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Positive Solutions of Second Order Discrete Problem on Infinite Intervals

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Abstract In this paper, by using the discrete Arzelá-Ascoli Lemma and the fixed-point theorem in cones, we discuss the existence of positive solutions of the following second order discrete Sturm-Liouville boundary value problem on infinite intervals

$$\begin{cases} -\Delta^2 u(x-1) = f(x, u(x), \Delta u(x-1)), & x \in \mathbb{N}, \\ u(0) - a\Delta u(0) = B, \quad \Delta u(\infty) = C, \end{cases}$$

where $\Delta u(x) = u(x+1) - u(x)$ is the forward difference operator, $\mathbb{N} = \{1, 2, \ldots, \infty\}$, $f : \mathbb{N}\times\mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $a > 0$, B and C are nonnegative constants, $\mathbb{R}_+ = [0, +\infty)$, $\Delta u(\infty) = \lim_{x\to\infty} \Delta u(x)$.

Keywords positive solutions; second order discrete problems; infinite intervals; fixed-point theorem in cones

MR(2020) Subject Classification 39A27; 47H10

1. Introduction

In this paper, we are concerned with the existence of positive solutions of the discrete problem

$$\begin{cases} -\Delta^2 u(x-1) = f(x, u(x), \Delta u(x-1)), & x \in \mathbb{N}, \\ u(0) - a\Delta u(0) = B, \quad \Delta u(\infty) = C, \end{cases} \tag{1.1}$$

where $\Delta u(x) = u(x+1) - u(x)$ is the forward difference operator, $\mathbb{N} = \{1, 2, \ldots, \infty\}$ and $f : \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $a > 0$, B and C are nonnegative constants, $\mathbb{R}_+ = [0, +\infty)$, $\Delta u(\infty) = \lim_{x\to\infty} \Delta u(x)$.

Problem (1.1) is related to the difference equation with a wide range of applications in physics, economics and biology. In recent years, many scholars have studied the existence of solutions for difference problems on finite intervals by using various methods of nonlinear analysis, we refer to [1–6]. However, these problems on infinite intervals are relatively rare [7–9]. It is well known that the nonlinear problem on the infinite intervals was first proposed by Gross [10], it was used to solve the turbulence problem of unsteady flow in porous media. It is more difficult to show the existence of solutions of problem (1.1) on the infinite intervals than that on the finite intervals

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because of the lack of compactness in the working space. Thus the fixed-point theory on the cone cannot be used directly. In this paper, we overcome these difficulties by constructing a suitable working space.

In 1993, Bobisud [11] used the diagonalization process on $[a, +\infty)$ to obtain the existence of unbounded solutions for the second order differential problem

$$\begin{cases} u''(x) + h(x)f(x, u(x)) = 0, & x \in [a, +\infty), \\ u(a) = 0, \quad u'(+\infty) = 0, \end{cases} \tag{1.2}$$

where a is a constant, $h : [a, +\infty) \to \mathbb{R}$ and $f : [a, +\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous. Moreover, Chen and Zhang [12] obtained necessary and sufficient conditions of the existence of positive solutions to (1.2).

Yan, O'Regan and Agarwal [13] applied the method of upper and lower solutions to show the existence of unbounded positive solutions of nonlinear differential problems

$$\begin{cases} u''(x) + \phi(x)f(x, u(x), u'(x)) = 0, & x \in [0, +\infty), \\ au(0) - bu'(0) = u(0) \geq 0, \quad u'(+\infty) = k > 0, \end{cases}$$

where $a, b > 0$, $\phi : (-1, 1) \to \mathbb{R}_+$ and $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ are continuous.

Lian [14] showed the existence of solutions and positive solutions for the following Sturm-Liouville problem under Nagumo conditions

$$\begin{cases} u''(x) + \phi(x)f(x, u(x), u'(x)) = 0, & x \in [0, +\infty), \\ u(0) - au'(0) = B, \quad u'(+\infty) = C, \end{cases}$$

where $a > 0$, $B, C \in \mathbb{R}$, $\phi : (-1, 1) \to \mathbb{R}_+$ and $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. Note that [15] is the discretization of differential problems in [14].

