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Sample Path Large Deviations for Independent Random Variables under Sub-Linear Expectations

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Abstract In this work, the sample path large deviations for independent, identically distributed random variables under sub-linear expectations are established. The results obtained in sub-linear expectation spaces extend the corresponding ones in probability space.

Keywords sample path large deviations; independent random variables; sub-linear expectation

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1. Introduction

In order to describe the uncertainty in probability, Peng [1, 2] introduced the concepts of the sub-linear expectations space. The works of Peng [1, 2] encouraged many scholars to try to extend the results in classic probability space to those under sub-linear expectations space. Zhang [3–5] obtained Donsker's invariance principle, exponential inequalities and Rosenthal's inequality under sub-linear expectations. Under sub-linear expectations, Gao and Xu [6] proved large deviations and moderate deviations for independent random variables under sub-linear expectations. Gao and Xu [7] got large deviation principle for the empirical measures for independent random variables under sub-linear expectations. Xu and Zhang [8] proved a three series theorem for independent random variables under sub-linear expectations. Zhong and Wu [9] investigated the complete convergence and complete moment convergence for weighted sums of extended negatively dependent random variables under sub-linear expectations. Zhang [10] established the strong law of large numbers and the law of the iterated logarithm for a sequence of extended independent random variables under sub-linear expectations. For more limit theorems under sub-linear expectations, the interested readers could refer to [11–28], and references therein.

In probability space, sample path large deviations for random walks hold in Dembo and Zeitouni [29]. For references on large deviations and moderate deviations in linear expectation space, the interested reader could refer to Dembo and Zeitouni [29], Deuschel and Stroock [30], and refercences therein. Encouraged by sample path large deviations for random walks in Dembo

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and Zeitouni [29], the works of Gao and Jiang [12], Gao and Xu [6, 7], we try to prove sample path large deviations for independent random variables under sub-linear expectations, extending the corresponding results of Dembo and Zeitouni [29] in classic probability space. Our main contribution is that we extend sample path large deviations for random walks in classic probability space to that under sub-linear expectations, and our method of proof here is inspired by that of Dembo and Zeitouni [29], Gao and Jiang [12], Gao and Xu [6].

The remainders of this paper are organized as follows. The basic notions, concepts and relevant properties under sub-linear expectations are introduced in the next section, our main result Theorem 3.1 and some necessary lemmas are presented in Section 3, and the proof of Theorem 3.1 is given in Section 4.

2. Preliminary

We use similar notations as in the work by Peng [2], Chen [16], Zhang [5]. Assume that $(\Omega, \mathcal{F})$ is a given measurable space. Suppose that $\mathcal{H}$ is a subset of all random variables on $(\Omega, \mathcal{F})$ such that $X_1, \ldots, X_n \in \mathcal{H}$ yields $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in \mathcal{C}_{l,\mathrm{Lip}}(\mathbb{R}^n)$, where $\mathcal{C}_{l,\mathrm{Lip}}(\mathbb{R}^n)$ represents the linear space of (local lipschitz) function φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \le C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)(|\mathbf{x} - \mathbf{y}|), \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ relying on φ.

Definition 2.1 *A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \mapsto \bar{\mathbb{R}} := [-\infty, \infty]$ fulfilling the following properties: for all $X, Y \in \mathcal{H}$*

(a) If $X \ge Y$, then $\mathbb{E}[X] \ge \mathbb{E}[Y]$;

(b) $\mathbb{E}[c] = c$, $\forall\, c \in \mathbb{R}$;

(c) $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall\, \lambda \ge 0$;

(d) $\mathbb{E}[X + Y] \le \mathbb{E}[X] + \mathbb{E}[Y]$ whenever $\mathbb{E}[X] + \mathbb{E}[Y]$ is not of the form $\infty - \infty$ or $-\infty + \infty$.

A set function $V : \mathcal{F} \mapsto [0, 1]$ is said to be a capacity if

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$;

(b) $V(A) \le V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.

A capacity V is called sub-additive if $V(A \cup B) \le V(A) + V(B)$, $A, B \in \mathcal{F}$.

