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Gabor Analysis on Finite Product of Cyclic Groups

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Abstract We present a complete characterization of Gabor frame operators on finite dimensional Hilbert space $L^2(\Gamma)$, where $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$ and $m_1, m_2, \ldots, m_p$ are positive integers. The notion of generalized B-modulation and generalized B-translation is introduced and some significant properties of the generalized pseudo B-Gabor like frames are discussed.

Keywords Gabor frame; Gabor frame operator; modulation; translation

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1. Introduction

One of the fascinating and attractive research area in mathematics is frame theory, which was developed by Duffin and Schaeffer [1] and exhibited mostly through [2–4]. Time-frequency analysis of signals in $L^2(\mathbb{R})$, as suggested by Gabor in Theory of Communication [5], aims at representing functions (signals) in $L^2(\mathbb{R})$ as superposition of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$. Excellent work of Janssen [6] in 1980's formulated frame theory as an independent topic of mathematical research. Daubechies, Grossmann and Meyer promoted the idea of combining Gabor analysis with frame theory in 1986 with the fundamental work [7]. A major component in frame theory is the frame operator associated with a given frame. In particular, Gabor frames and corresponding frame operators, which are very special in their construction, are flourished over the last few decades [8, 9].

Characterization of Gabor frame operators in $L^2(\mathbb{R})$ and in $l^2(\mathbb{Z}_{\mathbb{N}})$ was discussed in [9, 10]. Motivated from these works, here we discuss a characterization of Gabor frame operators in $L^2(\Gamma)$. This manuscript is arranged such a way that, Section 2 contains some basic definitions and results needed for the present work. In Section 3 we characterize Gabor frame operators on $L^2(\Gamma)$, where Γ represents the direct product of cyclic groups in the form $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$ for positive integers $m_1, m_2, \ldots, m_p$. Section 4 contains discussion about Gabor semi-frame operators on $L^2(\Gamma)$. The concept of generalized pseudo B-Gabor like frames and some remarkable results about this are given in Section 5. For a precise treatment of frame theory and the theory of Gabor frames we refer the text books [11] and [12].

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2. Preliminaries

A countable sequence of elements $\{u_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is said to be a frame in $\mathcal{H}$, if there are positive constants α, β such that

$$\alpha\|x\|^2 \leq \sum_{k=1}^{\infty} |\langle x, u_k\rangle|^2 \leq \beta\|x\|^2, \quad \forall\, x \in \mathcal{H}.$$

These constants α, β are called frame bounds and are not unique. A frame $\{u_k\}_{k=1}^{\infty}$ is called a tight frame, if the frame bounds are same and is called a Parseval frame or normalized tight frame when $\alpha = \beta = 1$.

If $\{u_k\}_{k=1}^{\infty}$ is a frame in a Hilbert space $\mathcal{H}$, then the map S defined by $Sx = \sum_{k=1}^{\infty}\langle x, u_k\rangle u_k$ for all $x \in \mathcal{H}$ is a bounded, invertible, self-adjoint and positive operator on $\mathcal{H}$ (see [11]). It is called the frame operator associated with the frame $\{u_k\}_{k=1}^{\infty}$.

If S is the frame operator of a frame $\{u_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, then the frame $\{S^{-1}u_k\}_{k=1}^{\infty}$ is called the (canonical) dual frame of $\{u_k\}_{k=1}^{\infty}$. As follows, every frame in $\mathcal{H}$ admits the frame decomposition in two ways [11].

Theorem 2.1 *Let $\{u_k\}_{k=1}^{\infty}$ be a frame with frame operator S in a Hilbert space $\mathcal{H}$. Then for all $x \in \mathcal{H}$, $x = \sum_{k=1}^{\infty}\langle x, S^{-1}u_k\rangle u_k$ and $x = \sum_{k=1}^{\infty}\langle x, u_k\rangle S^{-1}u_k$. Both the series converge unconditionally for all $x \in \mathcal{H}$.*

3. Gabor analysis on direct product of finite cyclic groups

Fourier transform, translation operator and modulation operator have major role in Gabor analysis and the study about interplay between these operators makes Gabor analysis more beautiful. Through out in this article, we consider the finite product $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$, where $m_1, m_2, \ldots, m_p$ are positive integers. The space $L^2(\Gamma)$, consisting of all complex valued functions on Γ, is a finite dimensional Hilbert space w.r.t. the standard inner product given by

$$\langle f, g\rangle = \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_p=1}^{m_p} f(i_1, i_2, \ldots, i_p)\overline{g(i_1, i_2, \ldots, i_p)}, \quad f, g \in L^2(\Gamma).$$

For each $k = (k_1, k_2, \ldots, k_p) \in \Gamma$, the translation operator $T_k : L^2(\Gamma) \to L^2(\Gamma)$ is defined by

$$T_k g(r_1, r_2, \ldots, r_p) = g(s_1, s_2, \ldots, s_p), \quad (r_1, r_2, \ldots, r_p) \in \Gamma,$$

where $s = (s_1, s_2, \ldots, s_p) \in \Gamma$ is such that $r_j - k_j \equiv s_j \pmod{m_j}$; for $j = 1, 2, \ldots, p$. Similarly for $l = (l_1, l_2, \ldots, l_p) \in \Gamma$, the modulation operator $M_l : L^2(\Gamma) \to L^2(\Gamma)$ is defined by

$$M_l g(r_1, r_2, \ldots, r_p) = e^{2\pi i[\frac{l_1 r_1}{m_1} + \frac{l_2 r_2}{m_2} + \cdots + \frac{l_p r_p}{m_p}]} g(r_1, r_2, \ldots, r_p), \quad (r_1, r_2, \ldots, r_p) \in \Gamma.$$

Definition 3.1 *For $g \in L^2(\Gamma) \setminus \{0\}$, the collection of elements $\{M_l T_k g; k, l \in \Lambda \subset \Gamma \times \Gamma\}$ is called a Gabor System generated by the window function g. A Gabor frame (also known as a Weyl-Heisenberg frame) in $L^2(\Gamma)$ is a Gabor system which spans $L^2(\Gamma)$. The frame operator of such a Gabor frame is given by $S(f) = \sum_{k,l\in\Lambda}\langle f, M_l T_k g\rangle M_l T_k g$.*

A comprehensive study about the Gabor frames and corresponding frame operators in the space $l^2(\mathbb{Z}_N)$ was carried out in [10]. One of the significant property about the Gabor frame operator on $l^2(\mathbb{Z}_N)$ says that; the frame operator of a Gabor frame $\{M_lT_kg : (k,l) \in \Lambda\}$ in $l^2(\mathbb{Z}_N)$, where $\Lambda = \Lambda_1 \times \Lambda_2$ and Λ_1, Λ_2 are subgroups of $\mathbb{Z}_N$ commuting with translation T_k and modulation M_l for all $(k,l) \in \Lambda_1 \times \Lambda_2$. We present similar result in the space $L^2(\Gamma)$.