Motivated by the above papers, we show the existence of unbounded positive solutions of the second order discrete problem (1.1) by excepting the Nagumo condition. In this article, we assume that the following conditions are satisfied.

(H1) $f : \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, for all $R > 0$,

$$\tilde{f}(x, R, z) = \max_{0 \leq w \leq R} f(x, (1 + x)w, z), \quad x \in \mathbb{N}, \ z \geq 0$$

and $\tilde{f}$ satisfies $1 \leq \sum_{x=1}^{\infty} \tilde{f}(x, R, z) < \infty$.

(H2) There exist two functions $a_1 : \mathbb{N} \to \mathbb{R}_+$, $b_2 : [1, m]_{\mathbb{Z}} \to \mathbb{R}_+$ such that

$$\limsup_{w,z \to 0} \max_{x \in \mathbb{N}} \frac{f(x, (1 + x)w, z)}{w + z} = a_1(x), \quad \liminf_{w,z \to +\infty} \min_{x \in [1,m]_{\mathbb{Z}}} \frac{f(x, (1 + x)w, z)}{w + z} = b_2(x)$$

and a_1, b_2 satisfy

$$2(C + 1)|a_1|_1 \leq 1, \quad \frac{a}{2m(1 + m)^2} \sum_{x=1}^{m} b_2(x) \geq 1,$$

where $|a_1|_1 = \sum_{x=1}^{\infty} a_1(x)$, $m \in \mathbb{N}$ and $m > 1$.

(H3) There exist two functions $a_2 : [1, m]_{\mathbb{Z}} \to \mathbb{R}_+$, $b_1 : \mathbb{N} \to \mathbb{R}_+$ such that

$$\liminf_{w,z \to 0} \min_{x \in [1,m]_{\mathbb{Z}}} \frac{f(x, (1 + x)w, z)}{w + z} = a_2(x), \quad \limsup_{w,z \to +\infty} \max_{x \in \mathbb{N}} \frac{f(x, (1 + x)w, z)}{w + z} = b_1(x)$$

and a_2, b_1 satisfy

$$\frac{a}{2m(1+m)^2}\sum_{x=1}^{m} a_2(x) \geq 1, \quad 2(C+1)|b_1|_1 \leq 1,$$

where $|b_1|_1 = \sum_{x=1}^{\infty} b_1(x)$, $m \in \mathbb{N}$ and $m > 1$.

To prove that any solutions of problem (1.1) are unbounded, we further assume

(H4) $\lim_{x\to\infty}\sum_{i=1}^{\infty} G(x,i)f(i,w,z) = +\infty$ uniformly on w, z in any compact intervals of $[0,+\infty)$.

Our main results are the following.

Theorem 1.1 *Assume (H1) and (H2) hold. Then Problem (1.1) has at least one positive solution. Moreover, if (H4) is true, then the positive solution is unbounded.*

Theorem 1.2 *Assume (H1) and (H3) hold. Then Problem (1.1) has at least one positive solution. Moreover, if (H4) is true, then the positive solution is unbounded.*

2. Preliminaries

Let $\mathbb{N}_0 = \{0, 1, \ldots, \infty\}$, $Y := \{u | u : \mathbb{N}_0 \to \mathbb{R}_+, \lim_{x\to\infty} \Delta u(x) = C\}$. Then Y is a Banach space under the norm $\|u\| = \max\{\|u\|_1, \|u\|_2\}$, where

$$\|u\|_1 = \sup_{x\in\mathbb{N}_0} \frac{|u(x)|}{1+x}, \quad \|u\|_2 = \sup_{x\in\mathbb{N}_0} |\Delta u(x)|.$$

Set $X := \{u \in Y | u(0) - a\Delta u(0) = B\}$. By [9], it is not difficult to establish $(X, \|\cdot\|)$ is a Banach space.