In this sequel, given a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, we assume that $\mathbb{E}$ is regular, i.e., for all $\{X_n, n \in \mathbb{N}\} \subset \mathcal{H}$, $X_n(\omega) \downarrow 0$, $\omega \in \Omega \Longrightarrow \lim_{n\to} \mathbb{E}(X_n) = 0$. Let $\mathcal{M}$ denote the collections of all probability meansures on Ω. By [2, Theorem 1.2.2], [11, Theorem 12], there exists a relatively compact subset $\mathscr{P} \subset \mathcal{M}$ such that

$$\mathbb{E}(X) = \sup_{P \in \mathscr{P}} \mathbf{E}_P(X), \text{ for all } X \in \mathcal{H},$$

where $\mathbf{E}_P(X)$ is the classic expectation of X under probability measure P. For any Borel measurable function X, define

$$\bar{\mathbb{E}}(X) = \sup_{P \in \mathscr{P}} \mathbf{E}_P(X), \ c(A) = \bar{\mathbb{E}}(I_A), \quad \forall\, A \in \mathscr{B}(\Omega), \tag{2.1}$$

where $I(A)$ or I_A represents the indicator function of A throughout this paper. Then $c(\cdot)$ is a capacity [11].

Let $\mathcal{L}^1(\Omega)$ be the topological completion space of $\mathcal{H}$ under the semi-norm $\mathbb{E}(|\cdot|)$. Then $\mathbb{E}$ can be only extended to continuous map from $\mathcal{L}^1(\Omega)$ to $\mathbb{R}$, denoted by $\mathbb{E}$. Let $\mathbb{L}_c^1$ be the quotient space of $\mathcal{L}^1(\Omega)$ with respect to quasi sure equivalence relation [11]. Then $(\mathbb{L}_c^1, \mathbb{E}(|\cdot|))$ is a Banach space, and for all $X \in \mathbb{L}_c^1$,

$$\mathbb{E}(X) = \bar{\mathbb{E}}(X).$$

Assume that $\mathbf{X} = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ are two random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$. $\mathbf{Y}$ is said to be independent of $\mathbf{X}$, if for each $\psi \in \mathcal{C}_{l,\mathrm{Lip}}(\mathbb{R}^{m+n})$, we have

$$\mathbb{E}[\psi(\mathbf{X}, \mathbf{Y})] = \mathbb{E}[\mathbb{E}[\psi(\mathbf{x}, \mathbf{Y})]|_{\mathbf{x}=\mathbf{X}}],$$

see [2, Definition 1.3.11]. $\{X_n\}_{n=1}^{\infty}$ is called a sequence of independent random variables, if X_{n+1} is independent of $(X_1, \ldots, X_n)$ for each $n \geq 1$.

Suppose that $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors defined, respectively, in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are named identically distributed if for every $\psi \in \mathcal{C}_{l,\mathrm{Lip}}(\mathbb{R}^n)$ such that $\psi(\mathbf{X}_1) \in \mathcal{H}_1, \psi(\mathbf{X}_2) \in \mathcal{H}_2$,

$$\mathbb{E}_1[\psi(\mathbf{X}_1)] = \mathbb{E}_2[\psi(\mathbf{X}_2)],$$

whenever the sub-linear expectations are finite. $\{X_n\}_{n=1}^{\infty}$ is said to be identically distributed if for each $i \geq 1$, X_i and X_1 are identically distributed.

In the paper we suppose that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \leq \sum_{n=1}^{\infty} \mathbb{E}(X_n)$, whenever $X \leq \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathcal{H}$, and $X \geq 0$, $X_n \geq 0$, $n = 1, 2, \ldots$. Let C stand for a positive constant which may differ from place to place.

Throughout this paper, let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be a sequence of independent, identically distributed (i.i.d.) random vectors in $\mathbb{R}^d$ under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, with

$$\mathbb{E}(\mathbf{X}_1) = \mathbb{E}(-\mathbf{X}_1) = \mathbf{0},$$

$$\bar{\Lambda}(\lambda) = \log \bar{\mathbb{E}}(\mathrm{e}^{\langle \lambda, \mathbf{X}_1 \rangle}) < \infty \text{ for all } \lambda \in \mathbb{R}^d.$$

