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Maps Preserving Semi-Fredholm Operators with Fixed Nullity

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Abstract Let $\mathcal{X}$ be an infinite-dimensional complex Banach space, and $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$. Given an integer $n \geq 1$, we characterize all the bijective maps on $\mathcal{B}(\mathcal{X})$ preserving the difference of semi-Fredholm operators with nullity equal to n in both directions, and establish the structure of the given maps.

Keywords operator algebra; adjacency; preserver map; semi-Fredholm operator

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1. Introduction

The study of preserver problems has a long history and has established many remarkable results in the past decades. Preserver problems aim to characterize those linear or nonlinear maps on operator algebras preserving certain properties, subsets, or relations. As we know, the subject of Fredholm operators has been an active area in operator theory, which plays a key role in the research of Weyl type theorems and its variants [1–4]. Thus, on the one hand, several authors have studied the linear maps which preserve the class of semi-Fredholm operators, Fredholm operators and the related operators in both directions, respectively [5–9]. It has been shown that such maps preserve the ideal of compact operators in both directions, and that the maps induced by them on the Calkin algebra are Jordan automorphisms. For example, Cao in [5] discussed the linear surjective maps preserving upper semi-Weyl operators, and show that their induced maps on the Calkin algebra are Jordan automorphisms. However, the problem of determining the structure of these maps itself has not been solved.

On the other hand, many authors are interested in nonlinear preserver problems about the class of Fredholm operators and the related operators recently [10–15]. In [12], Bourhim, Mashreghi and Stepanyan characterized the nonlinear map $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ which preserves the difference of bounded below operators in both directions. Note that the set of all bounded below operators is a subset of all semi-Fredholm operators with zero nullity. Given any positive

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integer n, Ji, Jiao and Shi in [10] investigated the nonlinear maps preserving semi-Fredholm operator with nullity less than n.

Inspired by these, we will further consider the structure of nonlinear maps preserving the semi-Fredholm operators with fixed nullity, and completely describe the structure of these maps. Throughout this paper, $\mathcal{X}$ denotes an infinite-dimensional Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{X}^*$ is called the dual space of $\mathcal{X}$, $\mathcal{B}(\mathcal{X})$ denotes the algebra of all bounded linear operators acting on $\mathcal{X}$, and $\mathcal{F}(\mathcal{X})$ denotes the subspace of all finite rank operators in $\mathcal{B}(\mathcal{X})$. The linear span of a set S is denoted by $\mathrm{Span}\{S\}$. For $T \in \mathcal{B}(\mathcal{X})$, write $\ker(T)$ for its kernel and $\mathrm{ran}(T)$ for its range. Recall that an operator T is called semi-Fredholm if $\mathrm{ran}(T)$ is closed and either $\dim\ker(T)$ or $\mathrm{codim}\,\mathrm{ran}(T)$ is finite. For such operator T, the index is defined by $\mathrm{ind}(T) = \dim\ker(T)$-$\mathrm{codim}\,\mathrm{ran}(T)$. Let Λ be any subset of $\mathcal{B}(\mathcal{X})$. We say that a map $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ preserves the difference in Λ in both directions if for every $T_1, T_2 \in \mathcal{B}(\mathcal{X})$,

$$\varphi(T_1) - \varphi(T_2) \in \Lambda \ \Leftrightarrow \ T_1 - T_2 \in \Lambda.$$

For each integer $n \geq 1$, let us introduce the following set:

$$\mathcal{A}_n(X) = \{T \in \mathcal{B}(\mathcal{X}) : T \text{ is semi-Fredholm and } \dim\ker(T) = n\}.$$

We can observe that the identity $I \notin \mathcal{A}_n(X)$. The purpose of this paper is to consider the bijective map φ on $\mathcal{B}(\mathcal{X})$ which preserves the difference of $\mathcal{A}_n(X)$. More precisely, the following theorem states the main result of this paper:

Theorem 1.1 *Let $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a bijective map. The following assertions are equivalent:*

(i) φ preserves the difference of $\mathcal{A}_n(X)$ in both directions.