Remark 2.1 For all $u \in X$, $\|u\| = \|u\|_2$. In fact,

$$\frac{|u(x)|}{1+x} = \frac{1}{1+x}\Big|\sum_{s=0}^{x-1}\Delta u(s)\Big| \leq \frac{x}{1+x}M_1,$$

where $M_1 = \sup_{x\in\mathbb{N}_0} |\Delta u(x)|$, this means that $\|u\|_1 \leq \|u\|_2$, and so $\|u\| = \|u\|_2$.

Lemma 2.2 ([16]) *Let E be a linear space, $\|\cdot\|_N$ and p be respectively a norm and a semi-norm on E such that $(E, \|\cdot\|)$ is a Banach space, where $\|e\| = \max\{\|e\|_N, p(e)\}$ for $e \in E$. Let P be a cone in E, r_1 and r_2 be two positive real numbers such that $r_1 < r_2$. Set $\Omega_i = \{u \in E : \|u\| < r_i\}$, $i = 1, 2$. Let $T : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ be a compact mapping. If one of the following conditions is satisfied:*

(i) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, $\|Tu\|_N \geq \|u\|_N$, $u \in P \cap \partial\Omega_2$;

(ii) $\|Tu\|_N \geq \|u\|_N$, $u \in P \cap \partial\Omega_1$, $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then T has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.3 ([15]) *The discrete problem (1.1) is equivalent to the operator equation*

$$Tu(x) = aC + B + Cx + \sum_{i=1}^{\infty} G(x,i)f(i, u(i), \Delta u(i-1)), \quad x \in \mathbb{N}_0,$$

where

$$G(x,i)=\begin{cases} a+i, & i\le x,\\ a+x, & i>x.\end{cases}$$

Define the cone K in X by

$$K=\{u|u\in X, u(x)\ge 0 \text{ on } x\in\mathbb{N}_0,\ \Delta^2 u(x-1)\le 0 \text{ on } x\in\mathbb{N}\}.$$

Lemma 2.4 *Assume that (H1) holds. Then the operator $T: K\to K$ is completely continuous.*

Proof For any $u\in K,\ 0\le u\le R$, it follows from (H1) that

$$\begin{aligned}
&\Big|\sum_{i=1}^{\infty}G(x,i)f(i,u(i),\Delta u(i-1))\Big|\le\sum_{i=1}^{\infty}\big|G(x,i)f(i,u(i),\Delta u(i-1))\big|\\
&=\sum_{i=1}^{\infty}\big|G(x,i)f\big(i,(1+i)\frac{u(i)}{1+i},\Delta u(i-1)\big)\big|\\
&\le\sum_{i=1}^{x}G(x,i)\tilde f(i,R,\Delta u(i-1))+\sum_{i=x+1}^{\infty}G(x,i)\tilde f(i,R,\Delta u(i-1))\\
&=\sum_{i=1}^{x}(a+i)\tilde f(i,R,\Delta u(i-1))+\sum_{i=x+1}^{\infty}(a+x)\tilde f(i,R,\Delta u(i-1))\\
&\le(a+x)\sum_{i=1}^{\infty}\tilde f(i,R,\Delta u(i-1))<\infty.
\end{aligned}$$

On the other hand, for all $u\in X, x\in\mathbb{N}_0$, we have

$$\begin{aligned}
&\lim_{x\to\infty}\Delta(Tu)(x)=\lim_{x\to\infty}(Tu(x+1)-Tu(x))\\
&=\lim_{x\to\infty}\Big(aC+B+C(x+1)+\sum_{i=1}^{\infty}G(x+1,i)f(i,u(i),\Delta u(i-1))\Big)-\\
&\quad\lim_{x\to\infty}\Big(aC+B+Cx+\sum_{i=1}^{\infty}G(x,i)f(i,u(i),\Delta u(i-1))\Big)\\
&=\lim_{x\to\infty}\Big(C+\sum_{i=1}^{\infty}\big(G(x+1,i)-G(x,i)\big)f(i,u(i),\Delta u(i-1))\Big)\\
&=\lim_{x\to\infty}\Big(C+\sum_{i=x+1}^{\infty}f(i,u(i),\Delta u(i-1))\Big)=C
\end{aligned}$$