Write

$$Z_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} \mathbf{X}_i, \quad 0 \leq t \leq 1, \tag{2.2}$$

and let c_n be the capacity induced by $Z_n(\cdot)$ in $L_\infty([0,1])$, i.e.,

$$c_n(A) \equiv c(Z_n(t)|_{t \in [0,1]} \in A), \quad A \subset L_\infty([0,1]).$$

Throughout, $|\mathbf{x}| \equiv \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ represents the Euclidean norm on $\mathbb{R}^d$, $\|f\|$ denotes the supremum norm on $L_\infty([0,1])$, and

$$\bar{\Lambda}^*(\mathbf{x}) \equiv \sup_{\lambda \in \mathbb{R}^d} [\langle \lambda, \mathbf{x} \rangle - \bar{\Lambda}(\lambda)]$$

stands for the Fenchel-Legendre transform of $\bar{\Lambda}(\cdot)$, where $\bar{\Lambda}(\lambda) \equiv \log \bar{\mathbb{E}}(\mathrm{e}^{\langle \lambda, \mathbf{X}_1 \rangle})$.

We conclude this section with some notions on large deviations under a sub-linear expectation space [7, 12].

Definition 2.2 *Assume that* (S,ρ) *is a Polish space. Suppose that* $\{Z_n, n\ge 1\}$ *is a family of measurable maps from* Ω *into* (S,ρ)*. A nonnegative function* I *on* (S,ρ) *is called (good) rate function if* $\{I\le l\}$ *is (compact) closed for all* $0\le l<\infty$*.*

$\{c(Z_n\in\cdot), n\ge 1\}$ *is said to satisfy large deviation principle (LDP) with rate function* $I(x)$ *if for any closed subset* $F\subset S$*,*

$$\limsup_{n\to\infty}\frac{1}{n}\log c(Z_n\in F)\le -\inf_{x\in F} I(x);$$

and for any open subset $O\subset S$*,*

$$\liminf_{n\to\infty}\frac{1}{n}\log c(Z_n\in O)\ge -\inf_{x\in O} I(x).$$

3. Main results

Our main result is the following.

Theorem 3.1 *The capacities* c_n *satisfy in* $L_\infty([0,1])$ *the LDP with the good rate function*

$$I(\phi)=\begin{cases}\int_0^1\bar\Lambda^*(\dot\phi(t))\mathrm{d}\,t, & \text{if } \phi\in\mathcal{AC}, \phi(0)=0,\\ \infty, & \text{otherwise,}\end{cases}\tag{3.1}$$

where

$$\mathcal{AC}\equiv\Big\{\phi\in C([0,1]):\sum_{\ell=1}^{k}|t_\ell-s_\ell|\to 0, s_\ell<t_\ell\le s_{\ell+1}<t_{\ell+1}\Longrightarrow\sum_{\ell=1}^{k}|\phi(t_\ell)-\phi(s_\ell)|\to 0\Big\}.$$

The proof of Theorem 3.1 relies on the following lemmas.

Lemma 3.2 *Let* $\tilde c_n$ *stand for the capacity induced by* $\tilde Z_n(\cdot)$ *in* $L_\infty([0,1])$*, where*

$$\tilde Z_n(t)\equiv Z_n(t)+(t-\frac{[nt]}{n})\mathbf{X}_{[nt]+1}\tag{3.2}$$

is the polygonal approximation of $Z_n(t)$*. Then the capacities* c_n *and* $\tilde c_n$ *are exponentially equivalent in* $L_\infty([0,1])$*, i.e., for any* $\delta>0$*,*

$$\limsup_{n\to\infty}\frac{1}{n}\log c\Big(\sup_{0\le t\le 1}|\tilde Z_n(t)-Z_n(t)|>\delta\Big)=-\infty.$$

Lemma 3.3 *Let* $\mathcal{X}$ *consist of all the maps from* $[0,1]$ *to* $\mathbb{R}^d$ *such that* $t=0$ *is mapped to the origin, and* $\mathcal{X}$ *is equipped with the topology of pointwise convergence on* $[0,1]$*. Then the capacities* $\tilde c_n$ *of Lemma 3.2 (defined on* $\mathcal{X}$ *by the natural embedding) satisfy LDP in this space with the good rate function* $I(\cdot)$ *of (3.1).*

