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On Ideal-Derived-Set Mappings in Uniform Spaces

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Abstract In this paper, we introduce some ideal-derived-set mappings in uniform spaces and investigate the images of those mappings. Meanwhile, we introduce a concept of $\mathcal{I}$-Hurewicz boundedness, study some basic topological operations of them. Finally, we obtain an equivalent characterization of $\mathcal{I}$-Hurewicz boundedness in uniform spaces.

Keywords uniform spaces; mappings; $\mathcal{I}$-cluster points; $\mathcal{I}$-Hurewicz boundedness

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1. Introduction

Convergence of sequences in a topological space is a basic and important concept in mathematics [1, 2]. In addition to the usual convergence of sequences, statistical convergence [3–5] and ideal convergence [6–8] have attracted extensive attention. In particular, the notion of ideals is a very useful in topology [6], analysis [3] and set theory [9], and have been studied for a long time.

Uniform spaces introduced by Weil [10] can be regarded as a type of space between topological spaces and metric spaces. One used additional structures to define uniform properties such as completeness, uniform continuity, and uniform convergence in uniform spaces. Maio and Kočinac [4] considered statistical convergence in topological spaces and uniform spaces and showed that this convergence can be applied to selection principles theory, function spaces and hyperspaces. Bilalov and Nazarova [11] considered statistical Cauchy sequences in uniform spaces and showed that if a uniform space $(X,\mathcal{U})$ is sequentially complete with a countable base, then each $\mathcal{I}$-Cauchy sequence in $(X,\mathcal{U})$ is $\mathcal{I}$-convergent. As we know, statistical convergence is a special $\mathcal{I}$-convergence [12]. Are some of conclusions in [4] valid in the sense of ideal convergence?

On this basis, we introduce some ideal-derived-set mappings in uniform spaces and investigate the images of those mappings. Meanwhile, we introduce a concept of $\mathcal{I}$-Hurewicz boundedness, study some basic topological operations and obtain an equivalent characterization of $\mathcal{I}$-Hurewicz boundedness in uniform spaces. Firstly, these results generalize some results on statistical convergence in uniform spaces [4, 11]. Secondly, we draw into $\mathcal{I}$-Hurewicz boundedness in uniform

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spaces which greatly enrich the applications of ideal convergence. Thirdly, these results prompt the study of the relationship between spaces and mappings.

In this paper, the set of all positive integers is denoted by $\mathbb{N}$. If no otherwise specified, we consider all mappings are surjection, all topological spaces are supposed to be Hausdorff.

2. Preliminaries

In this section we recall some concepts, notations and terminologies which will be used in the sequel. The readers may refer to [1, 2, 13] for the concepts, notations and terminologies not explicitly given here. The concept of $\mathcal{I}$-convergence in topological spaces is a generalization of statistical convergence and the usual convergence, which is based on the ideal of subsets of the set $\mathbb{N}$ of all positive integers. Let $\mathcal{A} = 2^{\mathbb{N}}$ be the family of all subsets of $\mathbb{N}$. An ideal $\mathcal{I} \subseteq \mathcal{A}$ is a hereditary family of subsets of $\mathbb{N}$ which is downward and stable under finite unions [14, p. 670], i.e., the following are satisfied: if $B \subset A \in \mathcal{I}$, then $B \in \mathcal{I}$; if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal $\mathcal{I}$ is said to be non-trivial, if $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I} \subseteq \mathcal{A}$ is called admissible if $\mathcal{I} \supseteq \{\{n\} : n \in \mathbb{N}\}$. The family of all finite subsets of $\mathbb{N}$ is denoted by $\mathcal{I}_{\text{fin}}$, which is the smallest non-trivial ideal contained in each admissible ideal. Next, we consider the case that $\mathcal{I}$ is an admissible ideal on $\mathbb{N}$.

Recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X is said to be $\mathcal{I}$-convergent to a point $x \in X$ provided for any neighborhood U of x, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$, which is denoted by $x_n \xrightarrow{\mathcal{I}} x$, and the point x is called the $\mathcal{I}$-limit point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ (see [14, Definition 3.1]). A point x is called an $\mathcal{I}$-cluster point of a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X if for each neighborhood U of x, the set $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$. Denote by $\mathcal{I}(x_n)$ the set of all $\mathcal{I}$-cluster points of a sequence $\{x_n\}_{n\in\mathbb{N}}$ (see [15, Definition 6]).