and

$$\begin{aligned}
&Tu(0)-aT\Delta u(0)=(a+1)Tu(0)-aTu(1)\\
&=(a+1)\Big(aC+B+\sum_{i=1}^{\infty}G(0,i)f(i,u(i),\Delta u(i-1))\Big)-\\
&\quad a\Big(aC+B+C+\sum_{i=1}^{\infty}G(1,i)f(i,u(i),\Delta u(i-1))\Big)\\
&=(a+1)\Big(aC+B+\sum_{i=1}^{\infty}af(i,u(i),\Delta u(i-1))\Big)-
\end{aligned}$$

$$a\Big(aC+B+C+\sum_{i=1}^{\infty}(a+1)f(i,u(i),\Delta u(i-1))\Big)$$
$$=a^2C+aC+aB+B-a^2C-aC-aB=B.$$

Since f and G are nonnegative, then $Tu\geq 0$, and so $Tu\in X$.

Next, we will prove the continuity of T. Suppose that $u^{(m)}\subset X$ and $u^{(m)}\to u$ as $m\to\infty$. Then it deduces from the continuity of f and Remark 2.1 that

$$\|Tu^{(m)}-Tu\|=\|Tu^{(m)}-Tu\|_2=\sup_{x\in\mathbb{N}_0}|\Delta Tu^{(m)}(x)-\Delta Tu(x)|$$
$$=\sup_{x\in\mathbb{N}_0}\Big|\sum_{i=1}^{\infty}(G(x+1,i)-G(x,i))(f(i,u^{(m)}(i),\Delta u^{(m)}(i-1))-f(i,u(i),\Delta u(i-1))\Big|$$
$$\leq\sum_{i=1}^{\infty}|f(i,u^{(m)}(i),\Delta u^{(m)}(i-1))-f(i,u(i),\Delta u(i-1))|\to 0,\quad m\to\infty,$$

therefore, T is continuous.

Secondly, we will show that T is compact. Set $B:=\{u|u\in X,|u|\leq R\}$. For all $u\in B$, we have

$$\|Tu\|_1=\sup_{x\in\mathbb{N}_0}\Big|\frac{aC+B+Cx}{1+x}+\sum_{i=1}^{\infty}\frac{G(x,i)}{1+x}f(i,u(i),\Delta u(i-1))\Big|$$
$$\leq\sup_{x\in\mathbb{N}_0}\big|\frac{aC+B}{1+x}+\frac{Cx}{1+x}\big|+\max\{a,1\}\sum_{i=1}^{\infty}\tilde{f}(i,R,\Delta u(i-1))$$
$$\leq\max\{aC+B,C\}+\max\{a,1\}\sum_{i=1}^{\infty}\tilde{f}(i,R,\Delta u(i-1))<\infty,$$
$$\|Tu\|_2=\sup_{x\in\mathbb{N}_0}\Big|C+\sum_{i=1}^{\infty}(G(x+1,i)-G(x,i))f(i,u(i),\Delta u(i-1))\Big|$$
$$\leq C+\sum_{i=1}^{\infty}f(i,u(i),\Delta u(i-1))\leq C+\sum_{i=1}^{\infty}\tilde{f}(i,R,\Delta u(i-1))<\infty.$$

Then $T(B)$ is bounded uniformly. Combining this with discrete Arzelá-Ascoli Lemma [14], we can prove that $T(B)$ is relatively compact. In fact,