Remark 3.4 Let $\mathcal{X}$ consist of all the maps from $[0,1]$ to $\mathbb{R}^d$ such that $t=0$ is mapped to the origin, and $\mathcal{X}$ is equipped with the topology of pointwise convergence on $[0,1]$. Let $\mathcal{J}$ represent the collection of all ordered finite subsets of $[0,1]$. For any $j=\{0<t_1<t_2<\cdots<t_{|j|}\le 1\}\in\mathcal{J}$

and any $f : [0,1] \mapsto \mathbb{R}^d$, let $p_j(f)$ denote the vector $(f(t_1), f(t_2), \ldots, f(t_{|j|})) \in (\mathbb{R}^d)^{|j|}$. In Lemma 3.3, with the condition that

$$\bar{\Lambda}(\lambda) = \log \bar{\mathbb{E}}(\mathrm{e}^{\langle \lambda, \mathbf{X}_1 \rangle}) < \infty \text{ for all } |\lambda| < \eta,$$

where $\eta > 0$, in place of that $\bar{\Lambda}(\lambda) = \log \bar{\mathbb{E}}(\mathrm{e}^{\langle \lambda, \mathbf{X}_1 \rangle}) < \infty$ for all $\lambda \in \mathbb{R}^d$, the other assumptions remain unchanged, with Remark 4.2 in place of Lemma 4.1, by the similar proof of Lemma 3.3, we see that the capacities c_n of Lemma 3.2 (defined on $\mathcal{X}$ by the natural embedding) satisfy LDP in this space with the good rate function $\tilde{I}(\cdot)$, where

$$\tilde{I}(\mathbf{x}) = \sup_{j \in \mathcal{J}} \{\bar{I}_j(p_j(\mathbf{x}))\}, \quad \mathbf{x} \in \mathcal{X},$$

$\bar{I}_j(\cdot)$ is defined as in Remark 4.2.

Lemma 3.5 ([12]) *The capacities $\tilde{c}_n$ are exponentially tight in the space $C_0([0,1])$ of all continuous functions $f : [0,1] \mapsto \mathbb{R}^d$ fulfilling that $f(0) = 0$, equipped with the supremum norm topology, i.e., for any $L > 0$, there exists a compact $K_L \subset C_0([0,1])$ such that*

$$\limsup_{n \to \infty} \frac{1}{n} \log \tilde{c}_n(K_L^c) \leq -L.$$

We also give some necessary theorems and lemmas for capacities as follows.

Lemma 3.6 *Assume that $\hat{\mu}_\epsilon$ is an exponentially tight family of capacities on $\mathcal{X}$ equipped with the topology τ_1. If $\hat{\mu}_\epsilon$ satisfies an LDP with respect to a Haudorff topology τ_2 coarser than τ_1, then the same LDP is valid with respect to the topology τ_1.*

Proof The proof is similar to that of [29, Corollary 4.2.6], and is omitted here. □

Lemma 3.7 *Assume that $\mathcal{E}$ is a measurable subset of $\mathcal{X}$ such that $\hat{\mu}_\epsilon(\mathcal{E}^c) = 0$ for all $\epsilon > 0$ and capacities $\hat{\mu}_\epsilon$. Assume that $\mathcal{E}$ inherits the topology of $\mathcal{X}$.*

(a) If $\mathcal{E}$ is a closed subset of $\mathcal{X}$ and $\{\hat{\mu}_\epsilon\}$ satisfies the LDP in $\mathcal{E}$ with the rate function I, then $\hat{\mu}_\epsilon$ satisfies the LDP in $\mathcal{X}$ with rate function I' fulfilling $I' = I$ on $\mathcal{E}$ and $I' = \infty$ on $\mathcal{E}^c$.

(b) If $\hat{\mu}_\epsilon$ satisfies the LDP in $\mathcal{X}$ with rate function I and $\mathcal{D}_I \equiv \{x : I(x) < \infty\} \subset \mathcal{E}$, then the same LDP is valid in $\mathcal{E}$. Particularly, if $\mathcal{E}$ is a closed subset of $\mathcal{X}$, then $\mathcal{D}_I \subset \mathcal{E}$ and therefore the LDP is valid in $\mathcal{E}$.