(ii) There is an operator $R \in \mathcal{B}(\mathcal{X})$ and there are two bijective continuous linear, or conjugate linear mappings $A : \mathcal{X} \to \mathcal{X}$ and $B : \mathcal{X} \to \mathcal{X}$, such that

$$\varphi(T) = ATB + R \text{ for all } T \in \mathcal{B}(\mathcal{X}),$$

or, two bijective continuous linear, or conjugate linear mappings $A : \mathcal{X}^* \to \mathcal{X}$ and $B : \mathcal{X} \to \mathcal{X}^*$ such that

$$\varphi(T) = AT^*B + R \text{ for all } T \in \mathcal{B}(\mathcal{X}).$$

In this case $\mathcal{X}$ is reflexive.

2. Main results

For $x \in \mathcal{X}$, $f \in \mathcal{X}^*$, we denote by $x \otimes f$ the rank one operator in $\mathcal{B}(\mathcal{X})$, and $(x \otimes f)(y) = f(y)x$ for all $y \in \mathcal{X}$. Recall that two operators are said to be adjacent if their difference is a rank one operator. We firstly give a characterization of the adjacency of operators in terms of operators in $\mathcal{A}_n(X)$.

Lemma 2.1 *Let $T, S \in \mathcal{B}(\mathcal{X})$ be two different operators. The following assertions are equivalent:*

(i) T and S are adjacent.

(ii) There exists an operator $R \in \mathcal{B}(\mathcal{X})\backslash\{T, S\}$ such that $\mathcal{A}_n(\mathcal{X}) + R \subseteq (\mathcal{A}_n(\mathcal{X}) + T) \cup (\mathcal{A}_n(\mathcal{X}) + S)$.

Proof (i)⇒(ii). Suppose that T and S are adjacent. Set $R = \frac{1}{2}(T+S)$, then $R \in \mathcal{B}(\mathcal{X})\backslash\{T, S\}$. Let $U \in \mathcal{B}(\mathcal{X})$ be such that $U - R \in \mathcal{A}_n(\mathcal{X})$. By [8, Lemma 5], we know that if $W \in \mathcal{B}(\mathcal{X})$. Then there exists a vector $a \in \mathcal{X}$ such that $\dim\ker(W + \lambda F) = \dim\ker(W)$ for every λ on $\mathbb{Q}\backslash\{-f(a)^{-1}\}$, where F is a rank one operator in $\mathcal{B}(\mathcal{X})$. It implies that

$$\dim\ker(W + F) = \dim\ker(W) \text{ or } \dim\ker(W - F) = \dim\ker(W)$$

for every rank one operator F in $\mathcal{B}(\mathcal{X})$. Notice that $T-S$ is a rank one operator. Let $W = U-R$. Then either $U - T = (U - R) - \frac{1}{2}(T - S) \in \mathcal{A}_n(\mathcal{X})$ or $U - S = (U - R) + \frac{1}{2}(T - S) \in \mathcal{A}_n(\mathcal{X})$.

(ii)⇒(i). Suppose that T and S are not adjacent, that is $\dim\operatorname{ran}(T - S) \geq 2$. Without loss of generality, we can assume that $R = 0$. So we need to prove that there exists $U \in \mathcal{A}_n(\mathcal{X})$ such that $U - T \notin \mathcal{A}_n(\mathcal{X})$ and $U - S \notin \mathcal{A}_n(\mathcal{X})$, i.e., $U \notin (\mathcal{A}_n(\mathcal{X}) + T) \cup (\mathcal{A}_n(\mathcal{X}) + S)$.