$$\big|\frac{Tu(x)}{1+x}-\lim_{x\to\infty}\frac{Tu(x)}{1+x}\big|=\Big|\frac{aC+B+Cx}{1+x}-C+\sum_{i=1}^{\infty}(\frac{G(x,i)}{1+x}-1)f(i,u(i),\Delta u(i-1))\Big|$$
$$\leq\big|\frac{aC+B+Cx}{1+x}-C\big|+\sum_{i=1}^{\infty}\big|\frac{G(x,i)}{1+x}-1\big|f(i,u(i),\Delta u(i-1))$$
$$\leq\big|\frac{aC+B+Cx}{1+x}-C\big|+\sum_{i=1}^{\infty}\big|\frac{G(x,i)}{1+x}-1\big|\tilde{f}(i,R,\Delta u(i-1))\to 0,\quad x\to\infty,$$
$$|\Delta(Tu)(x)-\lim_{x\to\infty}\Delta(Tu)(x)|=\Big|\sum_{i=1}^{\infty}(G(x+1,i)-G(x,i))f(i,u(i),\Delta u(i-1))\Big|$$
$$\leq\sum_{i=1}^{\infty}|(G(x+1,i)-G(x,i))|f(i,u(i),\Delta u(i-1))$$

$$= \sum_{i=x+1}^{\infty} f(i,u(i),\Delta u(i-1)) \to 0, \quad x\to\infty.$$

Thus, $T(B+)$ is relatively compact.

Consequently, $T: X\to X$ is completely continuous.

Finally, we prove that $Tu\in K$ for any $u\in K$. It is easy to see that $Tu\geq 0$. On the other hand,

$$\Delta(Tu)(x) = C + \sum_{i=1}^{\infty}(G(x+1,i)-G(x,i))f(i,u(i),\Delta u(i-1))$$
$$= C + \sum_{i=x+1}^{\infty} f(i,u(i),\Delta u(i-1)),$$

clearly, $\Delta^2 u(x-1)\leq 0$, then $Tu\in K$. □

Lemma 2.5 *For any $u\in K$, $u(x)\geq \rho(x)\|u\|_1$, $x\in\mathbb{N}$, where $\rho(x)=\frac{1}{2x}$.*

Proof For any $u\in K$ and $x\in\mathbb{N}_0$, then $\Delta^2 u(x)\leq 0$, and so $\Delta u(x)$ is nonincreasing. Combining this with $\lim_{x\to\infty}\Delta u(x)\geq 0$, we have $\Delta u(x)\geq 0$. Then $u(x)$ is nondecreasing. Thus

$$\|u\|_1 = \sup_{x\in\mathbb{N}_0}\frac{|u(x)|}{1+x} = \sup_{x\in\mathbb{N}_0}\frac{1}{1+x}\Big|u(0)+\sum_{i=0}^{x-1}\Delta u(i)\Big|$$
$$\leq u(0)+x(u(1)-u(0))\leq u(0)+xu(1)\leq 2xu(x)$$

and hence, $u(x)\geq \frac{1}{2x}\|u\|_1$. □

Lemma 2.6 *Assume (H1) and (H4) hold. Then u is unbounded.*

Proof Let $u\in K$ be a possible positive solution of problem (1.1). Suppose on the contrary that u is bounded, it follows from the proof of Lemma 2.5 that $\Delta u(x)\geq 0$, together with $u(\infty)=\lim_{x\to+\infty}u(x)>0$ imply that $u(x)$ achieves its maximum at $+\infty$. In addition,

$$u(x) = aC+B+xC+\sum_{i=1}^{\infty}G(x,i)f(i,u(i),\Delta u(i-1))$$
$$\geq \sum_{i=1}^{\infty}G(x,i)f(i,u(i),\Delta u(i-1)).$$

Taking the limit of x on both sides of the above formula yields

$$+\infty > u(\infty)\geq \lim_{x\to+\infty}\sum_{i=1}^{\infty}G(x,i)f(i,u(i),\Delta u(i-1)) = +\infty,$$

which is a contradiction. Thus u is unbounded. □

3. Proof of main results

We shall use Lemma 2.2 to prove our main results.