Let X be a nonempty set and A, B be subsets of $X \times X = X^2$. The set $\triangle = \{(x, x) : x \in X\}$ is called a diagonal of X^2. Denote by $A^{-1} = \{(y, x) : (x, y) \in A\}$, $A \circ B = \{(x, y) :$ there is $z \in X$ such that $(x, z) \in A$ and $(z, y) \in B\}$, $A[x] = \{y \in X : (x, y) \in A\}$. $A \circ A$ is also denoted by A^2. A is called symmetric if $A = A^{-1}$. For $A \subseteq X$, $U \subseteq X^2$, denote $U[A] = \{y \in X$: there is $x \in A$ such that $(x, y) \in U\}$.

Definition 2.1 ([16, Definition 4.1.1]) *Uniformity on the set X is a non-empty family $\mathcal{U}$, satisfying the following conditions.*

(1) If $U \in \mathcal{U}$, then $\triangle \subseteq U$;

(2) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(3) If $U \in \mathcal{U}$, then there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$;

(4) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;

(5) If $U \in \mathcal{U}$ and $U \subseteq V \subseteq X^2$, then $V \in \mathcal{U}$.

Pair $(X, \mathcal{U})$ is called a uniform space. Subfamily $\mathcal{B} \subseteq \mathcal{U}$ of the uniformity $\mathcal{U}$ is called its base if for each $U \in \mathcal{U}$, there is $B \in \mathcal{B}$ such that $B \subseteq U$.

Definition 2.2 ([16, Definition 4.1.13]) *Let $(X,\mathcal{U})$ be a uniform space. Put*

$$\tau_{\mathcal{U}} = \{G \subseteq X : \text{ for arbitrary } x \in G, \text{ there is } U \in \mathcal{U} \text{ such that } U[x] \subseteq G\}.$$

Then $\tau_{\mathcal{U}}$ is a topology on X induced by the uniformity $\mathcal{U}$ and $\tau_{\mathcal{U}}$ is called a uniform topology on X.

Put $\mathcal{U}_x = \{U[x] : U \in \mathcal{U}\}$, then $\mathcal{U}_x$ is a local base at x in $(X, \tau_{\mathcal{U}})$ (see [16, p. 122]).

Definition 2.3 ([16, p. 121]) *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two uniform spaces. A mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is said to be uniformly continuous with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$ if for each $F \in \mathcal{V}$, there is $M \in \mathcal{U}$ such that $\phi(M) \subseteq F$, where $\phi : X^2 \to Y^2$ is defined by $\phi(x, y) = (f(x), f(y))$.*

It is easy to see that $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous $\Leftrightarrow$ for each $F \in \mathcal{V}$, we have $\phi^{-1}(F) \in \mathcal{U} \Leftrightarrow$ for each $F \in \mathcal{V}$, we have the set $\{(x, z) \in X^2 : \phi(x, z) \in F\} \in \mathcal{U}$ (see [16, p. 122]); if $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous, then it is continuous [16, Lemma 4.1.4].

Lemma 2.4 ([15, Theorem 10]) *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then $\mathcal{I}(x_n)$ is closed for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X.*

Lemma 2.5 *Let K be a compact subset of a space X. Then for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\{n \in \mathbb{N} : x_n \in K\} \notin \mathcal{I}$, we have $K \cap \mathcal{I}(x_n) \neq \emptyset$.*

Proof Suppose, to the contrary, that $K \cap \mathcal{I}(x_n) = \emptyset$. Then for each $y \in K$, there is a neighborhood V_y of y such that $M_y = \{n \in \mathbb{N} : x_n \in V_y\} \in \mathcal{I}$. From the open cover $\{V_y : y \in K\}$ of K, there exists a finite subcover $\{V_{y_1}, V_{y_2}, \ldots, V_{y_j}\}$. Since $\{n \in \mathbb{N} : x_n \in K\} \subseteq M_{y_1} \cup M_{y_2} \cup \cdots \cup M_{y_j}$, it follows that $\{n \in \mathbb{N} : x_n \in K\} \in \mathcal{I}$. This is a contradiction. $\square$

3. Some mappings using ideals in uniform spaces

In this section, we consider some ideal-derived-set mappings in uniform spaces. Especially, the images of those mappings in $\mathcal{I}(x_n)$. Several results about $\mathcal{I}$-convergence on the real line and in metric spaces (see, for example, [14, 15]) are true also in uniform spaces.

Definition 3.1 ([4, p. 34]) *Let $(X,\mathcal{U})$ be a uniform space.*

(1) A subset $B \subseteq X$ is called bounded if there are $U \in \mathcal{U}$ and $x \in X$ such that $B \subseteq U[x]$;

(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is bounded if the set $\{x_n : n \in \mathbb{N}\}$ is bounded;

(3) $(X,\mathcal{U})$ is called boundedly compact if each closed, bounded set in X is compact (so, for each $U \in \mathcal{U}$ and each $x \in X$, the set $\overline{U[x]}$ is compact).