First, we claim that there exist $x_1, y_1 \in \mathcal{X}$ such that $\{Tx_1, Sy_1\}$ is a linearly independent set. Suppose on the contrary that $\{Tx\}$ and $\{Sy\}$ are linearly dependent for every $x, y \in \mathcal{X}$. Hence, we can easily get that $Sy \in \operatorname{Span}\{Tx\}$ for any $x \notin \ker(T)$ and any $y \in \mathcal{X} \setminus \operatorname{Span}\{x\}$. From linearity it follows that $\operatorname{ran}(S) \subseteq \operatorname{Span}\{Tx\}$, and in the same way, $\operatorname{ran}(T) \subseteq \operatorname{Span}\{Sy\}$ for some $y \notin \ker(S)$. This clearly contradicts non-adjacency.

Note that we may assume that x_1 and y_1 are linearly independent. Indeed, otherwise there exists a vector $u \in \mathcal{X}$ forming with y_1 a linearly independent set. Since $\theta : \lambda \mapsto S(y_1 + \lambda u)$ is continuous on $\mathbb{K}$ and $\operatorname{Span}\{Tx_1\}$ is a closed subset which does not contain $\theta(0) = Sy_1$ because $\{Tx_1, Sy_1\}$ is linearly independent, there exists a non-zero scalar α such that

$$S(y_1 + \alpha u) \notin \operatorname{Span}\{Tx_1\}.$$

Therefore, the sets $\{x_1, y_1 + \alpha u\}$ and $\{Tx_1, Sy_1\}$ are linearly independent.

We shall distinguish three cases.

Case 1. $\operatorname{ran}(T)$ and $\operatorname{ran}(S)$ are infinite-dimensional.

Since $\dim\operatorname{ran}(T) = \infty$ and $\dim\operatorname{ran}(S) = \infty$, we can find a linearly independent set $\{x_i, y_i \in \mathcal{X}, 2 \leq i \leq n+1\}$ such that $\{x_i, y_i : 1 \leq i \leq n+1\}$ and $\{Tx_i, Sy_i : 1 \leq i \leq n+1\}$ are linearly independent, respectively. Then there exists a subspace $\mathcal{M}_0 \subseteq \mathcal{X}$ satisfying $\dim\mathcal{M}_0 = n$ and $\mathcal{M}_0 \cap \{x_i, y_i : 1 \leq i \leq n+1\} = \{0\}$. Let $F : \mathcal{X}_1 \to \mathcal{X}_2$ be an isomorphism, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are closed subspaces of codimension $3n+2$, moreover,

$$\operatorname{Span}\{x_i, y_i : 1 \leq i \leq n+1\} \cap \mathcal{M}_0 \cap \mathcal{X}_1 = \{0\},$$

$$\operatorname{Span}\{Tx_i, Sy_i : 1 \leq i \leq n+1\} \cap \mathcal{X}_2 = \{0\}.$$

We define $U \in \mathcal{B}(\mathcal{X})$ by

$$\begin{cases} Ux_i = Tx_i, & \forall\, 1 \leq i \leq n+1; \\ Uy_i = Sy_i, & \forall\, 1 \leq i \leq n+1; \\ Ux = 0, & \forall\, x \in \mathcal{M}_0; \\ Ux = Fx, & \forall\, x \in \mathcal{X}_1. \end{cases}$$

Clearly, we have $\dim\ker(U)=n$, thus $U\in\mathcal{A}_n(\mathcal{X})$. On the other hand, note that

$$\dim\ker(U-T)\geq n+1 \text{ and } \dim\ker(U-S)\geq n+1.$$

It implies that $U-T$, $U-S\notin\mathcal{A}_n(\mathcal{X})$.

Case 2. Exactly one of $\mathrm{ran}(T)$ and $\mathrm{ran}(S)$ is finite-dimensional.