Proof of Theorem 1.1 Since $\limsup_{w,z\to 0}\max_{x\in\mathbb{N}}\frac{f(x,(1+x)w,z)}{w+z}=a_1(x)$, we may choose $R_1>0$, for all $0<w,\ z\le R_1$, and we have

$$\max_{x\in\mathbb{N}} f(x,(1+x)w,z)\le a_1(x)(w+z). \tag{3.1}$$

Now, set $\Omega_1=\{u\in X:\|u\|<R_1\}$. For any $u\in K\cap\partial\Omega_1$, it deduces from (H1), (3.1) and $2(C+1)|a_1|_1\le 1$ that

$$\begin{aligned}
\|Tu\|&=\|Tu\|_2=\sup_{x\in\mathbb{N}_0}|\Delta(Tu)(x)|\\
&=\sup_{x\in\mathbb{N}_0}\Big|aC+B+C(x+1)+\sum_{i=1}^{\infty}G(x+1,i)f(i,u(i),\Delta u(i-1))-\\
&\quad aC-B-Cx-\sum_{i=1}^{\infty}G(x,i)f(i,u(i),\Delta u(i-1))\Big|\\
&=\sup_{x\in\mathbb{N}_0}\Big|C+\sum_{i=1}^{\infty}(G(x+1,i)-G(x,i))f(i,u(i),\Delta u(i-1))\Big|\\
&\le C+\sup_{x\in\mathbb{N}_0}\sum_{i=1}^{\infty}|G(x+1,i)-G(x,i)|f(i,u(i),\Delta u(i-1))\\
&\le C+\sum_{i=1}^{\infty}\tilde{f}(i,R_1,\Delta u(i-1))\\
&=C+\sum_{i=1}^{\infty}\max_{i\in\mathbb{N}} f(i,(1+i)\frac{u(i)}{1+i},\Delta u(i-1))\\
&\le C+|a_1|_1(\frac{u(i)}{1+i}+\Delta u(i-1))\le C+2|a_1|_1\|u\|.
\end{aligned}$$

Moreover, it follows from (H1) and

$$\sum_{i=1}^{\infty}\tilde{f}(i,R_1,\Delta u(i-1))\ge 1,\quad \sum_{i=1}^{\infty}\tilde{f}(i,R_1,\Delta u(i-1))\le 2|a_1|_1\|u\|$$

that $2|a_1|_1\|u\|\ge 1$. Then, by (H2), we have

$$C+2|a_1|_1\|u\|\le 2C|a_1|_1\|u\|+2|a_1|_1\|u\|\le\|u\|.$$

On the other hand, since $\liminf_{w,z\to+\infty}\min_{x\in[1,m]_{\mathbb{Z}}}\frac{f(x,(1+x)w,z)}{w+z}=b_2(x)$, there exist $R_2>R_1>0$ such that

$$\min_{x\in[1,m]_{\mathbb{Z}}} f(x,(1+x)w,z)\ge b_2(x)(w+z),\quad w,z\ge R_2,\ x\in[1,m]_{\mathbb{Z}}. \tag{3.2}$$

Set $\Omega_2=\{u\in X:\|u\|_1<2m(1+m)R_2\}$. For any $u\in K\cap\partial\Omega_2$, according to (H1), (3.2), Lemma 2.5 and $\frac{a}{2m(1+m)^2}\sum_{x=1}^{m}b_2(x)\ge 1$, we have

$$\begin{aligned}
\|Tu\|_1&=\sup_{x\in\mathbb{N}_0}\Big|\frac{aC+B+xC}{1+x}+\sum_{i=1}^{\infty}\frac{G(x,i)}{1+x}f(i,u(i),\Delta u(i-1))\Big|\\
&\ge\sum_{i=1}^{\infty}\frac{G(x,i)}{1+x}f(i,u(i),\Delta u(i-1))
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{a}{1+m}\sum_{i=1}^{m} f\big(i,(1+i)\frac{u(i)}{1+i},\Delta u(i-1)\big)\\
&\geq \frac{a}{1+m}\sum_{i=1}^{m} b_2(i)\big(\frac{u(i)}{1+i}+\Delta u(i-1)\big)\\
&\geq \frac{a}{1+m}\sum_{i=1}^{m} b_2(i)\frac{u(i)}{1+i} \geq \frac{a}{1+m}\sum_{i=1}^{m} b_2(i)\frac{\|u\|_1}{2i(1+i)}\\
&\geq \frac{a}{2m(1+m)^2}\sum_{i=1}^{m} b_2(i)\|u\|_1 \geq \|u\|_1.
\end{aligned}
$$