Lemma 3.2 *The frame operator of a Gabor frame $\{M_lT_kg : (k,l) \in \Lambda\}$ in $L^2(\Gamma)$, where $\Lambda = \Lambda_1 \times \Lambda_2$ and Λ_1, Λ_2 are subgroups of Γ, commutes with all translations T_k and all modulations M_l for $(k,l) \in \Lambda_1 \times \Lambda_2$.*

Proof For a non zero $g \in L^2(\Gamma)$, consider a Gabor frame $\{M_lT_kg : (k,l) \in \Lambda\}$ in $L^2(\Gamma)$ with frame operator $S(f) = \sum_{(k,l)\in\Lambda} \langle f, M_lT_kg\rangle M_lT_kg$. Now for any $l' = (l'_1, l'_2, \dots, l'_p) \in \Lambda_2$, we have

$$\begin{aligned} S(M_{l'}f) &= \sum_{(k,l)\in\Lambda} \langle M_{l'}f, M_lT_kg\rangle M_lT_kg = \sum_{(k,l)\in\Lambda} \langle f, M_qM_lT_kg\rangle M_lT_kg \\ &= \sum_{(k,r)\in\Lambda} \langle f, M_rT_kg\rangle M_lT_kg = \sum_{(k,r)\in\Lambda} \langle f, M_rT_kg\rangle M_{l'}M_rT_kg \\ &= M_{l'} \sum_{(k,r)\in\Lambda} \langle f, M_rT_kg\rangle M_rT_kg = M_{l'}S(f). \end{aligned}$$

Where $q = (q_1, q_2, \dots, q_p) \in \Lambda_2$ is such that $m_j - l'_j \equiv q_j \pmod{m_j}$, for $j = 1, 2, \dots, p$ and $r = (r_1, r_2, \dots, r_p) \in \Lambda_2$ is such that $q_j + l_j \equiv r_j \pmod{m_j}$, for $j = 1, 2, \dots, p$. So that $M_l = M_{l'}M_r$. Now for any $k' = (k'_1, k'_2, \dots, k'_p) \in \Lambda_1$, we have by using commutator relation

$$\begin{aligned} S(T_{k'}f) &= \sum_{(k,l)\in\Lambda} \langle T_{k'}f, M_lT_kg\rangle M_lT_kg = \sum_{(k,l)\in\Lambda} \langle T_{k'}f, T_kM_lg\rangle T_kM_lg \\ &= \sum_{(k,l)\in\Lambda} \langle f, T_tT_kM_lg\rangle T_kM_lg = \sum_{(t',l)\in\Lambda} \langle f, T_{t'}M_lg\rangle T_kM_lg \\ &= \sum_{(t',l)\in\Lambda} \langle f, T_{t'}M_lg\rangle T_{k'}T_{t'}M_lg = T_{k'} \sum_{(t',l)\in\Lambda} \langle f, M_lT_{t'}g\rangle M_lT_{t'}g \\ &= T_{k'}S(f). \end{aligned}$$

Where $m_j - k'_j \equiv t_j \pmod{m_j}$, for $j = 1, 2, \dots, p$ and $m_j - (t_j + k_j) \equiv t'_j \pmod{m_j}$, for $j = 1, 2, \dots, p$. So that $T_k = T_{k'}T_{t'}$. This completes the proof. □

The existence and construction of Gabor frames in $L^2(\Gamma)$ are very usual in the field of numerical harmonic analysis. Following result ensures the existence and it gives a systematic procedure for constructing such frames in the finite dimensional space $L^2(\Gamma)$.

Proposition 3.3 *If $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_p$ and $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_p$ are subgroups of $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$ with the property that Ω_j and Λ_j are subgroups of $\mathbb{Z}_{m_j}$ with $\mathrm{card}(\Omega_j \times \Lambda_j) \geq m_j$, for $j = 1, 2, \dots, p$, then there is a Gabor frame $\{M_lT_kg : k \in \Omega,\ l \in \Lambda\}$ for $L^2(\Gamma)$.*

Proof Let $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_p$ and $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_p$ are subgroups of $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$ such that Ω_j and Λ_j are subgroups of $\mathbb{Z}_{m_j}$ with $\mathrm{card}(\Omega_j \times \Lambda_j) \geq m_j$, for

$j = 1, 2, \ldots, p$. Now let $\text{card}(\Omega_j) = r_j$ and $\text{card}(\Lambda_j) = s_j$, for $j = 1, 2, \ldots, p$. Then $\Omega_j = \frac{m_j}{r_j}\mathbb{Z}_{r_j}$ and $\Lambda_j = \frac{m_j}{s_j}\mathbb{Z}_{s_j}$, furthermore, $r_j s_j \geq m_j$, for $j = 1, 2, \ldots, p$. Therefore, fundamental domain of Ω_j is $\mathbb{Z}_{\frac{m_j}{r_j}}$ for each j and hence the fundamental domain of Ω is $\Delta = \mathbb{Z}_{\frac{m_1}{r_1}} \times \mathbb{Z}_{\frac{m_2}{r_2}} \times \cdots \times \mathbb{Z}_{\frac{m_p}{r_p}}$. Also it is easy to see that $\Lambda_j^{\perp} = s_j\mathbb{Z}_{\frac{m_j}{s_j}}$ for each j and hence its fundamental domain is $\mathbb{Z}_{s_j}$. Thus the fundamental domain of $\Lambda^{\perp}$ is $\mathbb{Q} = \mathbb{Z}_{s_1} \times \mathbb{Z}_{s_2} \times \cdots \times \mathbb{Z}_{s_p}$. Since $\frac{m_j}{r_j} \leq s_j$ for each j it is clear that $\Delta \subseteq \mathbb{Q}$. Hence by [11, Theorem 21.5.4] for any $c > 0$ and $g = c\chi_{\Delta}$, the family $\{M_l T_k g : k \in \Omega, l \in \Lambda\}$ is a Gabor frame in $L^2(\Gamma)$. □

Remark 3.4 A subgroup H of $\Gamma \times \Gamma$ is said to be definite subgroup, if H is of the form $H = \Omega \times \Lambda$, where Ω and Λ are subgroups of Γ as in Proposition 3.3. It is easy to see that $H = \Gamma \times \Gamma$ is a definite subgroup of $\Gamma \times \Gamma$. If H is a definite subgroup of $\Gamma \times \Gamma$, a Gabor frame in $L^2(\Gamma)$ of the form $\{M_l T_k g : k, l \in H\}$ is called a regular Gabor frame in $L^2(\Gamma)$.