Note that $\mathrm{ran}(T)$ is infinite-dimensional, and so there exists $\{x_i:1\leq i\leq n+1\}$, such that $\{Sy_1,Tx_i:1\leq i\leq n+1\}$ is a linearly independent set. Let $y_i\in\ker(S)$, $2\leq i\leq n+1$ be such that $\{x_i,y_i:1\leq i\leq n+1\}$ and $\{Sy_1,Tx_i:1\leq i\leq n+1\}$ are linearly independent, respectively. Let $F:\mathcal{X}_1\to\mathcal{X}_2$ be an isomorphism, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are closed subspaces of codimension $2n+2$, moreover,

$$\mathrm{Span}\{x_i,y_i:1\leq i\leq n+1\}\cap\mathcal{X}_1=\{0\},$$

$$\mathrm{Span}\{Sy_1,Tx_i:1\leq i\leq n+1\}\cap\mathcal{X}_2=\{0\}.$$

Let $U\in\mathcal{B}(\mathcal{X})$ be expressed by

$$\begin{cases} Ux_i=Tx_i, & \forall\, 1\leq i\leq n+1;\\ Uy_1=Sy_1; & \\ Uy_i=0, & \forall\, 2\leq i\leq n+1;\\ Ux=Fx, & \forall\, x\in\mathcal{X}_1.\end{cases}$$

Clearly, we have $\dim\ker(U)=n$. It implies that $U\in\mathcal{A}_n(\mathcal{X})$. On the other hand, we have

$$\dim\ker(U-T)\geq n+1 \text{ and } \dim\ker(U-S)\geq n+1.$$

Thus $U-T$, $U-S\notin\mathcal{A}_n(\mathcal{X})$.

Case 3. Both $\mathrm{ran}(T)$ and $\mathrm{ran}(S)$ are finite-dimensional.

There exists a subspace $\mathcal{M}_0\subseteq\mathcal{X}$ satisfying $\dim\mathcal{M}_0=n$, $T\mid_{M_0}=S\mid_{M_0}=0$ and $\mathcal{M}_0\cap\mathrm{Span}\{x_1,y_1\}=\{0\}$. Let $F:\mathcal{X}_1\to\mathcal{X}_2$ be an isomorphism, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are closed subspaces of codimension $n+2$, moreover

$$\mathrm{Span}\{x_1,y_1\}\cap\mathcal{M}_0\cap\mathcal{X}_1=\{0\},$$

$$\mathrm{Span}\{Tx_1,Sy_1\}\cap\mathcal{X}_2=\{0\}.$$

Let $U\in\mathcal{B}(\mathcal{X})$ be expressed by

$$\begin{cases} Ux_1=Tx_1; & \\ Uy_1=Sy_1; & \\ Ux=0, & \forall\, x\in\mathcal{M}_0;\\ Ux=Fx, & \forall\, x\in\mathcal{X}_1.\end{cases}$$

Clearly, we have $\dim\ker(U)=n$. It implies that $U\in\mathcal{A}_n(\mathcal{X})$. On the other hand, we have

$$\dim\ker(U-T)\geq n+1 \text{ and } \dim\ker(U-S)\geq n+1.$$

Thus, $U-T$, $U-S\notin\mathcal{A}_n(\mathcal{X})$. This completes the proof. □

The following proposition establishes that, if the nonlinear map preserves semi-Fredholm operators with fixed nullity, then the map preserves adjacency in both directions.

Proposition 2.2 *Let $\varphi: \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a bijective map preserving the difference of $\mathcal{A}_n(\mathcal{X})$ in both directions. Then φ preserves adjacency in both directions.*

Proof Suppose that $T, S \in \mathcal{B}(\mathcal{X})$ are adjacent. It follows from Lemma 2.1 that there exists an operator $R \in \mathcal{B}(\mathcal{X})\backslash\{T, S\}$ such that $\mathcal{A}_n(\mathcal{X}) + R \subseteq (\mathcal{A}_n(\mathcal{X}) + T) \cup (\mathcal{A}_n(\mathcal{X}) + S)$. Then $\varphi(R) \in \mathcal{B}(\mathcal{X})\backslash\{\varphi(T), \varphi(S)\}$ because φ is bijective. Let $F \in \mathcal{B}(\mathcal{X})$ be such that $F - \varphi(R) \in \mathcal{A}_n(\mathcal{X})$. Since φ is bijective, there is $U \in \mathcal{B}(\mathcal{X})$ such that $F = \varphi(U)$. Thus, $U - R \in \mathcal{A}_n(\mathcal{X})$, which implies that $U \in (\mathcal{A}_n(\mathcal{X}) + T) \cup (\mathcal{A}_n(\mathcal{X}) + S)$. Then