(4) $(X,\mathcal{U})$ is said to have nice closed sections if for each $U \in \mathcal{U}$ and each $x \in X$, the set $\overline{U[x]}$ is compact whenever it is a proper subset of X.

It is obvious that each boundedly compact uniform space has nice closed sections, and each uniform space with nice closed sections is locally compact.

Denote by $bs(X)$ the set of all bounded sequences in $(X,\mathcal{U})$, and by $cs(X)$ the set of all sequences $\{x_n\}_{n\in\mathbb{N}}$ in X with $\mathcal{I}(x_n) \neq \emptyset$. For a space X, denote by $\mathrm{CL}(X)$ and $\mathbb{K}(X)$ the set of all nonempty closed and the set of all nonempty compact subsets of X, respectively.

Remark 3.2 (1) By Lemma 2.5, we have $\mathcal{I}(x_n) \neq \emptyset$ for each bounded sequence $\{x_n\}_{n\in\mathbb{N}}$, i.e., $bs(X) \subseteq cs(X)$.

(2) Let $(X,\mathcal{U})$ be a boundedly compact uniform space. It follows from Lemma 2.4 that $\mathcal{I}(x_n)$ is compact.

Definition 3.3 *(1) Let $(X,\mathcal{U})$ be a boundedly compact uniform space. We define a mapping $\theta : bs(X) \to \mathbb{K}(X)$ by $\theta(\{x_n\}_{n\in\mathbb{N}}) = \mathcal{I}(x_n)$.*

(2) Define a uniformity $\widetilde{\mathcal{U}}$ on the set $cs(X)$, induced by $\mathcal{U}$, as follows:

$$\widetilde{\mathcal{U}} = \{\widetilde{U} = (\{x_n\}, \{y_n\}) : x_n \in U[y_n] \text{ and } y_n \in U[x_n] \text{ for each } n \in \mathbb{N}\}.$$

(3) We equip $\mathbb{K}(X)$ with the Hausdorff-Bourbaki uniformity $\mathcal{U}_H$ inherited from the space $\mathrm{CL}(X)$, namely

$$\mathcal{U}_H = \{\underline{U} = (A, B) \in \mathrm{CL}(X) \times \mathrm{CL}(X) : U \in \mathcal{U}, A \subseteq U[B] \text{ and } B \subseteq U[A]\}.$$

Theorem 3.4 *Let $(X,\mathcal{U})$ be a boundedly compact uniform space. Then the mapping θ : $(bs(X),\widetilde{\mathcal{U}}) \to (\mathbb{K}(X),\mathcal{U}_H)$ is uniformly continuous.*

Proof Let $U \in \mathcal{U}$. Select symmetric $V \in \mathcal{U}$ such that $V^3 \subseteq U$. Let $(\{x_n\}, \{y_n\}) \in \widetilde{V}$ and $z \in \mathcal{I}(x_n)$. Then $A = \{n \in \mathbb{N} : x_n \in V[z]\} \notin \mathcal{I}$. For each $n \in A$, since $(x_n, y_n) \in V$ and $(x_n, z) \in V$, it follows that $y_n \in V^2[z]$. Thus $B = \{n \in \mathbb{N} : y_n \in V^2[z]\} \supseteq A$, and then $B \notin \mathcal{I}$. By Lemma 2.5, there is $t \in \overline{V^2[z]} \cap \mathcal{I}(y_n)$, and so $V[t] \cap V^2[z] \neq \emptyset$. Take $y \in V[t] \cap V^2[z]$. It follows that $(t, y) \in V$ and $(y, z) \in V^2$, and that $(t, z) \in V^3$. Therefore, $z \in V^3[\mathcal{I}(y_n)]$. This means that $\mathcal{I}(x_n) \subseteq V^3[\mathcal{I}(y_n)] \subseteq U[\mathcal{I}(y_n)]$. Similarly, we can show that $\mathcal{I}(y_n) \subseteq U[\mathcal{I}(x_n)]$. Hence $(\mathcal{I}(x_n), \mathcal{I}(y_n)) \in \underline{U} \in \mathcal{U}_H$. □

From Lemma 2.5, one obtains directly the following lemma.

Lemma 3.5 *Let $(X,\mathcal{U})$ be a locally compact uniform space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then $\mathcal{I}(x_n) \neq \emptyset$ if and only if there is a compact set $K \subseteq X$ such that $\{n \in \mathbb{N} : x_n \in K\} \notin \mathcal{I}$.*

For a space X, the lower Vietoris topology on the set $\mathrm{CL}(X)$ is generated by the sets U^-, U open in X, where $U^- = \{F \in \mathrm{CL}(X) : F \cap U \neq \emptyset\}$.