An immediate consequence of Proposition 3.3 is the following.

Proposition 3.5 *There exists a Parseval Gabor frame in $L^2(\Gamma)$ with identity operator as its frame operator.*

Proof As observed in previous Proposition, for any subgroups $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_p$ and $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_p$ of Γ with $\text{card}(\Omega_j \times \Lambda_j) \geq m_j$, for $j = 1, 2, \ldots, p$; there is always a Gabor frame $\mathcal{G} = \{M_l T_k g : k \in \Omega, l \in \Lambda\}$ in $L^2(\Gamma)$. It is admitted that the frame operator S of this Gabor frame is bounded, positive, invertible and by Lemma 3.2, S commutes with involved modulations and translations. Hence $S^{-1/2}$ also has these properties. Therefore, $S^{-1/2}(\mathcal{G}) = S^{-1/2}\{M_l T_k g : k \in \Omega, l \in \Lambda\} = \{M_l T_k S^{-1/2} g : k \in \Omega, l \in \Lambda\}$ is a Gabor frame in $L^2(\Gamma)$. Now for each $f \in L^2(\Gamma)$,

$$
\begin{aligned}
f &= S^{-1/2} S S^{-1/2} f \\
&= S^{-1/2} \sum_{(k,l) \in \Omega \times \Lambda} \langle S^{-1/2} f, M_l T_k g \rangle M_l T_k g \\
&= \sum_{(k,l) \in \Omega \times \Lambda} \langle f, S^{-1/2} M_l T_k g \rangle S^{-1/2} M_l T_k g \\
&= \sum_{(k,l) \in \Omega \times \Lambda} \langle f, M_l T_k S^{-1/2} g \rangle M_l T_k S^{-1/2} g,
\end{aligned}
$$

hence the frame operator of the Gabor frame $\{M_l T_k S^{-1/2} g : k \in \Omega, l \in \Lambda\}$ is the identity operator. □

Construction of frames with a given operator as their frame operator is an interesting theme having practical relevance in the context of frames. As shown in [8], positive and invertible operators can become frame operators on separable Hilbert spaces. However, in view of Lemma 3.1, those operators on $L^2(\Gamma)$ which are positive and invertible but fail to commute with certain translation and modulation operators cannot correspond to any Gabor frame in $L^2(\Gamma)$. Here, we present a complete characterization of frame operators of Gabor frames in $L^2(\Gamma)$.

Proposition 3.6 *A bounded linear operator on $L^2(\Gamma)$ can be realized as a frame operator of a* *regular Gabor frame in $L^2(\Gamma)$ if and only if it is positive, invertible and commutes with translation operator $T_{k'}$ and modulation operator $M_{l'}$ for $k' \in \Omega$ and $l' \in \Lambda$.*

Proof Let $\{M_l T_k g : (k,l) \in \Omega \times \Lambda\}$ be a regular Gabor frame in $L^2(\Gamma)$ with S as its frame operator. It is known that S is positive, invertible and by Lemma 3.2, S commutes with translation operator $T_{k'}$ and modulation operator $M_{l'}$ for $k' \in \Omega$ and $l' \in \Lambda$.

Conversely, if S is a linear positive and invertible operator on $L^2(\Gamma)$ that commutes with translation operator $T_{k'}$ and modulation operator $M_{l'}$ for $k' \in \Omega$ and $l' \in \Lambda$, then so is its positive square root $S^{1/2}$. By Proposition 3.5, there is always a Parseval Gabor frame $\{M_l T_k g : (k,l) \in \Omega \times \Lambda\}$ in $L^2(\Gamma)$ with identity operator I as its frame operator. The invertibility and commutativity of $S^{1/2}$ ensure that the image of this frame under $S^{1/2}$ is a frame. This frame is a Gabor frame in $L^2(\Gamma)$, since

$$\begin{aligned} S^{1/2}(\{M_l T_k g : (k,l) \in \Omega \times \Lambda\}) &= \{S^{1/2} M_l T_k g : (k,l) \in \Omega \times \Lambda\} \\ &= \{M_l T_k S^{1/2} g : (k,l) \in \Omega \times \Lambda\}. \end{aligned}$$

Now for all $f \in L^2(\Gamma)$,

$$\begin{aligned} \sum_{(k,l)\in\Omega\times\Lambda} \langle f, M_l T_k S^{1/2} g\rangle M_l T_k S^{1/2} g &= S^{1/2} \sum_{(k,l)\in\Omega\times\Lambda} \langle S^{1/2} f, M_l T_k g\rangle M_l T_k g \\ &= S^{1/2} I S^{1/2}(f) = S(f). \end{aligned}$$

Thus S is the frame operator of the regular Gabor frame $\{M_l T_k S^{1/2} g : (k,l) \in \Omega \times \Lambda\}$. □

4. Gabor semi-frame operators on $L^2(\Gamma)$

Any finite sequence of elements in $L^2(\Gamma)$ can be considered as a Bessel sequence in $L^2(\Gamma)$. Let $\{u_k\}_{k\in\Lambda}$, $|\Lambda| < \infty$ be such a sequence. Then there is a bounded linear positive operator S on $L^2(\Gamma)$ defined by $S(f) = \sum_{k\in\Lambda} \langle f, u_k\rangle u_k$ for all $f \in L^2(\Gamma)$. We call this operator as the semi-frame operator associated to $\{u_k\}_{k\in\Lambda}$.

Analogously, the family $\mathcal{G}(g,\Lambda) = \{M_l T_k g : (k,l) \in \Lambda\}$ where $g \in L^2(\Gamma)$ and $\Lambda \subseteq \Gamma \times \Gamma$, is a Bessel sequence in $L^2(\Gamma)$. Hence there is an associated semi-frame operator $S_{\mathcal{G},\Lambda}$ corresponding to $\mathcal{G}(g,\Lambda)$, called the Gabor semi-frame operator associated with the generating set Λ and generating function g.