$$\mathcal{A}_n(\mathcal{X}) + \varphi(R) \subseteq (\mathcal{A}_n(\mathcal{X}) + \varphi(T)) \cup (\mathcal{A}_n(\mathcal{X}) + \varphi(S)).$$

By Lemma 2.1 again, we obtain that $\varphi(T)$ and $\varphi(S)$ are adjacent. Moreover, since φ^{-1} satisfies the same properties as φ, we also have that φ preserves adjacency in both directions. □

Let $h : \mathbb{C} \to \mathbb{C}$ be a ring homomorphism. And an additive map $A : \mathcal{X} \to \mathcal{X}$ satisfying $A(\alpha x) = h(\alpha)A(x)$ $(x \in \mathcal{X}, \alpha \in \mathbb{C})$ is called an *h-quasilinear* operator. We now need the following result.

Lemma 2.3 ([16, Theorem 3.3]) *Let $\varphi : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X})$ be a bijective additive map preserving rank one operators in both directions. Then there exists a ring automorphism $h : \mathbb{C} \to \mathbb{C}$, and either there exist bijective h-quasilinear maps $A : \mathcal{X} \to \mathcal{X}$ and $C : \mathcal{X}^* \to \mathcal{X}^*$ such that*

$$\varphi(x \otimes f) = Ax \otimes Cf \text{ for all } x \in \mathcal{X} \text{ and } f \in \mathcal{X}^*,$$

or there exist bijective h-quasilinear maps $B : \mathcal{X}^* \to \mathcal{X}$ and $D : \mathcal{X} \to \mathcal{X}^*$ such that

$$\varphi(x \otimes f) = Bf \otimes Dx \text{ for all } x \in \mathcal{X} \text{ and } f \in \mathcal{X}^*.$$

Let $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a bijective map preserving the difference of $\mathcal{A}_n(\mathcal{X})$ in both directions and $\varphi(0) = 0$. By Proposition 2.2, we have φ preserves adjacency in both directions. Since $\varphi(0) = 0$, we obtain φ preserves rank one operators in both directions. Note that every rank two operator is adjacent to some rank one operator. Then repeating this discussion, we have φ preserves finite rank operators in both directions. That is, $\varphi(\mathcal{F}(\mathcal{X})) \subseteq \mathcal{F}(\mathcal{X})$. By [17, Theorem 1.5], we see that $\varphi|_{\mathcal{F}(\mathcal{X})}$ is an additive map. Therefore, from Lemma 2.3, we establish the forms of the given map φ on the set of rank one operators. On this basis, we obtain the following proposition.

Proposition 2.4 *Let $\varphi: \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a bijective map preserving the difference of $\mathcal{A}_n(\mathcal{X})$ in both directions and $\varphi(0) = 0$. Then $\varphi(I)$ is invertible.*

Proof Set $S = \varphi(I)$. Note that I is a semi-Fredholm operator. Then for every rank n idempotent operator P, $I - P$ is a semi-Fredholm operator. Since $\dim\ker(I - P) = \dim \operatorname{ran}(P) = n$, it follows that $I - P \in \mathcal{A}_n(\mathcal{X})$. Moreover, we know φ is a bijective map preserving the difference of $\mathcal{A}_n(\mathcal{X})$. Thus $S - \varphi(P) \in \mathcal{A}_n(\mathcal{X})$ for every rank n idempotent operator P.

Since $S - \varphi(P)$ is semi-Fredholm operator for rank n idempotent operator P, it follows that S is also a semi-Fredholm operator. On the other hand, $I \notin \mathcal{A}_n(\mathcal{X})$, then $S \notin \mathcal{A}_n(\mathcal{X})$. Thus $\dim\ker(S) \neq n$.

