}{125}f^{-\frac{3}{5}}\omega\langle\nabla\omega, \nabla f\rangle - \frac{18}{25}f^{\frac{2}{5}}\langle\nabla\omega, \nabla\omega\rangle - \frac{2}{5}f^{-1}\langle\nabla\omega, \nabla f\rangle + \frac{1}{3}f^{-\frac{8}{5}}\omega^3, \tag{2.1}$$

where $A_{ij} = \delta_{ij} + u_iu_j = \delta_{ij} + \frac{36}{25}f^{\frac{2}{5}}f_if_j$ and $\omega = |\nabla f|^2$.

Remark 2.2 Let $f = u^\varepsilon$, and $\frac{2}{3} < \varepsilon < 1$. We also get a similar inequality of (2.1). And the last term is replaced by

$$\frac{-6\varepsilon^2 + 10\varepsilon - 4}{\varepsilon^4}\frac{\omega^3}{f^{4-2/\varepsilon}}.$$

Proof For $\omega = |\nabla f|^2$, using (1.5), we have

$$\omega_t = 2\langle\nabla f,\ \nabla f_t\rangle = 2\langle\nabla f,\ \nabla[\triangle f + \frac{18}{25}f^{\frac{2}{5}}\langle\nabla|\nabla f|^2, \nabla f\rangle + \frac{1}{5}f^{-1}|\nabla f|^2 + \frac{36}{125}f^{-\frac{3}{5}}|\nabla f|^4]\rangle. \tag{2.2}$$

Using local orthonormal system, we have

$$\omega_t = 2\langle\nabla f, \nabla\triangle f\rangle + \frac{216}{125}\frac{|\nabla f|^2}{f^{3/5}}\langle\nabla|\nabla f|^2, \nabla f\rangle + \frac{36}{25}f^{\frac{2}{5}}\langle\nabla|\nabla f|^2, \nabla|\nabla f|^2\rangle + \frac{72}{25}f^{\frac{2}{5}}f_if_jf_kf_{ijk} + \frac{2}{5}f^{-1}\langle\nabla|\nabla f|^2, \nabla f\rangle - \frac{2}{5}\frac{|\nabla f|^4}{f^2} - \frac{72}{125} \times \frac{3}{5}\frac{|\nabla f|^6}{f^{8/5}}. \tag{2.3}$$

Now we calculate ω_j and ω_{ij}.

$$\omega_j = 2f_if_{ij}, \quad \omega_{ij} = 2f_{ki}f_{kj} + 2f_kf_{kij}. \tag{2.4}$$

And using

$$A_{ij} = \delta_{ij} + u_iu_j = \delta_{ij} + \frac{36}{25}f^{\frac{2}{5}}f_if_j,$$

we have

$$\begin{aligned} A_{ij}\omega_{ij} &= (\delta_{ij} + \frac{36}{25}f^{\frac{2}{5}}f_if_j)(2f_{ki}f_{kj} + 2f_kf_{kij}) \\ &= 2f_{ki}^2 + 2f_kf_{kii} + \frac{72}{25}f^{\frac{2}{5}}f_if_jf_{kj}f_{ki} + \frac{72}{25}f^{\frac{2}{5}}f_if_jf_kf_{kij} \\ &= 2|D^2f|^2 + 2f_kf_{kii} + \frac{18}{25}f^{\frac{2}{5}}\langle\nabla|\nabla f|^2, \nabla|\nabla f|^2\rangle + \frac{72}{25}f^{\frac{2}{5}}f_if_jf_kf_{kij}. \end{aligned} \tag{2.5}$$