Proposition 4.1 *Let $\Delta = \Delta_1 \times \Delta_2 \subseteq \Gamma \times \Gamma$ be such that $\Omega = \Delta_1 - (r_1, r_2, \ldots, r_p)$ and $\Lambda = \Delta_2 - (t_1, t_2, \ldots, t_p)$ are subgroups of Γ for some $(r_1, r_2, \ldots, r_p)$, $(t_1, t_2, \ldots, t_p) \in \Gamma$ with $\operatorname{card}(\Omega_j \times \Lambda_j) \geq m_j$, for $j = 1, 2, \ldots, p$. If S is the Gabor semi-frame operator on $L^2(\Gamma)$ associated with $\mathcal{G}(g,\Delta)$, then there are Gabor semi-frame operators S_r and S_t on $L^2(\Gamma)$ such that $ST_r = T_r S_r$ and $SM_t = M_t S_t$. Moreover, $S_r T_h = T_h S_r$ for all $h \in \Omega$ and $S_t M_p = M_p S_t$ for all $p \in \Lambda$.*

Proof Let S be the semi-frame operator of $\mathcal{G}(g,\Lambda)$ as given in the statement. Then for $f \in L^2(\Gamma)$,

$S(f) = \sum_{(k,l)\in\Delta_1\times\Delta_2}\langle f, M_lT_kg\rangle M_lT_kg$.

$$\begin{aligned} ST_r(f) &= \sum_{(k,l)\in\Delta_1\times\Delta_2}\langle T_r(f), M_lT_kg\rangle M_lT_kg \\ &= \sum_{(k,l)\in\Delta_1\times\Delta_2}\langle f, T_{-r}M_lT_kg\rangle M_lT_kg \\ &= \sum_{(q,l)\in\Omega\times\Delta_2}\langle f, e^{-2\pi i[t_1+t_2+\cdots+t_p]}M_lT_qg\rangle M_lT_rT_qg, \end{aligned}$$

where $t=(t_1,t_2,\ldots,t_p)\in\Gamma$ is such that $(m_j-l_j)r_j\equiv t_j \pmod{m_j}$; and $q=(q_1,q_2,\ldots,q_p)\in\Omega$ satisfies $k_j-r_j\equiv q_j \pmod{m_j}$, for $j=1,2,\ldots,p$, and hence $T_k=T_rT_q$,

$$ST_r(f) = T_r\sum_{(q,l)\in\Omega\times\Delta_2}\langle f, M_lT_qg\rangle M_lT_qg = T_rS_r(f),$$

where $S_r(f)=\sum_{(q,l)\in\Omega\times\Delta_2}\langle f, M_lT_qg\rangle M_lT_qg$. Also,

$$\begin{aligned} SM_t(f) &= \sum_{(k,l)\in\Delta_1\times\Delta_2}\langle M_t(f), M_lT_kg\rangle M_lT_kg \\ &= \sum_{(k,l)\in\Delta_1\times\Delta_2}\langle f, M_{-t}M_lT_kg\rangle M_lT_kg \\ &= \sum_{(k,q)\in\Delta_1\times\Lambda}\langle f, M_qT_kg\rangle M_tM_qT_kg, \end{aligned}$$

where $q=(q_1,q_2,\ldots,q_p)\in\Lambda$ is such that $l_j-t_j\equiv q_j \pmod{m_j}$, for $j=1,2,\ldots,p$, and hence $M_l=M_tM_q$,

$$SM_t(f) = M_t\sum_{(k,q)\in\Delta_1\times\Lambda}\langle f, M_qT_kg\rangle M_qT_kg = M_tS_t(f),$$

where $S_t(f)=\sum_{(k,q)\in\Delta_1\times\Lambda}\langle f, M_qT_kg\rangle M_qT_kg$. Hence there are Gabor semi-frame operators S_r and S_t on $L^2(\Gamma)$ such that $ST_r=T_rS_r$ and $SM_t=M_tS_t$. Now for $h\in\Omega$,

$$\begin{aligned} S_rT_h(f) &= \sum_{(k',l)\in\Omega\times\Delta_2}\langle T_h(f), M_lT_{k'}g\rangle M_lT_{k'}g \\ &= \sum_{(k',l)\in\Omega\times\Delta_2}\langle f, T_{-h}M_lT_{k'}g\rangle M_lT_{k'}g \\ &= \sum_{(k',l)\in\Omega\times\Delta_2}\langle f, e^{-2\pi i[t_1+t_2+\cdots+t_p]}M_lT_{-h}T_{k'}g\rangle M_lT_{k'}g \\ &= \sum_{(q,l)\in\Omega\times\Delta_2}\langle f, e^{-2\pi i[t_1+t_2+\cdots+t_p]}M_lT_qg\rangle M_lT_hT_qg, \end{aligned}$$

where $t=(t_1,t_2,\ldots,t_p)\in\Gamma$ is such that $(m_j-l_j)h_j\equiv t_j \pmod{m_j}$; and $q=(q_1,q_2,\ldots,q_p)\in\Omega$ satisfies $q_j\equiv k'_j-h_j \pmod{m_j}$, for $j=1,2,\ldots,p$, and hence $T_{k'}=T_hT_q$,

$$S_rT_h(f) = T_h\sum_{(q,l)\in\Omega\times\Delta_2}\langle f, M_lT_qg\rangle M_lT_qg = T_hS_r(f).$$

Similarly, for each $p\in\Lambda$, $S_tM_p=M_pS_t$. Thus S_r commutes with all translations T_h for every $h\in\Omega$ and S_t commutes with all modulations M_p for every $p\in\Lambda$. □

Remark 4.2 It can be noted that if S is a Gabor semi-frame operator as in Proposition 4.1, then the invertibility of S, S_r and S_t are equivalent.

5. Generalized pseudo B-Gabor like frames and operators in finite Hilbert spaces

In [11, Corollary 5.3.2], about the interplay of bounded linear operators between two separable Hilbert spaces says that, if $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces, then every surjective bounded linear operator $A : \mathcal{H} \to \mathcal{K}$ maps frames in $\mathcal{H}$ to frames in $\mathcal{K}$. In particular, every invertible bounded linear operator between two separable Hilbert spaces maps frames in one to frames in the other.

This motivates us to look at the images of Gabor frames under invertible bounded linear operators $B : L^2(\Gamma) \to \mathcal{H}$, since

$$\begin{aligned} B(\{M_l T_k g : k, l \in \Gamma\}) &= \{BM_l T_k g : k, l \in \Gamma\} \\ &= \{BM_l B^{-1} BT_k B^{-1} Bg : k, l \in \Gamma\} \\ &= \{(BM_l B^{-1})(BT_k B^{-1})(Bg) : k, l \in \Gamma\}, \end{aligned}$$

they are generated by the action of a family of operators $\{M_l^B T_k^B : k, l \in \Gamma\}$ on a single generator Bg, where $M_l^B = BM_l B^{-1}$ and $T_k^B = BT_k B^{-1}$. Thus, such image frames are structured frames in $\mathcal{H}$. We will formulate the following definitions which will be useful in our further discussions.