By virtue of Lemma 2.2, we show that problem (1.1) has at least one positive solution. □

Proof of Theorem 1.2 Since $\liminf_{w,z\to 0}\min_{x\in[1,m]_{\mathbb{Z}}}\frac{f(x,(1+x)w,z)}{w+z}=a_2(x)$, there exists $R_3>0$ such that

$$
\min_{x\in[1,m]_{\mathbb{Z}}} f(x,(1+x)w,z) \geq a_2(x)(w+z), \quad w,z\geq R_3. \tag{3.3}
$$

For any $u\in K$, $x\in[1,m]_{\mathbb{Z}}$, we get

$$
\frac{u(x)}{1+x} \geq \frac{\|u\|_1}{2m(1+m)}.
$$

Set $\Omega_3=\{u\in X:\|u\|_1<2m(1+m)R_3\}$. For all $u\in K\cap\partial\Omega_3$, according to (H1), (3.3), Lemma 2.5 and $\frac{a}{2m(1+m)^2}\sum_{x=1}^{m}a_2(x)\geq 1$, we have

$$
\begin{aligned}
\|Tu\|_1 &= \sup_{x\in\mathbb{N}_0}\Big|\frac{aC+B+Cx}{1+x}+\sum_{i=1}^{\infty}\frac{G(x,i)}{1+x}f(i,u(i),\Delta u(i-1))\Big|\\
&\geq \Big|\frac{aC+B+Cx}{1+x}\Big|+\Big|\sum_{i=1}^{\infty}\frac{G(x,i)}{1+x}f(i,u(i),\Delta u(i-1))\Big|\\
&\geq \frac{1}{1+x}\sum_{i=1}^{\infty}G(x,i)f(i,u(i),\Delta u(i-1))\\
&\geq \frac{a}{1+m}\sum_{i=1}^{m} f\big(i,(1+i)\frac{u(i)}{1+i},\Delta u(i-1)\big)\\
&\geq \frac{a}{1+m}\sum_{i=1}^{m} a_2(i)\big(\frac{u(i)}{1+i}+\Delta u(i-1)\big) \geq \frac{a}{1+m}\sum_{i=1}^{m} a_2(i)\frac{u(i)}{1+i}\\
&\geq \frac{a}{2m(1+m)^2}\sum_{i=1}^{m} a_2(i)\|u\|_1 \geq \|u\|_1.
\end{aligned}
$$

On the other hand, since $\limsup_{w,z\to+\infty}\max_{x\in\mathbb{N}}\frac{f(x,(1+x)w,z}{w+z}=b_1(x)$, there exists $R_\epsilon>0$ such that

$$
\max_{x\in\mathbb{N}} f(x,(1+x)w,z) \leq b_1(x)(w+z), \quad 0<w,\ z\leq R_\epsilon. \tag{3.4}
$$

Let $R_4>\max\{2m(1+m)R_3,R_2,R_\epsilon\}$. Set $\Omega_4=\{u\in X:\|u\|<R_4\}$. For any $u\in K\cap\partial\Omega_4$,

from (H1), (3.4) and $2(C+1)|b_1|_1 \le 1$, it follows

$$\|Tu\| = \|Tu\|_2 = \sup_{x\in\mathbb{N}_0}\Big|C + \sum_{i=1}^{\infty}(G(x+1,i) - G(x,i))f(i,u(i),\Delta u(i-1))\Big|$$

$$\le C + \sum_{i=1}^{\infty}\widetilde{f}(i,R_4,\Delta u(i-1))$$

$$= C + \sum_{i=1}^{\infty}\max_{i\in\mathbb{N}} f\big(i,(1+i)\frac{u(i)}{1+i},\Delta u(i-1)\big)$$

$$\le C + |b_1|_1\big(\frac{u(i)}{1+i} + \Delta u(i-1)\big) \le C + 2|b_1|_1\|u\|.$$