Using (2.3) and (2.5), we have

$$\begin{aligned} A_{ij}\omega_{ij}-\omega_t=&2|D^2f|^2+2f_k(f_{kii}-f_{iik})-\frac{216}{125}\frac{|\nabla f|^2}{f^{3/5}}\langle\nabla|\nabla f|^2,\nabla f\rangle- \\ &\frac{18}{25}f^{\frac{2}{5}}\langle\nabla|\nabla f|^2,\nabla|\nabla f|^2\rangle-\frac{2}{5}f^{-1}\langle\nabla|\nabla f|^2,\nabla f\rangle+\frac{2}{5}\frac{|\nabla f|^4}{f^2}+\frac{216}{625}\frac{|\nabla f|^6}{f^{8/5}} \\ \geq&2f_kf_lR_{kl}-\frac{216}{125}\frac{\omega}{f^{3/5}}\langle\nabla\omega,\nabla f\rangle-\frac{18}{25}f^{\frac{2}{5}}\langle\nabla\omega,\nabla\omega\rangle-\frac{2}{5}f^{-1}\langle\nabla\omega,\nabla f\rangle+\frac{216}{625}\frac{\omega^3}{f^{8/5}} \\ \geq&-2(n-1)k_1\omega-\frac{216}{125}\frac{\omega}{f^{3/5}}\langle\nabla\omega,\nabla f\rangle-\frac{18}{25}f^{\frac{2}{5}}\langle\nabla\omega,\nabla\omega\rangle- \\ &\frac{2}{5}f^{-1}\langle\nabla\omega,\nabla f\rangle+\frac{216}{625}\frac{\omega^3}{f^{8/5}}, \end{aligned} \tag{2.6}$$

where we have used

$$f_{kii}-f_{iik}=R_{kl}f_l,\ \omega=|\nabla f|^2 \text{ and } \mathrm{Ric}(M)\geq-(n-1)k_1.$$

For (2.6) and $\frac{216}{625}>\frac{1}{3}$, so we arrive at the inequality (2.1). This completes the proof. □

As in [4], we will use the well-known cut-off function of Li-Yau (or see [12]) to get our result. Now we give the proof of Theorem 1.3.

Proof Let $\psi=\psi(x,t)$ be a smooth cut-off function supported in $Q_{R,T}$, satisfying the following properties.

(1) $\psi=\psi(d(x,x_0),t)\equiv\psi(r,t)$; $\psi(r,t)=1$ in $Q_{R/2,T/2}, 0\leq\psi\leq1$.

(2) ψ is a decreasing as a radial function in the spatial variables.

(3) $|\frac{\partial\psi}{\partial r}|\frac{1}{\psi^a}\leq\frac{C_a}{R}$ and $|\frac{\partial^2\psi}{\partial r^2}|\frac{1}{\psi^a}\leq\frac{C_a}{R^2}$, when $0<a<1$.

(4) $|\frac{\partial\psi}{\partial t}|\frac{1}{\psi^{1/3}}\leq\frac{C}{T}$.

Using Lemma 2.1, by a straightforward calculation, we can get

$$\begin{aligned} A_{ij}(\psi\omega)_{ij}-(\psi\omega)_t\geq&A_{ij}\psi_{ij}\omega+2A_{ij}\psi_i\omega_j-\psi_t\omega-2(n-1)k_1\psi\omega- \\ &\frac{216}{125}f^{-\frac{3}{5}}\omega\langle\psi\nabla\omega,\nabla f\rangle-\frac{18}{25}f^{\frac{2}{5}}\langle\psi\nabla\omega,\nabla\omega\rangle- \\ &\frac{2}{5}f^{-1}\langle\psi\nabla\omega,\nabla f\rangle+\frac{1}{3}f^{-\frac{8}{5}}\psi\omega^3. \end{aligned} \tag{2.7}$$