We shall distinguish two cases.

Case 1. $0 < \dim\ker(S) < n$.

Let $\{x_i : 1 \le i \le m, m < n\}$ be a linearly independent set such that $\ker(S) = \mathrm{Span}\{x_i : 1 \le i \le m\}$. Take a linearly independent set $\{y_i : 1 \le i \le n-m\}$ such that the set $\{x_i, y_j : 1 \le i \le m,\ 1 \le j \le n-m\}$ is linearly independent. Write

$$\mathcal{X} = \mathrm{Span}\{x_1, x_2, \ldots, x_m\} \oplus \mathrm{Span}\{y_1, y_2, \ldots, y_{n-m}\} \oplus \mathcal{X}_1.$$

Now we define an operator $F \in \mathcal{B}(\mathcal{X})$ by

$$\begin{cases} Fx_i = 0, & \forall\, 1 \le i \le m; \\ Fy_i = Sy_i, & \forall\, 1 \le i \le n-m; \\ Fx = 0, & \forall\, x \in \mathcal{X}_1. \end{cases}$$

Clearly, we have $\dim\mathrm{ran}(F) = n - m$. Moreover,

$$\begin{cases} (S-F)x_i = 0, & \forall\, 1 \le i \le m; \\ (S-F)y_i = 0, & \forall\, 1 \le i \le n-m; \\ (S-F)x \ne 0, & \forall\, x \in \mathcal{X}_1. \end{cases}$$

Then $\dim\ \ker(S-F) = n$. Thus $S - F \in \mathcal{A}_n(\mathcal{X})$. Since φ is a bijective map that preserves finite rank, there exists a rank $n-m$ operator K such that $\varphi(K) = F$. Hence $I - K \in \mathcal{A}_n(\mathcal{X})$ and $\dim\ker(I-K) = n$. However, $\ker(I-K) \subseteq \mathrm{ran}(K)$, a contradiction.

Case 2. $\dim\ker(S) > n$.

For every rank one operator $x \otimes f$, by Lemma 2.3, we have that there exists a ring automorphism $h : \mathbb{C} \to \mathbb{C}$ satisfying that either there exist bijective h-quasilinear maps $A : \mathcal{X} \to \mathcal{X}$ and $C : \mathcal{X}^* \to \mathcal{X}^*$ such that $\varphi(x \otimes f) = Ax \otimes Cf$ for all $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, or there exist bijective h-quasilinear maps $B : \mathcal{X}^* \to \mathcal{X}$ and $D : \mathcal{X} \to \mathcal{X}^*$ such that $\varphi(x \otimes f) = Bf \otimes Dx$ for all $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$.

Suppose that $\varphi(x \otimes f) = Ax \otimes Cf$ for all $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. The annihilator of $\ker(S)$ is denoted by

$$\ker(S)^\perp = \{h \in \mathcal{X}^*, h(z) = 0, z \in \ker(S)\}.$$

Since $\dim\ker(S) > n$, we can find linearly independent vectors

$$h_1, h_2, \ldots, h_n \in \ker(S)^\perp.$$

Consider $f_1, f_2, \ldots, f_n \in \mathcal{X}^*$ such that $Cf_i = h_i$, $1 \le i \le n$. By the Hahn-Banach Theorem, there exist $x_1, x_2, \ldots, x_n \in \mathcal{X}$ such that $f_i(x_j) = \delta_{ij}$ for $1 \le i, j \le n$.