Theorem 3.6 *Let $(X,\mathcal{U})$ be a locally compact uniform space. Then the mapping θ : $(cs(X),\widetilde{\mathcal{U}}) \to (\mathrm{CL}(X), \tau_{V^-})$ is continuous.*

Proof It suffices to prove that for each G^- is open in $(\mathrm{CL}(X), \tau_{V^-})$, the set $\theta^{-1}(G^-)$ is open in $(bs(X),\widetilde{\mathcal{U}})$. Let $\{x_n\}_{n\in\mathbb{N}} \in \theta^{-1}(G^-)$. By Lemma 2.4 and the definition of G^-, there is $z \in \mathcal{I}(x_n) \cap G \neq \emptyset$. Since X is locally compact there exists $U \in \mathcal{U}$ such that $\overline{U[z]} \subseteq G$ and $\overline{U[z]}$ is compact. Select symmetric $W \in \mathcal{U}$ such that $W^2 \subseteq U$. And Since $z \in \mathcal{I}(x_n)$, it follows that $A = \{n \in \mathbb{N} : x_n \in W[z]\} \notin \mathcal{I}$. Choose $\{y_n\}_{n\in\mathbb{N}} \in \widetilde{W}[\{x_n\}]$. For each $n \in A$, since $x_n \in W[z]$, $y_n \in W[x_n]$, it follows that $y_n \in W^2[z] \subseteq U[z]$. Hence $B = \{n \in \mathbb{N} : y_n \in \overline{U[z]}\} \supseteq \{n \in \mathbb{N} : y_n \in U[z]\} \supseteq A$, and further $B \notin \mathcal{I}$. Because $\overline{U[z]}$ is compact and $\overline{U[z]} \subseteq G$, it follows from Lemma 2.5 that $\mathcal{I}(y_n) \cap G \neq \emptyset$. Thus $\{y_n\}_{n\in\mathbb{N}} \in \theta^{-1}(G^-)$ and $\widetilde{W}[\{x_n\}] \subseteq \theta^{-1}(G^-)$. □

The following topologies on the set $\mathrm{CL}(X)$ are well known.

The upper Fell topology τ_{F^+} has as a base the sets $(X \setminus K)^+$, K a compact subset of X, where for $A \subseteq X$, $A^+ = \{F \in \mathrm{CL}(X) : F \subseteq A\}$. The Fell topology τ_F is the join $\tau_{F^+} \vee \tau_{V^-} = \{A \cup B : A \in \tau_{F^+}, B \in \tau_{V^-}\}$.

If $(X,\mathcal{U})$ is a uniform space, then the upper proximal topology τ_{P^+} is generated by the sets G^{++}, G open in X, where $G^{++} = \{F \in \mathrm{CL}(X)$: there is $U \in \mathcal{U}$ such that $U[F] \subseteq G\}$. The proximal topology τ_P is the join of τ_{P^+} and the lower Vietoris topology τ_{V^-}, i.e., $\tau_P = \tau_{P^+} \vee \tau_{V^-}$.

Theorem 3.7 *Let $(X,\mathcal{U})$ be a noncompact uniform space with nice closed sections. Then the mappings are continuous.*

(1) $\theta : (cs(X),\widetilde{\mathcal{U}}) \to (\mathrm{CL}(X), \tau_{F^+})$;

(2) $\theta : (cs(X),\widetilde{\mathcal{U}}) \to (\mathrm{CL}(X), \tau_{P^+})$.

Proof Since the proofs of (1) and (2) are quite similar, we show the case for the upper proximal topology.

Let $\{x_n\}_{n\in\mathbb{N}} \in cs(X)$, G be open in X and G^{++} be a neighborhood of $\theta(\{x_n\}_{n\in\mathbb{N}}) = \mathcal{I}(x_n)$. Hence there exists $U \in \mathcal{U}$ such that $U[\mathcal{I}(x_n)] \subseteq G$. Select symmetric $V \in \mathcal{U}$ such that $V^4 \subseteq U$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence such that $(\{x_n\}, \{y_n\}) \in \widetilde{V}$. Then for each $n \in \mathbb{N}$, we have $(x_n, y_n) \in V$. Take $t \in \mathcal{I}(y_n)$. Then the set $A = \{n \in \mathbb{N} : y_n \in V[t]\} \in \mathcal{I}$. For each $n \in A$, since $x_n \in V^2[t]$, it follows that the set $B = \{n \in \mathbb{N} : x_n \in \overline{V^2[t]}\} \supseteq A$, and that $B \notin \mathcal{I}$. Since $\overline{V^2[t]}$ is compact and $B \notin \mathcal{I}$, it follows from Lemma 3.5 that $\overline{V^2[t]} \cap \mathcal{I}(x_n) \neq \emptyset$. Pick $z \in \overline{V^2[t]} \cap \mathcal{I}(x_n)$. Then $t \in V^3[z]$. This implies that $\mathcal{I}(y_n) \subseteq V^3[\mathcal{I}(x_n)]$. Therefore, $V[\mathcal{I}(y_n)] \subseteq V^4[\mathcal{I}(x_n)] \subseteq U[\mathcal{I}(x_n)] \subseteq G$, i.e., $\theta(\widetilde{V})[\{x_n\}] \subseteq G^{++}$. □

Combining Theorems 3.6 and 3.7, we have the following Corollary.