So suppose that the maximum of ψw is reached at (x_1,t_1). As in [12], we can assume, without loss of generality, that x_1 is not on the cut-locus of x_0. For the matrix A_{ij} is positive definite matrix, so at (x_1,t_1), we have $A_{ij}(\psi\omega)_{ij}\leq0$, $(\psi\omega)_t\geq0$ and $\nabla(\psi\omega)=0$. Therefore, using (2.7), at (x_1,t_1), we can get

$$\begin{aligned} (2\psi\omega^3)(x_1,t_1)\leq&\big(-6f^{\frac{8}{5}}A_{ij}\psi_{ij}\omega-12f^{\frac{8}{5}}A_{ij}\psi_i\omega_j+6f^{\frac{8}{5}}\psi_t\omega+12(n-1)k_1f^{\frac{8}{5}}\psi\omega+ \\ &\frac{216\times6}{125}f\omega\langle\psi\nabla\omega,\nabla f\rangle+\frac{108}{25}f^2\langle\psi\nabla\omega,\nabla\omega\rangle+\frac{12}{5}f^{\frac{3}{5}}\langle\psi\nabla\omega,\nabla f\rangle\big) \\ =&\big(-6f^{\frac{8}{5}}A_{ij}\psi_{ij}\omega-12f^{\frac{8}{5}}A_{ij}\psi_i\omega_j+6f^{\frac{8}{5}}\psi_t\omega+12(n-1)k_1f^{\frac{8}{5}}\psi\omega- \\ &\frac{216\times6}{125}f\omega^2\langle\nabla\psi,\nabla f\rangle+\frac{108}{25}f^2\omega^2\psi^{-1}\langle\nabla\psi,\nabla\psi\rangle-\frac{12}{5}f^{\frac{3}{5}}\omega\langle\nabla\psi,\nabla f\rangle\big). \end{aligned} \tag{2.8}$$

Now we need to find an upper bound for each term of the right-hand side of (2.8).

For the first term on the right-hand side of (2.8), we need the Laplacian comparison theorem [12], that is, if $\mathrm{Ric}(M) \geq -(n-1)k_1$, then

$$\triangle r \leq \frac{n-1}{r}(1+\sqrt{k_1}r).$$

And Hessian comparison theorem [13], that is, if $K_M \geq -k_2^2$, then

$$r_{ij} \leq \frac{1}{r}(1+k_2 r)g_{ij}.$$

And then using the properties of ψ, we have

$$\begin{aligned}\triangle\psi &= (\frac{\partial\psi}{\partial r})\triangle r + (\frac{\partial^2\psi}{\partial r^2})|\nabla r|^2 \geq (\frac{\partial\psi}{\partial r})\frac{n-1}{R}(2+\sqrt{k_1}R) + (\frac{\partial^2\psi}{\partial r^2})|\nabla r|^2\\ &\geq -\psi^{1/3}c(n)(\frac{1}{R^2}+\frac{\sqrt{k_1}}{R})\end{aligned}$$

and

$$\begin{aligned}\psi_{11} &= (\frac{\partial\psi}{\partial r})r_{11} + (\frac{\partial^2\psi}{\partial r^2})(r_1)^2 \geq (\frac{\partial\psi}{\partial r})\frac{1}{R}(2+\sqrt{k_2}R) + (\frac{\partial^2\psi}{\partial r^2})(r_1)^2\\ &\geq -\psi^{2/3}c(\frac{1}{R^2}+\frac{\sqrt{k_2}}{R}).\end{aligned}$$

Therefore,

$$\begin{aligned}-6f^{\frac{8}{5}}A_{ij}\psi_{ij}\omega &= -6f^{\frac{8}{5}}\omega\triangle\psi - \frac{216}{25}f^2\omega^2\psi_{11}\\ &\leq c(n)f^{\frac{8}{5}}(\frac{1}{R^2}+\frac{\sqrt{k_1}}{R})\psi^{1/3}\omega + cf^2(\frac{1}{R^2}+\frac{\sqrt{k_2}}{R})\psi^{2/3}\omega^2\\ &\leq c(n)\frac{\mathcal{D}^{12/5}}{R^3} + c(n)\mathcal{D}^{12/5}\frac{(k_1)^{3/4}}{R^{3/2}} + c\frac{\mathcal{D}^6}{R^6} + c\mathcal{D}^6\frac{(k_2)^3}{R^3} + \frac{2}{9}\psi\omega^3. \qquad (2.9)\end{aligned}$$