Let $P = \sum_{i=1}^n x_i \otimes f_i$. Clearly, P is a rank n idempotent operator. It follows that $S - \varphi(P) \in \mathcal{A}_n(\mathcal{X})$, which implies that $\dim\ker(S - \varphi(P)) = n$. Note that $\varphi\,|_{\mathcal{F}(\mathcal{X})}$ is an additive map. Then we get that

$$\begin{aligned} \varphi(P) &= \varphi(x_1 \otimes f_1 + x_2 \otimes f_2 + \cdots + x_n \otimes f_n) \\ &= Ax_1 \otimes h_1 + Ax_2 \otimes h_2 + \cdots + Ax_n \otimes h_n. \end{aligned}$$

Clearly, $\varphi(P)\mid_{\ker(S)}=0$. Then for every $z\in\ker(S)$ we have $\varphi(P)z=0$. That is $(S-\varphi(P))z=0$. It follows that $\ker(S)\subseteq\ker(S-\varphi(P))$. It implies that

$$\dim\ker(S)\leq\dim\ker(S-\varphi(P))=n,$$

a contradiction.

Suppose that $\varphi(x\otimes f)=Bf\otimes Dx$. Then using a similar argument, we again get a contradiction.

Therefore, we get that $\dim\ker(S)=0$, that is, S is injective. Now we will prove that S is surjective.

Let y be an arbitrary non-zero vector in $\mathcal{X}$, there exist $g\in\mathcal{X}^*, x\in\mathcal{X}$ and $f\in\mathcal{X}^*$ such that $x\otimes f$ is a rank one idempotent operator and $\varphi(x\otimes f)=y\otimes g$. For the operator $x\otimes f$, there exist $x_1,x_2,\ldots,x_{n-1}\in\mathcal{X}$ and $f_1,f_2,\ldots,f_{n-1}\in\mathcal{X}^*$ such that

$$P=x\otimes f+x_1\otimes f_1+x_2\otimes f_2+\cdots+x_{n-1}\otimes f_{n-1}$$

is a rank n idempotent operator. Thus we have

$$\begin{aligned}\varphi(P)&=\varphi(x\otimes f+x_1\otimes f_1+x_2\otimes f_2+\cdots+x_{n-1}\otimes f_{n-1})\\&=y\otimes g+\sum_{i=1}^{n-1}\varphi(x_i\otimes y_i).\end{aligned}$$

Since $S-\varphi(P)\in\mathcal{A}_n(\mathcal{X})$, we have $\dim\ker(S-\varphi(P))=n$. For any non-zero vectors $z\in\ker(S-\varphi(P))$, it follows from that S is injective that $\varphi(P)(z)=Sz\neq0$. Then we have $\dim S(\ker(S-\varphi(P)))=n$. Thus $\mathrm{ran}(\varphi(P))=S(\ker(S-\varphi(P)))\subseteq\mathrm{ran}(S)$, which implies that $y\in\mathrm{ran}(S)$. Since y is arbitrary, we obtain S is surjective. □

Now, we are ready to prove our main result:

Proof (ii)⇒(i). This is obvious.

(i)⇒(ii). Assume that φ preserves the differences in $\mathcal{A}_n(\mathcal{X})$ in both directions. Note that $\varphi-\varphi(0)$ has the same properties as φ. Thus, after replacing φ by $\varphi-\varphi(0)$, we may assume that $\varphi(0)=0$. First we will show φ preserves the invertibility in both directions. Let $W\in\mathcal{B}(\mathcal{X})$ be an arbitrary invertible operator. We consider a map ψ given by $\psi(T)=\varphi(WT)$ for all $T\in\mathcal{B}(\mathcal{X})$. Clearly, ψ is a bijection and preserves the difference in $\mathcal{A}_n(\mathcal{X})$ in both directions. It follows from Proposition 2.4 that $\psi(I)=\varphi(W)$ is invertible. Moreover, since φ^{-1} satisfies the same properties as φ. Consequently, φ preserves the set of invertible operators in both directions.