Definition 5.1 *Let $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$. For an invertible bounded linear operator $B : L^2(\Gamma) \to \mathcal{H}$ and $k \in \Gamma$, generalized B-translation T_k^B on $\mathcal{H}$ is defined by $T_k^B = BT_k B^{-1}$ and for $l \in \Gamma$, generalized B-modulation M_l^B on $\mathcal{H}$ is defined by $M_l^B = BM_l B^{-1}$, where T_k and M_l are respectively the translation and modulation operators on $L^2(\Gamma)$.*

The family $\{M_l^B T_k^B g : k, l \in \Gamma\}$ generated by $g \in \mathcal{H}$ is called a generalized pseudo B-Gabor like system in $\mathcal{H}$. Such a system is called a generalized pseudo B-Gabor like frame (generalized pseudo B-Gabor like Bessel sequence) if it forms a frame (Bessel sequence) in $\mathcal{H}$. A frame $\mathcal{G}$ in $\mathcal{H}$ is called a generalized pseudo Gabor like frame, if $\mathcal{G}$ is a generalized pseudo B-Gabor like frame for some invertible operator $B : L^2(\Gamma) \to \mathcal{H}$.

Gabor type unitary systems discussed in [14] were defined by generalizing the remarkable property $T_a E_b = e^{-2\pi i a b} E_b T_a$ of the pair (T_a, E_b) of translation and modulation operators on $L^2(\mathbb{R})$. Interestingly, for the generalization of the system of operators generated by the combination $T_a E_b$, we need not stick on to the unitary system. Instead, a system of invertible operators can be considered, as Proposition 5.2 below suggests.

Proposition 5.2 *Let $B : L^2(\Gamma) \to \mathcal{H}$ be invertible. Then the following statements hold.*

(i) $M_l^B T_k^B = e^{2\pi i [t_1 + t_2 + \cdots + t_p]} T_k^B M_l^B$ *for all* $k, l \in \Gamma$ *and* $t_j = \frac{(m_j - l_j) k_j}{m_j}$ *for* $j = 1, 2, \ldots, p$.

(ii) $\{M_l^B T_k^B g : k, l \in \Gamma\}$ *is a generalized pseudo B-Gabor like frame in $\mathcal{H}$ if and only if the family* $\{T_k^B M_l^B g : k, l \in \Gamma\}$ *is also a frame in $\mathcal{H}$.*

Proof First we establish the commutator relation between M_l and T_k analogues to $T_aE_b = e^{-2\pi iab}E_bT_a$ of the pair (T_a, E_b) of translation and modulation operators on $L^2(\mathbb{R})$ (see [11]). For all $k, l \in \Gamma$ and $g \in L^2(\Gamma)$

$$\begin{aligned} M_lT_kg(r_1, r_2, \dots, r_p) &= M_lg(s_1, s_2, \dots, s_p) \text{ where}, r_j - k_j \equiv s_j \pmod{m_j}, \text{ for } j = 1, 2, \dots, p \\ &= e^{2\pi i[\frac{l_1s_1}{m_1} + \frac{l_2s_2}{m_2} + \cdots + \frac{l_ps_p}{m_p}]} g(s_1, s_2, \dots, s_p), \end{aligned}$$

but then we have $l_j(r_j - k_j) \equiv l_js_j \pmod{m_j}$, for $j = 1, 2, \dots, p$. Hence,

$$\begin{aligned} M_lT_kg(r_1, r_2, \dots, r_p) &= e^{2\pi i[\frac{l_1r_1}{m_1} + \cdots + \frac{l_pr_p}{m_p}]} e^{2\pi i[\frac{(m_1-l_1)k_1}{m_1} + \cdots + \frac{(m_p-l_p)k_p}{m_p}]} g(s_1, s_2, \dots, s_p) \\ &= e^{2\pi i[\frac{l_1r_1}{m_1} + \cdots + \frac{l_pr_p}{m_p}]} e^{2\pi i[\frac{(m_1-l_1)k_1}{m_1} + \cdots + \frac{(m_p-l_p)k_p}{m_p}]} T_kg(r_1, r_2, \dots, r_p) \\ &= e^{2\pi i[\frac{(m_1-l_1)k_1}{m_1} + \cdots + \frac{(m_p-l_p)k_p}{m_p}]} T_k(e^{2\pi i[\frac{l_1r_1}{m_1} + \cdots + \frac{l_pr_p}{m_p}]} g(r_1, r_2, \dots, r_p)) \\ &= e^{2\pi i[t_1 + t_2 + \cdots + t_p]} T_kM_lg(r_1, r_2, \dots, r_p), \end{aligned}$$

where $t_j = \frac{(m_j-l_j)k_j}{m_j}$, for $j = 1, 2, \dots, p$. Thus $M_l^BT_k^B = e^{2\pi i[t_1+t_2+\cdots+t_p]}T_k^BM_l^B$ for all $k, l \in \Gamma$ and $t_j = \frac{(m_j-l_j)k_j}{m_j}$, for $j = 1, 2, \dots, p$, proving (i).

Since $\{M_l^BT_k^Bg : k, l \in \Gamma\}$ is a frame in $\mathcal{H}$ and $e^{2\pi i[t_1+\cdots+t_p]}$ is of absolute value 1, the necessary frame inequality for the collection $\{T_k^BM_l^Bg : k, l \in \Gamma\}$ follows immediately from that of $\{M_l^BT_k^Bg : k, l \in \Gamma\}$. The reverse implication follows likewise. □

Forthcoming proposition gives a connection between generalized pseudo B-Gabor like frame in $\mathcal{H}$ and Gabor frame in $L^2(\Gamma)$.

Proposition 5.3 *The family $\{M_l^BT_k^Bg : k, l \in \Gamma\}$ forms a generalized pseudo B-Gabor like frame in $\mathcal{H}$ if and only if $\{M_lT_kB^{-1}g : k, l \in \Gamma\}$ forms a Gabor frame in $L^2(\Gamma)$.*

Proof Since

$$\begin{aligned} \{M_l^BT_k^Bg : k, l \in \Gamma\} &= \{BM_lB^{-1}BT_kB^{-1}g : k, l \in \Gamma\} \\ &= B(\{M_lT_kB^{-1}g : k, l \in \Gamma\}), \end{aligned}$$

the proof follows for both the cases of implications from the fact that, surjective maps takes frames into frames. □

For each invertible bounded linear map $B : L^2(\Gamma) \to \mathcal{H}$, the bounded linear operator BB^* on $\mathcal{H}$ is positive and invertible. Hence it becomes a frame operator of some frame in $\mathcal{H}$ (see [8]). Interestingly, this frame operator corresponds to a generalized pseudo B-Gabor like frame in $\mathcal{H}$.