Corollary 3.8 *Let $(X,\mathcal{U})$ be a noncompact uniform space with nice closed sections. Then the mappings are continuous.*

(1) $\theta : (cs(X),\widetilde{\mathcal{U}}) \to (\mathrm{CL}(X), \tau_F)$;

(2) $\theta : (cs(X),\widetilde{\mathcal{U}}) \to (\mathrm{CL}(X), \tau_P)$.

At the end of this section, we briefly introduce $\mathcal{I}$-Cauchy sequences in uniform spaces and give a counterexample. Let $(X,\mathcal{U})$ be a uniform space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called an $\mathcal{I}$-Cauchy sequence if for each $U \in \mathcal{U}$, there is $n_0 \in \mathbb{N}$ such that $\{n \in \mathbb{N} : (x_n, x_{n_0}) \notin U\} \in \mathcal{I}$. In other words, it means that for each $U \in \mathcal{U}$, we have $\{n \in \mathbb{N} : x_n \notin U[x_{n_0}]\} \in \mathcal{I}$.

Theorem 3.9 *If a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a uniform space $(X,\mathcal{U})$ is $\mathcal{I}$-convergent, then it is an $\mathcal{I}$-Cauchy sequence.*

Proof Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a uniform space $(X,\mathcal{U})$ and $\{x_n\}_{n\in\mathbb{N}}$ be $\mathcal{I}$-convergent to $x \in X$. Select symmetric $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Then the set $A = \{n \in \mathbb{N} : x_n \notin U[x]\} \in \mathcal{I}$. Pick $n_0 \in \mathbb{N}$ such that $x_{n_0} \in V[x]$. Then for each $n \in \mathbb{N} \setminus A$, we have $(x_n, x) \in V$ and

$(x, x_{n_0}) \in V$. Thus for each $n \in \mathbb{N} \setminus A$, we have $(x_n, x_{n_0}) \in V^2 \subseteq U$. This means that $\{n \in \mathbb{N} : (x_n, x_{n_0}) \notin U\} \subseteq A \in \mathcal{I}$. Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-Cauchy sequence. □

The converse of Theorem 3.9 need not be true.

Example 3.10 There is an $\mathcal{I}$-Cauchy sequence in a uniform space $(X,\mathcal{U})$ which is not $\mathcal{I}$-convergent.

Proof Let $X = \mathbb{R}$. Define $d : X \times X \to \mathbb{R}$ by $d(x, y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$, for each $x, y \in X$. It is easy to see that d is a metric on X. Put $U_r = \{(x, y) \in X^2 : d(x, y) < r\}$. Set $\mathcal{U} = \{U \subseteq X^2$: there is $r > 0$ such that $U_r \subseteq U\}$. Then $(X,\mathcal{U})$ is a uniform space and $U_r[x] = B(x, r) = \{y \in X : d(x, y) < r\}$. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ generated by all subsets of the set $\{n = m^2 : m \in \mathbb{N}\}$ and all finite subsets. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by

$$x_n \in \begin{cases} n, & n \neq m^2, \\ \frac{1}{n}, & n = m^2. \end{cases}$$

Claim 1. $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-Cauchy sequence in the uniform space $(X,\mathcal{U})$.

In fact, for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}, n_0 = m^2$ such that $n_0 > \frac{1}{\varepsilon}$. For $n > n_0$ and $n \neq m^2$, we have $d(x_n, x_{n_0}) = |\frac{n}{1+|n|} - \frac{n_0}{1+|n_0|}| = \frac{n-n_0}{(1+n)(1+n_0)} < \frac{1}{n_0} < \varepsilon$. Hence for each $U \in \mathcal{U}$ there is $r > 0$ such that $\{n \in \mathbb{N} : x_n \notin U[x_{n_0}]\} \subseteq \{n \in \mathbb{N} : x_n \notin U_r[x_{n_0}]\} = \{n \in \mathbb{N} : x_n \notin B(x_{n_0}, \varepsilon)\} \subseteq \{1, 2, \ldots, n_0\} \cup \{n = m^2 : m \in \mathbb{N}\} \in \mathcal{I}$.

Claim 2. $\{x_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}$-convergent in the uniform space $(X,\mathcal{U})$.