In the following, we will show φ preserves the invertibility of the difference of operators in both directions. Fix an arbitrary operator $T\in\mathcal{B}(\mathcal{X})$ and define

$$\phi(S)=\varphi(S+T)-\varphi(T)\text{ for all }S\in\mathcal{B}(\mathcal{X}).$$

Clearly, ϕ is a bijective map which preservers the difference of $\mathcal{A}_n(\mathcal{X})$ in both directions. Thus ϕ preserves the invertibility in both directions. In particular, for every $S\in\mathcal{B}(\mathcal{X})$, we have

$$S-T\text{ is invertible }\Leftrightarrow\phi(S-T)=\varphi(S)-\varphi(T)\text{ is invertible.}$$

Thus φ preserves the invertibility of the difference of operators in both directions. Therefore, by [12, Theorem 1.1], we have that φ takes the desired forms. □

References

[1] Junli SHEN, Guoxing JI. *Property (t) and perturbations*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2019, **113**(2): 1417–1427.

[2] Weijuan SHI. *Topological uniform descent and compact perturbations*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2019, **113**(3): 2221–2233.

[3] Xiaohong CAO, Jiong DONG, Junhui LIU. *Weyl's theorem and its perturbations for the functions of operators*. Oper. Matrices, 2018, **12**(4): 1145–1157.

[4] Junli SHEN, Alatancang CHEN. *A note on property (gaR) and perturbations*. Ukraïn. Mat. Zh., 2017, **69**(7): 974–983.

[5] Xiaohong CAO, Shizhao CHEN. *Linear maps between operator algebras preserving certain spectral functions*. Banach J. Math. Anal., 2014, **8**(1): 39–46.

[6] T. AGHASIZADEH, S. HEJAZIAN. *Maps preserving semi-Fredholm operators on Hilbert C^*-modules*. J. Math. Anal. Appl., 2009, **354**(2): 625–629.

[7] M. BENDAOUD, A. BOURHIM, M. BURGOS, et al. *Linear maps preserving Fredholm and Atkinson elements of C^*-algebras*. Linear Multilinear Algebra, 2009, **57**(8): 823–838.

[8] M. OUDGHIRI. *Linear mapping preserving the kernel or the range of operators*. Extracta Math., 2009, **24**(3): 251–258.

[9] B. AUPETIT. *Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras*. J. London Math. Soc. (2), 2000, **62**(3): 917–924.

[10] Guoxing JI, Xia JIAO, Weijuan SHI. *Nonlinear maps preserving semi-Fredholm operators with bounded nullity*. Quaest. Math., 2023, **46**(7): 1415–1421.

[11] M. OUDGHIRI, K. SOUILAH. *Nonlinear maps preserving Drazin invertible operators of bounded index*. Quaest. Math., 2020, **43**(3): 209–320.

[12] A. BOURHIM, J. MASHREGHI, A. STEPANYAN. *Nonlinear maps preserving the minimum and surjectivity moduli*. Linear Algebra Appl., 2014, **463**: 171–189.

[13] C. COSTARA, D. REPOVŠ. *Nonlinear mappings preserving at least one eigenvalue*. Studia Math., 2010, **200**(1): 79–89.

[14] Jinchuan HOU, Li HUANG. *Characterizing isomorphisms in terms of completely preserving invertibility or spectrum*. J. Math. Anal. Appl., 2009, **359**(1): 81–87.

[15] H. HAVLICEK, P. ŠEMRL. *From geometry to invertibility preservers*. Studia Math., 2006, **174**(1): 99–109.

[16] M. OMLADIČ, P. ŠEMRL. *Additive mappings preserving operators of rank one*. Linear Algebra Appl., 1993, **182**: 239–256.

[17] T. PETEK, P. ŠEMRL. *Adjacency preserving maps on matrices and operators*. Proc. Roy. Soc. Edinburgh Sect. A, 2002, **132**(3): 661–684.

[18] M. GONZALEZ, M. MBEKHTA. *Linear maps that preserve semi-Fredholm operators acting on Banach spaces*. Acta Sci. Math. (Szeged), 2018, **84**(1-2): 137–149.

[19] M. MBEKHTA, V. MÜLLER, M. OUDGHIRI. *Additive preservers of the ascent, descent and related subsets*. J. Operator Theory, 2014, **71**(1): 63–83.