Apart from the positivity and invertibility of the Gabor frame operators on $L^2(\Gamma)$, their commutativity with some specific modulation and translation operators were significant in characterizing the Gabor frame operators [9,10]. Here we look at the similar situation in the context of generalized pseudo B-Gabor like frames.

Theorem 5.4 *The following are equivalent for a given invertible bounded linear operator $B : L^2(\Gamma) \to \mathcal{H}$.*

(i) B^*B is a Gabor frame operator on $L^2(\Gamma)$.

(ii) Every generalized pseudo B-Gabor like frame operator on $\mathcal{H}$ satisfies, $SM_l^B = M_l^B S$ and $ST_k^B = T_k^B S$.

(iii) There exists a generalized pseudo B-Gabor like frame $\mathcal{G}$ in $\mathcal{H}$ with frame operator as identity operator.

Proof (i) $\Rightarrow$(ii). Suppose that S is the frame operator of a generalized pseudo B-Gabor like frame $\{M_l^B T_k^B g : k, l \in \Gamma\}$ in $\mathcal{H}$. Then $B^{-1} : \mathcal{H} \to L^2(\Gamma)$ maps this frame to the Gabor frame $\{M_l T_k B^{-1} g : k, l \in \Gamma\}$ whose frame operator is $B^{-1}S(B^{-1})^*$. Hence the operator $B^{-1}S(B^{-1})^*$ commutes with M_l and T_k for all $k, l \in \Gamma$. Now, assuming (i), we obtain

$$\begin{aligned} SM_l^B &= SBM_l B^{-1} = S(B^{-1})^*(B^*B)M_l B^{-1} \\ &= S(B^{-1})^* M_l (B^*B)B^{-1} = S(B^{-1})^* M_l B^* \\ &= BB^{-1}S(B^{-1})^* M_l B^* = BM_l(B^{-1}S(B^{-1})^*)B^* \\ &= BM_l B^{-1} S = M_l^B S \text{ for all } l \in \Gamma. \end{aligned}$$

Similarly, $ST_k^B = T_k^B S$. This proves (ii).

(ii)$\Rightarrow$(iii). By Remark 3.4, there is always a Gabor frame $\mathcal{G}$ in $L^2(\Gamma)$ and hence there is a generalized pseudo B-Gabor like frame $\mathcal{P} = B(\mathcal{G})$ in $\mathcal{H}$. Since by (ii), the frame operator S of such a generalized pseudo B-Gabor like frame $\mathcal{P}$ commutes with its involved B-modulations and B-translations, so does the operator $S^{-1/2}$. Hence the image frame $S^{-1/2}(\mathcal{P})$ will be a generalized pseudo B-Gabor like frame in $\mathcal{H}$ with frame operator as identity operator.

(iii)$\Rightarrow$(i). Let $\mathcal{G}$ be a generalized pseudo B-Gabor like frame in $\mathcal{H}$ with frame operator as identity operator. Then $B^{-1}(\mathcal{G})$ will be a Gabor frame in $L^2(\Gamma)$ with frame operator $B^{-1}I(B^{-1})^* = (B^*B)^{-1}$. Then it is obvious that $(B^*B)^{-1}$ commutes with M_l and T_k for all $k, l \in \Gamma$ and hence its inverse B^*B also has this property. □

Thus, each B as above has a specific control in terms of the bounded linear operator B^*B on $L^2(\Gamma)$ for yielding Parseval generalized pseudo B-Gabor like frames in $\mathcal{H}$ as well as generalized pseudo B-Gabor like frames having canonical dual frames with same structure. Such frames are more similar to Gabor frames in $L^2(\Gamma)$. In view of the above discussions we give a new definition which is suitable for identifying the structures more specifically.

Definition 5.5 *A generalized pseudo B-Gabor like frame $\{M_l^B T_k^B Bg : k, l \in \Gamma\}$ in a separable Hilbert space $\mathcal{H}$ is said to be a generalized pseudo B-Gabor frame if B^*B is a Gabor frame operator on $L^2(\Gamma)$. The frame operator of a generalized pseudo B-Gabor frame is called a generalized pseudo B-Gabor frame operator.*

Now we look at the canonical dual frame of generalized pseudo B-Gabor frames in $\mathcal{H}$. An important consequence of Theorem 5.4 is the following.

Theorem 5.6 *Let $B : L^2(\Gamma) \to \mathcal{H}$ be an invertible map such that B^*B is a Gabor frame operator on $L^2(\Gamma)$. Then for any given Gabor frame $\{M_l T_k g : k, l \in \Gamma\}$ in $L^2(\Gamma)$ with frame operator S, the canonical dual frame of the generalized pseudo B-Gabor frame $\{M_l^B T_k^B Bg : k, l \in \Gamma\}$ in $\mathcal{H}$ is again a generalized pseudo B-Gabor frame with generator $CS^{-1}g$, where $C = (B^*)^{-1}$. Further,* *this dual is a generalized pseudo C-Gabor frame with same generator $CS^{-1}g$.*

Proof Let $B : L^2(\Gamma) \to \mathcal{H}$ be an invertible map. Take $C = (B^*)^{-1}$, then $C : L^2(\Gamma) \to \mathcal{H}$ is also an invertible map and $C^*C = ((B^*)^{-1})^*(B^*)^{-1} = B^{-1}(B^*)^{-1} = (B^*B)^{-1}$. Since B^*B is a Gabor frame operator on $L^2(\Gamma)$ so is its inverse $(B^*B)^{-1}$. Thus C^*C is a Gabor frame operator on $L^2(\Gamma)$.