For given $a \in X$ and $n \neq m^2$, $d(x_n, a) = |\frac{n}{1+|n|} - \frac{a}{1+|a|}| = \frac{|n-a|}{(1+n)(1+|a|)} \to \frac{1}{1+|a|} \neq 0$ $(n \to \infty)$. Then there is $n' \in \mathbb{N}$ and $r > 0$ such that $\{n \in \mathbb{N} : x_n \notin U_r[a]\} = \{n \in \mathbb{N} : x_n \notin B(a, r)\} \subseteq \{n : n > n'\} \cup \{n \neq m^2 : m \in \mathbb{N}\} \in \mathbb{N} \setminus \mathcal{I}$. This implies that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}$-convergent. □

Bilalov and Nazarova showed that if a uniform space $(X,\mathcal{U})$ is sequentially complete with a countable base, then each $\mathcal{I}$-Cauchy sequence in X is $\mathcal{I}$-convergent [11, Theorem 3.1]. It is known that X is T_0 and uniformity $\mathcal{U}$ has a countable base if and only if X is metrizable [16, Theorem 4.1.9]. Hence the space in Example 3.10 is not sequentially complete.

4. $\mathcal{I}$-Hurewicz bounded uniform spaces

In this section, we study $\mathcal{I}$-Hurewicz bounded uniform spaces, obtain a characterization of $\mathcal{I}$-Hurewicz boundedness in terms of $\mathcal{I}$-groupability and some basic topological properties.

In [17], a uniform space $(X,\mathcal{U})$ is called Hurewicz bounded if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of entourages of the diagonal of X there is a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many sets $U_n[A_n]$.

Definition 4.1 *A uniform space $(X,\mathcal{U})$ is called $\mathcal{I}$-Hurewicz bounded if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$, there is a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of X such that for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin U_n[A_n]\} \in \mathcal{I}$.*

It is easy to verify that finite unions of $\mathcal{I}$-Hurewicz bounded uniform spaces are also $\mathcal{I}$-Hurewicz bounded.

Theorem 4.2 *$\mathcal{I}$-Hurewicz bounded uniform spaces are preserved by uniformly continuous mappings.*

Proof Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces, $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ be a uniformly continuous mapping. Suppose that $\{V_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{V}$. Since f is uniformly continuous, it follows that $\phi^{-1}(V_n)=U_n\in\mathcal{U}$, where $\phi: X^2\to Y^2$ is defined by $\phi(x,y)=(f(x),f(y))$. And since $(X,\mathcal{U})$ is $\mathcal{I}$-Hurewicz bounded, there is a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of X such that for each $x\in X$ the set $\{n\in\mathbb{N}: x\notin U_n[A_n]\}\in\mathcal{I}$. Because f is surjective, for each $y\in Y$ there is $x\in X$ such that $y=f(x)$. For each $n\in\mathbb{N}$, take finite subsets B_n of Y such that $A_n=f^{-1}(B_n)$. Then $\{n\in\mathbb{N}: y\notin V_n[B_n]\}=\{n\in\mathbb{N}: x\notin U_n[A_n]\}\in\mathcal{I}$. This implies that $(Y,\mathcal{V})$ is $\mathcal{I}$-Hurewicz bounded. □

It is known that each continuous mapping from compact uniform spaces to uniform spaces is uniformly continuous [16, Corollary 4.1.13]. So we have the following corollary.

Corollary 4.3 *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces, $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ be continuous and $(X,\mathcal{U})$ be a compact space with $\mathcal{I}$-Hurewicz bounded. Then $(Y,\mathcal{V})$ is $\mathcal{I}$-Hurewicz bounded.*

Definition 4.4 *A countable cover $\mathcal{U}=\{U_n: n\in\mathbb{N}\}$ of X is said to be $\mathcal{I}$-groupable cover if it can be represented as a countable union of finite, pairwise disjoint subfamilies $\mathcal{V}_n, n\in\mathbb{N}$ such that for each $x\in X$ the set $\{n\in\mathbb{N}: x\notin\bigcup\mathcal{V}_n\}\in\mathcal{I}$.*

The following theorem gives a characterization of $\mathcal{I}$-Hurewicz boundedness in terms of $\mathcal{I}$-groupability.

Theorem 4.5 *For a uniform space $(X,\mathcal{U})$ the following statements are equivalent:*

(1) X is $\mathcal{I}$-Hurewicz bounded;

(2) For each sequence $\{U_n\}_{n\in\mathbb{N}}$ of $\mathcal{U}$, there is a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite subsets of X such that $\{U_n[B_n]\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-groupable cover of X.