Moreover, it deduces from (H1) and

$$\sum_{i=1}^{\infty}\tilde{f}(i,R_4,\Delta u(i-1)) \ge 1, \quad \sum_{i=1}^{\infty}\tilde{f}(i,R_4,\Delta u(i-1)) \le 2|b_1|_1\|u\|$$

that $2|b_1|_1\|u\| \ge 1$, then, by (H3), we have

$$C + 2|b_1|_1\|u\| \le 2C|b_1|_1\|u\| + 2|b_1|_1\|u\| \le \|u\|.$$

Thus it follows from Lemma 2.2 that problem (1.1) has at least one positive solution. □

Example 3.1 Set

$$f(x,u,v) = \frac{1}{16(1+x)^{\frac{3}{2}}}\Big[\frac{\frac{u}{1+x}+v}{e^{\frac{u}{1+x}+v}} + 16\sqrt{2}\Big].$$

Assume $m \in \mathbb{N}$, $m > 1$, $a > 0$ and satisfy

$$\frac{1}{8}\Big(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2+m}}\Big) \ge \frac{2m(1+m)^2}{a}. \tag{3.5}$$

Then problem (1.1) has at least one positive solution. In fact, for all $x, w, z > 0$ with $w < R$,

$$f(x,(1+x)w,z) = \frac{1}{16(1+x)^{\frac{3}{2}}}\Big[\frac{w+z}{e^{w+z}} + 16\sqrt{2}\Big]$$

and $\tilde{f}(x,R,z) \ge \frac{\sqrt{2}}{(1+x)^{\frac{3}{2}}}$. Hence,

$$\sum_{x=1}^{\infty}\tilde{f}(x,R,z) > \sum_{k=1}^{\infty}\int_k^{k+1}\frac{\sqrt{2}}{(x+1)^{\frac{3}{2}}}\mathrm{d}x = \int_1^{\infty}\frac{\sqrt{2}}{(1+x)^{\frac{3}{2}}}\mathrm{d}x = 2 > 1.$$

It is not difficult to prove $\sum_{x=1}^{\infty}\tilde{f}(x,R,z) < \infty$ for any $z \ge 0$, and so (H1) holds. Moreover,

$$\limsup_{w,z\to+\infty}\max_{x\in\mathbb{N}}\frac{f(x,(1+x)w,z)}{w+z} = \frac{1}{8(1+x)^{\frac{3}{2}}} = b_1(x), \tag{3.6}$$

$$\liminf_{w,z\to 0}\min_{x\in[1,m]_{\mathbb{N}}}\frac{f(x,(1+x)w,z)}{w+z} = \frac{1}{16(1+x)^{\frac{3}{2}}} = a_2(x). \tag{3.7}$$

By (3.6), (3.7), for any $0 \le C \le 1$, we can get

$$\sum_{x=1}^{+\infty}b_1(x) = \sum_{x=1}^{+\infty}\frac{1}{8(1+x)^{\frac{3}{2}}} \le \sum_{k=1}^{+\infty}\int_{k-1}^{k}\frac{1}{8(1+x)^{\frac{3}{2}}}\mathrm{d}x \le \int_0^{+\infty}\frac{1}{8(1+x)^{\frac{3}{2}}}\mathrm{d}x = \frac{1}{4} \le \frac{1}{2(C+1)},$$

$$\sum_{x=1}^{m} a_2(x) = \sum_{x=1}^{m} \frac{1}{16(1+x)^{\frac{3}{2}}} \geq \sum_{x=1}^{m} \int_{k}^{k+1} \frac{1}{16(1+x)^{\frac{3}{2}}} \mathrm{d}x \geq \int_{1}^{m+1} \frac{1}{16(1+x)^{\frac{3}{2}}} \mathrm{d}x$$
$$= \frac{1}{8}\Big(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2+m}}\Big).$$

Combining with (3.5), the condition (H3) is true. According to Theorem 1.1, (1.1) has at least one positive solution. □

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