Now, for a given Gabor frame $\mathcal{G} = \{M_lT_kg : k, l \in \Gamma\}$ in $L^2(\Gamma)$ with frame operator S, the frame operator of the generalized pseudo B-Gabor frame $B(\mathcal{G}) = \{M_l^BT_k^BBg : k, l \in \Gamma\}$ is BSB^*. Hence the canonical dual frame of $B(\mathcal{G})$ is

$$\begin{aligned}(BSB^*)^{-1}(B(\mathcal{G})) &= (BSB^*)^{-1}(\{M_l^BT_k^BBg : k, l \in \Gamma\} \\ &= \{M_l^BT_k^B(BSB^*)^{-1}Bg : k, l \in \Gamma\}, \quad \text{by Theorem 5.4 (ii)} \\ &= \{M_l^BT_k^B(B^*)^{-1}S^{-1}g : k, l \in \Gamma\} \\ &= \{M_l^BT_k^BCS^{-1}g : k, l \in \Gamma\}, \text{ since } C = (B^*)^{-1}.\end{aligned}$$

Thus, the canonical dual frame of the generalized pseudo B-Gabor frame $B(\mathcal{G})$ in $\mathcal{H}$ is again a generalized pseudo B-Gabor frame with generator $CS^{-1}g$ and same generating set.

Now, mapping the canonical dual Gabor frame $S^{-1}(\mathcal{G}) = \{M_lT_kS^{-1}g : k, l \in \Gamma\}$ of $\mathcal{G}$ by C, we obtain the generalized pseudo C-Gabor frame $\{M_l^CT_k^CCS^{-1}g : k, l \in \Gamma\}$. Frame operator of this frame is $CS^{-1}C^* = (B^*)^{-1}S^{-1}((B^*)^{-1})^* = (B^*)^{-1}S^{-1}(B^{-1}) = (BSB^*)^{-1}$, the canonical dual frame operator of $B(\mathcal{G})$.

Hence, both the frames $\{M_l^BT_k^BCS^{-1}g : (k, l) \in \Gamma\}$ and $\{M_l^CT_k^CCS^{-1}g : (k, l) \in \Gamma\}$ are dual frames of $B(\mathcal{G})$ with common generator $CS^{-1}g$. Clearly, if the map $B : L^2(\Gamma) \to \mathcal{H}$ is unitary, we have $C = (B^*)^{-1} = B$ so that the above frames are precisely the same. □

The following example is a particular situation of Theorem 5.6. If $B, C : L^2(\Gamma) \to \mathcal{H}$ are invertible with $C = (B^*)^{-1}$, then $\{M_l^BT_k^BCS^{-1}g : k, l \in \Gamma\}$ and $\{M_l^CT_k^CCS^{-1}g : k, l \in \Gamma\}$ are respectively, generalized pseudo B-Gabor frame and generalized pseudo C-Gabor frame on $\mathcal{H}$ with same generator $CS^{-1}g$ and same frame operator $(BSB^*)^{-1}$ and with same generating set.

Example 5.7 Fourier transform $\mathcal{F} : L^2(\Gamma) \to L^2(\Gamma)$ given by

$$\mathcal{F}(g(r_1, r_2, \ldots, r_p)) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_p=1}^{m_p} g(r_1, \ldots, r_p)e^{-2\pi i[\frac{i_1r_1}{m_1}+\cdots+\frac{i_pr_p}{m_p}]}$$

satisfies the commutator relations; $\mathcal{F}M_l = T_l\mathcal{F}$ and $\mathcal{F}T_k = E_{-k}\mathcal{F}$, for $k, l \in \Gamma$, and M_l, T_k are the modulation and translation in $L^2(\Gamma)$, respectively [13]. From literature it is a known fact that $\mathcal{F}^* = \mathcal{F}^{-1}$.

Choose non zero complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\beta_1, \beta_2, \ldots, \beta_p$ with $|\alpha_j| \neq |\beta_j|$, for $j = 1, 2, \ldots, p$. Let us denote $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ so that $\frac{1}{\alpha} = (\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_p})$, $\overline{\alpha} = (\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_p})$ and $|\alpha|^2 = (|\alpha_1|^2, |\alpha_2|^2, \ldots, |\alpha_p|^2)$. Now define $\phi_{(\alpha,\beta)} : \Gamma \to \mathbb{C}$ by $\phi_{(\alpha,\beta)}(r_1, r_2, \ldots, r_p) = \phi_{1(\alpha_1,\beta_1)}(r_1)\phi_{2(\alpha_2,\beta_2)}(r_2)\cdots\phi_{p(\alpha_p,\beta_p)}(r_p)$, where $\phi_{j(\alpha_j,\beta_j)} : \mathbb{Z}_{m_j} \to \mathbb{C}$

is given by, for all $r_j \in \mathbb{Z}_{m_j}$,

$$\phi_{j(\alpha_j,\beta_j)}(r_j) = \begin{cases} \alpha_j, & \text{if } r_j \text{ is odd,} \\ \beta_j, & \text{if } r_j \text{ is even,} \end{cases} \quad \text{for } j = 1, 2, \ldots, p.$$

The multiplication operator $M_{\phi_{(\alpha,\beta)}} : L^2(\Gamma) \to L^2(\Gamma)$ is defined by, for all $(r_1, r_2, \ldots, r_p) \in \Gamma$, $M_{\phi_{(\alpha,\beta)}}(f)(r_1, r_2, \ldots, r_p) = (\phi_{(\alpha,\beta)}.f)(r_1, r_2, \ldots, r_p) = \phi_{(\alpha,\beta)}(r_1, \ldots, r_p).f(r_1, \ldots, r_p)$, is invertible with inverse $M^{-1}_{\phi_{(\alpha,\beta)}} = M_{\phi_{(\frac{1}{\alpha},\frac{1}{\beta})}}$. Further the adjoint of the multiplication operator is $M^*_{\phi_{(\alpha,\beta)}} = M_{\phi_{(\overline{\alpha},\overline{\beta})}}$, also $M^*_{\phi_{(\alpha,\beta)}} M_{\phi_{(\alpha,\beta)}} = M_{\phi_{(\overline{\alpha},\overline{\beta})}} M_{\phi_{(\alpha,\beta)}} = M_{\phi_{(|\alpha|^2,|\beta|^2)}}$. Again we can see that multiplication operator $M_{\phi_{(\alpha,\beta)}}$ commutes with translation T_k and modulation M_l for $k, l \in \Gamma$. For $b = (b_1, b_2, \ldots, b_p) \in \Gamma$ and for all $f \in L^2(\Gamma)$, define $B : L^2(\Gamma) \to L^2(\Gamma)$ by $B(f) = (\mathcal{F} M_{\phi_{(\alpha,\beta)}} \mathcal{F}^{-1} M_b)(f)$. Since $\mathcal{F}$, M_b and $M_{\phi_{(\alpha,\beta)}}$ are invertible, B is invertible with inverse $B^{-1} = M_{-b} \mathcal{F} M_{\phi_{(\frac{1}{\alpha},\frac{1}{\beta})}} \mathcal{F}^{-1}$.