Proof (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{U}$. For each $n\in\mathbb{N}$, select $V_n\in\mathcal{U}$ such that $V_n\subseteq\bigcap_{i\le n}U_i$. By condition (2), for each $n\in\mathbb{N}$, choose a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite subsets of X such that $\{V_n[B_n]: n\in\mathbb{N}\}$ is an $\mathcal{I}$-groupable cover of X. Thus there is a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that for each $x\in X$ the set $\{k\in\mathbb{N}: x\notin\bigcup_{n_{k-1}\le i<n_k}V_i[B_i]\}\in\mathcal{I}$, where $n_0=1$. Put $A_k=\bigcup_{n_{k-1}\le i<n_k}B_i$. Then each A_k is a finite subset of X and $\bigcup_{n_{k-1}\le i<n_k}V_i\subseteq U_k$ for $n_{k-1}\le i<n_k$ $(k\le i)$. For each $x\in X$, we have $\{k\in\mathbb{N}: x\notin U_k[A_k]\}\subseteq\{k\in\mathbb{N}: x\notin\bigcup_{n_{k-1}\le i<n_k}V_i[B_i]\}\in\mathcal{I}$. This implies that X is $\mathcal{I}$-Hurewicz bounded. □

Next, we will describe the behavior of $\mathcal{I}$-Hurewicz boundedness under basic topological operations.

Theorem 4.6 *Every subspace of an $\mathcal{I}$-Hurewicz bounded uniform space $(X,\mathcal{U})$ is $\mathcal{I}$-Hurewicz* *bounded.*

Proof Let $(Y,\mathcal{U}_Y)$ be a subspace of $(X,\mathcal{U})$ and $\{W_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{U}_Y$. For each $n \in \mathbb{N}$, let U_n be an element in $\mathcal{U}$ such that $W_n = U_n \cap (Y \times Y)$. Select symmetric $V_n \in \mathcal{U}$ such that $V_n^2 \subseteq U_n$. Apply to the sequence $\{V_n\}_{n\in\mathbb{N}}$ the fact that $(X,\mathcal{U})$ is $\mathcal{I}$-Hurewicz bounded and choose a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of X such that for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin V_n[A_n]\} \in \mathcal{I}$. For each $n \in \mathbb{N}$, put $B_n = \{a \in A_n$: there is $y \in Y$ with $y \in V_n[a]\}$ and for each $b \in B_n$, choose an element y_b with $y_b \in V_n[b]$. Put $C_n = \{y_b : b \in B_n\}$. We claim that $(Y,\mathcal{U}_Y)$ is $\mathcal{I}$-Hurewicz bounded. In fact, let $y \in Y$. Then $M_y = \{n \in \mathbb{N}$: there is $a \in A_n$ with $y \in V_n[a]\} \in \mathbb{N} \setminus \mathcal{I}$. By definition of B_n each such a belongs to B_n and so $M_y = \{n \in \mathbb{N}$: there is $y_a \in C_n$ satisfying $y_a \in V_n[a]\}$. By symmetric V_n, we have $a \in V_n[y_a]$. Since for each $n \in M_y$, $y \in V_n[a]$ it follows that $y \in V_n^2[y_a] \subseteq U_n[y_a]$. Thus, for each $n \in M_y$, we have $y \in W_n[y_a] \subseteq W_n[C_n]$, hence $\{n \in \mathbb{N} : y \in W_n[C_n]\} \supseteq M_y$, i.e., $\{n \in \mathbb{N} : y \notin W_n[C_n]\} \in \mathcal{I}$. Therefore, $(Y,\mathcal{U}_Y)$ is $\mathcal{I}$-Hurewicz bounded. □

Theorem 4.7 *If an $\mathcal{I}$-Hurewicz bounded uniform space $(X,\mathcal{U}_X)$ is dense in a uniform space $(Y,\mathcal{U})$, then Y is also $\mathcal{I}$-Hurewicz bounded.*

Proof Let $\{W_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{U}$ and $U_n = W_n \cap (X \times X) \in \mathcal{U}_X$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, select symmetric $O_n \in \mathcal{U}$ such that $O_n^2 \subseteq W_n$ and set $V_n = O_n \cap (X \times X)$. Since X is $\mathcal{I}$-Hurewicz bounded, there is a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of X such that for each $x \in X$ the set $M_x = \{n \in \mathbb{N} : x \in V_n[A_n]\} \in \mathbb{N} \setminus \mathcal{I}$. Next, we will show that Y is $\mathcal{I}$-Hurewicz bounded. Let $y \in Y$. Since $(X,\mathcal{U}_X)$ is dense in a uniform space $(Y,\mathcal{U})$, it follows that $X \cap O_n[y] \neq \emptyset$ for each $n \in \mathbb{N}$. For each $m \in M_x$, take $x \in X \cap O_m[y]$. Then there is $a_m \in A_m$ such that $x \in V_m[a_m] \subseteq O_m[a_m]$. And because $x \in O_m[y]$, it follows that $y \in O_m^2[a_m] \subseteq W_m[a_m]$ for each $m \in M_x$. Thus, $M_y = \{n \in \mathbb{N} : y \in W_n[A_n]\} \supseteq M_x$. Therefore, $(Y,\mathcal{U})$ is $\mathcal{I}$-Hurewicz bounded. □