For the second term on the right-hand side of (2.8), one has

$$\begin{aligned}-12f^{\frac{8}{5}}A_{ij}\psi_i\omega_j &= -12f^{\frac{8}{5}}(\delta_{ij}+\frac{36}{25}f^{2/5}f_if_j)\psi_i\omega_j\\ &= -12f^{8/5}(\langle\nabla\psi,\nabla\omega\rangle + \frac{36}{25}f^{2/5}\langle\nabla\psi,\nabla f\rangle\langle\nabla f,\nabla\omega\rangle)\\ &\leq 12\mathcal{D}^{8/5}\frac{|\nabla\psi|^2}{\psi^{4/3}}\psi^{1/3}\omega + \frac{36\times 12}{25}\mathcal{D}^2\frac{|\nabla\psi|^2}{\psi^{5/3}}\psi^{2/3}\omega^2\\ &\leq c\mathcal{D}^{12/5}\frac{1}{R^3} + c\mathcal{D}^6\frac{1}{R^6} + \frac{2}{9}\psi\omega^3. \qquad (2.10)\end{aligned}$$

For the third term on the right-hand side of (2.8), we obtain

$$6f^{8/5}\psi_t\omega \leq 6\mathcal{D}^{8/5}|\frac{\partial\psi}{\partial t}|\frac{1}{\psi^{1/3}}\psi^{1/3}\omega \leq c\frac{\mathcal{D}^{12/5}}{T^{3/2}} + \frac{1}{9}\psi\omega^3. \qquad (2.11)$$

Now we estimate $12(n-1)k_1f^{\frac{8}{5}}\psi\omega$ of (2.8),

$$12(n-1)k_1f^{8/5}\psi\omega \leq 12(n-1)k_1\mathcal{D}^{8/5}\psi^{1/3}\omega \leq c(n)k_1^{3/2}\mathcal{D}^{12/5} + \frac{1}{9}\psi\omega^3. \qquad (2.12)$$

For the fifth term of (2.8), we can get

$$-\frac{216\times 6}{125}f\omega^2\langle\nabla\psi,\nabla f\rangle \leq \frac{216\times 6}{125}f\omega^{5/2}|\nabla\psi| \leq c\mathcal{D}\frac{|\partial_r\psi|}{\psi^{5/6}}\psi^{5/6}\omega^{5/2} \leq c\frac{\mathcal{D}^6}{R^6} + \frac{1}{9}\psi\omega^3. \qquad (2.13)$$

For the term $\frac{108}{25}f^2\omega^2\psi^{-1}\langle\nabla\psi,\nabla\psi\rangle$ of (2.8), we obtain

$$\frac{108}{25}f^2\omega^2\psi^{-1}\langle\nabla\psi,\nabla\psi\rangle\le c\mathcal{D}^2\frac{|\nabla\psi|^2}{\psi^{5/3}}\psi^{2/3}\omega^2\le c\mathcal{D}^2\frac{|\frac{\partial}{\partial r}\psi|^2}{\psi^{5/3}}\psi^{2/3}\omega^2\le c\mathcal{D}^6\frac{1}{R^6}+\frac{1}{9}\psi\omega^3. \tag{2.14}$$

For the last term on the right-hand side of (2.8), we have

$$-\frac{12}{5}f^{\frac{3}{5}}\omega\langle\nabla\psi,\nabla f\rangle\le\frac{12}{5}\mathcal{D}^{3/5}\frac{|\frac{\partial}{\partial r}\psi|}{\psi^{1/2}}\psi^{1/2}\omega^{3/2}\le c\frac{\mathcal{D}^{6/5}}{R^2}+\frac{1}{9}\psi\omega^3. \tag{2.15}$$