It follows from direct calculation that, for $k, l \in \Gamma$, B commutes with modulation M_l but B does not commute with translation T_k. Observe that, $B^* = (\mathcal{F} M_{\phi_{(\alpha,\beta)}} \mathcal{F}^{-1} M_b)^* = M_{-b} \mathcal{F} M_{\phi_{(\overline{\alpha},\overline{\beta})}} \mathcal{F}^{-1}$ and

$$\begin{aligned} B^*B &= (M_{-b} \mathcal{F} M_{\phi_{(\overline{\alpha},\overline{\beta})}} \mathcal{F}^{-1})(\mathcal{F} M_{\phi_{(\alpha,\beta)}} \mathcal{F}^{-1} M_b) \\ &= M_{-b} \mathcal{F} M_{\phi_{(\overline{\alpha},\overline{\beta})}} M_{\phi_{(\alpha,\beta)}} \mathcal{F}^{-1} M_b \\ &= \mathcal{F} M_{\phi_{(|\alpha|^2,|\beta|^2)}} \mathcal{F}^{-1}, \text{ using commutator relation.} \end{aligned}$$

Evidently, B^*B is self-adjoint, positive and invertible and a simple computation proves that B^*B commutes with modulation M_l and translation T_k for $k, l \in \Gamma$. Hence B^*B is a Gabor frame operator on $L^2(\Gamma)$. Again by using suitable commutator relations we see that,

$$\begin{aligned} (BM_lB^{-1})(f) &= (\mathcal{F} M_{\phi_{(\alpha,\beta)}} \mathcal{F}^{-1} M_b) M_l (M_{-b} \mathcal{F} M_{\phi_{(\frac{1}{\alpha},\frac{1}{\beta})}} \mathcal{F}^{-1})(f) \\ &= \mathcal{F} M_{\phi_{(\alpha,\beta)}} T_{-l} \mathcal{F}^{-1} \mathcal{F} M_{\phi_{(\frac{1}{\alpha},\frac{1}{\beta})}} \mathcal{F}^{-1})(f) \\ &= \mathcal{F} T_{-l} \mathcal{F}^{-1}(f) = M_l(f) \\ (BT_kB^{-1})(f) &= (\mathcal{F} M_{\phi_{(\alpha,\beta)}} \mathcal{F}^{-1} M_b) T_k (M_{-b} \mathcal{F} M_{\phi_{(\frac{1}{\alpha},\frac{1}{\beta})}} \mathcal{F}^{-1})(f) \\ &= (\mathcal{F} M_{\phi_{(\alpha,\beta)}} \mathcal{F}^{-1} e^{2\pi i [t_1 + t_2 + \cdots + t_p]} T_k M_b M_{-b} \mathcal{F} M_{\phi_{(\frac{1}{\alpha},\frac{1}{\beta})}} \mathcal{F}^{-1})(f) \\ &\quad \text{where } t_j \equiv (m_j - b_j) k_j \pmod{m_j} \quad \text{for } j = 1, 2, \ldots, p \\ &= e^{2\pi i [t_1 + t_2 + \cdots + t_p]} \mathcal{F} M_k \mathcal{F}^{-1}(f) \\ &= e^{2\pi i [t_1 + t_2 + \cdots + t_p]} T_k(f). \end{aligned}$$

Taking $C = (B^*)^{-1} = \mathcal{F} M_{\phi_{(\frac{1}{\overline{\alpha}},\frac{1}{\overline{\beta}})}} \mathcal{F}^{-1} M_b$ then $C^{-1} = M_{-b} \mathcal{F} M_{\phi_{(\overline{\alpha},\overline{\beta})}} \mathcal{F}^{-1}$ and by using appropriate commutator relations, we obtain

$$\begin{aligned} (CM_lC^{-1})(f) &= (\mathcal{F} M_{\phi_{(\frac{1}{\overline{\alpha}},\frac{1}{\overline{\beta}})}} \mathcal{F}^{-1} M_b) M_l (M_{-b} \mathcal{F} M_{\phi_{(\overline{\alpha},\overline{\beta})}} \mathcal{F}^{-1})(f) \\ &= \mathcal{F} M_{\phi_{(\frac{1}{\overline{\alpha}},\frac{1}{\overline{\beta}})}} T_{-l} \mathcal{F}^{-1} \mathcal{F} M_{\phi_{(\overline{\alpha},\overline{\beta})}} \mathcal{F}^{-1})(f) \\ &= \mathcal{F} T_{-l} \mathcal{F}^{-1}(f) = M_l(f) \end{aligned}$$

$$
\begin{aligned}
(CT_kC^{-1})(f) &= (\mathcal{F}M_{\phi_{(\frac{1}{\alpha},\frac{1}{\beta})}}\mathcal{F}^{-1}M_b)T_k(M_{-b}\mathcal{F}M_{\phi_{(\overline{\alpha},\overline{\beta})}}\mathcal{F}^{-1})(f) \\
&= \mathcal{F}M_{\phi_{(\frac{1}{\alpha},\frac{1}{\beta})}}\mathcal{F}^{-1}e^{2\pi i[t_1+t_2+\cdots+t_p]}T_kM_bM_{-b}\mathcal{F}M_{\phi_{(\overline{\alpha},\overline{\beta})}}\mathcal{F}^{-1})(f) \\
&\quad \text{where } t_j \equiv (m_j-b_j)k_j \pmod{m_j} \quad \text{for } j=1,2,\ldots,p \\
&= e^{2\pi i[t_1+t_2+\cdots+t_p]}\mathcal{F}M_k\mathcal{F}^{-1}(f) \\
&= e^{2\pi i[t_1+t_2+\cdots+t_p]}T_k(f).
\end{aligned}
$$

Thus $BM_lB^{-1} = CM_lC^{-1}$, and $BT_kB^{-1} = CT_kC^{-1}$. Therefore, $M_l^BT_k^B = M_l^CT_k^C$. Hence for a given Gabor frame $\mathcal{G} = \{M_lT_kg : k,l \in \Gamma\}$ in $L^2(\Gamma)$ with frame operator S, C image of the canonical dual of $\mathcal{G}$, $C\{M_lT_kS^{-1}g : k,l \in \Gamma\} = \{M_l^CT_k^CCS^{-1}g : k,l \in \Gamma\}$ and the canonical dual frame $(BSB^*)^{-1}(B(\mathcal{G})) = \{M_l^BT_k^BCS^{-1}g : k,l \in \Gamma\}$ of $B(\mathcal{G})$ are exactly same frames in $L^2(\Gamma)$ with same generator $CS^{-1}g$ and same frame operator $(BSB^*)^{-1}$.

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