Corollary 4.8 *A uniform space X is $\mathcal{I}$-Hurewicz bounded if and only if its completion $\widetilde{X}$ is $\mathcal{I}$-Hurewicz bounded.*

Theorem 4.9 *The product $Z = (X \times Y,\mathcal{U}_X \times \mathcal{U}_Y)$ of $\mathcal{I}$-Hurewicz bounded uniform spaces $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ is $\mathcal{I}$-Hurewicz bounded.*

Proof Let $\{W_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{U}_X \times \mathcal{U}_Y$. Suppose that each $W_n = \{(z_1, z_2) \in Z \times Z, (x_1, x_2) \in U_n \in \mathcal{U}_X, (y_1, y_2) \in V_n \in \mathcal{U}_Y\}$, where $z_i = (x_i, y_i)$, $i = 1, 2$. For each $n \in \mathbb{N}$, select symmetric $G_n \in \mathcal{U}_X$ and symmetric $H_n \in \mathcal{U}_Y$ such that $G_n^2 \subseteq U_n$ and $H_n^2 \subseteq V_n$. Since $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ are $\mathcal{I}$-Hurewicz bounded, it follows that there are a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of X and a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite subsets of Y such that for each $x \in X$ and $y \in Y$ the sets $M_x = \{n \in \mathbb{N} : x \in G_n[A_n]\} \in \mathbb{N}\setminus\mathcal{I}$ and $M_y = \{n \in \mathbb{N} : y \in H_n[A_n]\} \in \mathbb{N}\setminus\mathcal{I}$. Next, we will show that $X\times Y$ is $\mathcal{I}$-Hurewicz bounded. Let $(x, y) \in X\times Y$. Since $M_x \in \mathbb{N}\setminus\mathcal{I}$ and $M_y \in \mathbb{N}\setminus\mathcal{I}$, it follows that $M_1 = M_x \cap M_y \in \mathbb{N}\setminus\mathcal{I}$. Therefore, for each $m \in M_1$, there are $a_m \in A_m$ and $b_m \in B_m$ with $x \in G_m[a_m]$, $y \in H_m[b_m]$. Similarly, the set $M_2 = \{n \in \mathbb{N} : a_n \in G_n[A_n], b_n \in H_n[B_n]\} \in \mathbb{N} \setminus \mathcal{I}$.

Set $K = M_1 \cap M_2$. Then $K \in \mathbb{N} \setminus \mathcal{I}$. Since $G_n^2 \subseteq U_n$ and $H_n^2 \subseteq V_n$ (for each $n \in \mathbb{N}$), we have that $x \in U_k[A_k]$ and $y \in V_k[B_k]$ for each $k \in K$. Thus, $\{k \in \mathbb{N} : (x, y) \in W_k[A_k \times B_k]\} \supseteq K$. Therefore, $Z = (X \times Y, \mathcal{U}_X \times \mathcal{U}_Y)$ is $\mathcal{I}$-Hurewicz bounded. $\square$

Theorem 4.10 *If a uniform space $(X, \mathcal{U})$ is $\mathcal{I}$-Hurewicz bounded, then the hyperspace $(\mathbb{F}(X), \mathcal{U}_H)$ is also $\mathcal{I}$-Hurewicz bounded.*

Proof Let $\{\underline{U_n}\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{U}_H$ and let $\{U_n\}_{n\in\mathbb{N}}$ be the corresponding sequence of elements of $\mathcal{U}$. Since X is $\mathcal{I}$-Hurewicz bounded, there is a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of X such that for each $x \in X$ the set $M_x = \{n \in \mathbb{N} : x \in U_n[A_n]\} \in \mathbb{N} \setminus \mathcal{I}$. Take an element $E \in \mathbb{F}(X)$. For each $e \in E$, take $a_n(e) \in A_n$ with $n \in M_e$. Set $B_n = \{a_n(e) : e \in E\}$ and $M = \bigcap_{e\in E} M_e$. Then $M \in \mathbb{N} \setminus \mathcal{I}$ and $E \subseteq U_m[B_m]$ for each $m \in M$. On the other hand, for each $b \in B_m$, $b \in U_m[E]$. Hence $B_m \subseteq U_m[E]$. Therefore, for each $m \in M$, $(E, B_m) \in \underline{U_m}$. Thus, $(\mathbb{F}(X), \mathcal{U}_H)$ is $\mathcal{I}$-Hurewicz bounded. $\square$

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