Substituting (2.9)–(2.15) in the right-hand side of (2.8), we can get

$$\begin{aligned}2\psi\omega^3(x_1,t_1)\le&\psi\omega^3(x_1,t_1)+c(n)\frac{\mathcal{D}^{12/5}}{R^3}+c(n)\mathcal{D}^{12/5}\frac{(k_1)^{3/4}}{R^{3/2}}+\\&c\frac{\mathcal{D}^6}{R^6}+c\mathcal{D}^6\frac{(k_2)^3}{R^3}+c\frac{\mathcal{D}^{6/5}}{R^2}+c(n)k_1^{3/2}\mathcal{D}^{12/5}+c\frac{\mathcal{D}^{12/5}}{T^{3/2}}.\end{aligned} \tag{2.16}$$

Therefore, we obtain

$$\begin{aligned}\psi\omega^3(x_1,t_1)&\le c(n)\mathcal{D}^6\Big(\frac{1}{R^3}+\frac{(k_1)^{3/4}}{R^{3/2}}+\frac{1}{R^6}+\frac{(k_2)^3}{R^3}+\frac{1}{R^2}+k_1^{3/2}+\frac{1}{T^{3/2}}\Big)\\&\le c(n)\mathcal{D}^6\Big(\frac{1}{R^2}+\frac{1}{R^3}+\frac{1}{R^6}+\frac{(k_2)^3}{R^3}+k_1^{3/2}+\frac{1}{T^{3/2}}\Big),\end{aligned} \tag{2.17}$$

where we have used $\mathcal{D}\ge 1$.

Then, for all $(x,t)\in Q_{R,T}$, we have

$$\begin{aligned}\psi^3\omega^3(x,t)&\le\psi^3\omega^3(x_1,t_1)\le\psi\omega^3(x_1,t_1)\\&\le c(n)\mathcal{D}^6\Big(\frac{1}{R^6}+\frac{1}{R^3}+\frac{1}{R^2}+\frac{(k_2)^3}{R^3}+k_1^{3/2}+\frac{1}{T^{3/2}}\Big).\end{aligned}$$

Notice that $\psi(x,t)=1$ in $Q_{R/2,T/2}$ and $w=|\nabla f|^2$. We have

$$|\nabla f|\le c(n)\mathcal{D}\Big(\frac{1}{R}+\frac{1}{\sqrt{R}}+\frac{1}{\sqrt[3]{R}}+\frac{\sqrt{k_2}}{\sqrt{R}}+\sqrt[4]{k_1}+\frac{1}{\sqrt[4]{T}}\Big).$$

So we arrive at the inequality (1.4). This ends the proof. □

By the local estimate, we deduce the proof of Corollary 1.4.

Proof Suppose that u is a positive ancient solution to the equation (1.1) such that

$$u(x,t)=o([\sqrt[3]{d(x,x_0)}+\sqrt[4]{|t|}]^{\frac{6}{5}})$$

near infinity, then f is a positive ancient solution to the equation (1.3) such that

$$f(x,t)=o([\sqrt[3]{d(x,x_0)}+\sqrt[4]{|t|}])$$

near infinity.

Fixing $(x_0,\ t_0)$ in space-time and using Theorem 1.1 for f on the cube $B(x_0,\ R)\times[t_0-R^{\frac{4}{3}},t_0]$, we have

$$|\nabla f|(x_0,t_0)\le c(n)\Big(\frac{o(\sqrt[3]{R})}{\sqrt[3]{R}}\frac{1}{\sqrt[3]{R^2}}+\frac{o(\sqrt[3]{R})}{\sqrt[3]{R}}+(1+\sqrt{k_2})\frac{o(\sqrt[3]{R})}{\sqrt[3]{R}}\frac{1}{\sqrt[6]{R}}\Big).$$

Letting $R\to\infty$, it follows that $|\nabla f|(x_0,t_0)=0$. Since $(x_0,\ t_0)$ is arbitrary, f is a constant. Notice that $f=u^{\frac{5}{6}}$, so u is a constant. This completes the proof. □

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