Proof The proof is similar to that of [29, Lemma 4.1.5], and is omitted here. □

Definition 3.8 *Assume that $(\mathcal{Y}, d)$ is a metric space. The capacities $\hat{\mu}_\epsilon$ and $\tilde{\mu}_\epsilon$ on $\mathcal{Y}$ are named exponentially equivalent if there exist two families $\mathcal{Y}$-valued random variables $\{Z_\epsilon\}$ and $\{\tilde{Z}_\epsilon\}$ such that $\hat{\mu}_\epsilon$ and $\tilde{\mu}_\epsilon$ are capacities induced by $\{Z_\epsilon\}$ and $\{\tilde{Z}_\epsilon\}$, respectively, and the following holds: For each $\delta > 0$,*

$$\limsup_{\epsilon \to 0} \epsilon \log c(d(Z_\epsilon, \tilde{Z}_\epsilon) > \delta) = -\infty.$$

Theorem 3.9 *Suppose that an LDP with a good rate function $I(\cdot)$ holds for capacities $\hat{\mu}_\epsilon$, which are exponentially equivalent to $\tilde{\mu}_\epsilon$. Then the same LDP is valid for $\tilde{\mu}_\epsilon$.*

Proof The proof is similar to that of [29, Theorem 4.2.13], and is omitted here. □

Theorem 3.10 (Contraction principle for capacities) *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hausdorff topological spaces and $f : \mathcal{X} \mapsto \mathcal{Y}$ is a continuous function. Consider a good rate function $I : \mathcal{X} \mapsto [0, \infty]$.*

(a) For each $y \in \mathcal{Y}$, write

$$I'(y) \equiv \inf\{I(x) : x \in \mathcal{X}, y = f(x)\}.$$

Then I' is a good rate function on $\mathcal{Y}$, where the infimum over the empty set is interpreted as ∞.

(b) If I controls the LDP corresponding to a family of capacities $\{c_\epsilon\}$ on $\mathcal{X}$, then I' controls the LDP corresponding to the family of capacities $\{c_\epsilon \circ f^{-1}\}$ on $\mathcal{Y}$.

Proof The proof is similar to that of [29, Theorem 4.2.1], and so it is omitted here. □

Lemma 3.11 *Assume that $\{\hat{\mu}_\epsilon\}$ is an exponentially tight capacities on the Banach space $\mathcal{X}$. Assume that $\bar{\Lambda}(\cdot) \equiv \lim_{\epsilon\to 0} \epsilon\bar{\Lambda}_{\hat{\mu}_\epsilon}(\cdot/\epsilon)$ is finite valued, Gateaux differentiable, and lower semicontinuous in $\mathcal{X}^*$($\mathcal{X}^*$ is the space of all continuous linear functionals on $\mathcal{X}$) with respect to the weak* topology. Then $\{\hat{\mu}_\epsilon\}$ satisfies the LDP with the good rate function $\bar{\Lambda}^*$.*

Proof The proof is similar to that of [29, Corollary 4.5.27], and is omitted here. □

4. Proofs of main results

The following is the proof of Theorem 3.1 in detail.

Proof of Theorem 3.1 By Lemma 3.3, $\{\tilde{c}_n\}$ satisfies LDP in $\mathcal{X}$. Observe that

$$\mathcal{D}_I \equiv \{\phi : I(\phi) < \infty\} \subset C_0([0,1]),$$

and by (2.2) and (3.2), $\tilde{c}_n(C_0([0,1])^c) = 0$ for all n. By Lemma 3.6, the LDP for $\{\tilde{c}_n\}$ also holds in $C_0([0,1])$ when equipped with the relative topology induced by $\mathcal{X}$. The latter is the pointwise convergence topology, produced by

$$V_{t,\mathbf{x},\delta} \equiv \{g \in C_0([0,1]) : |g(t) - \mathbf{x}| < \delta\}$$

with $t \in [0,1]$, $\mathbf{x} \in \mathbb{R}^d$ and $\delta > 0$. Because every $V_{t,x,\delta}$ is an open set under the supremum norm, the latter topology is stronger than the pointwise convergence topology. Thus, the exponential tightness of $\{\tilde{c}_n\}$ as established in Lemma 3.5 yields, by Lemma 3.6 the LDP with respect to the supremum norm topology on $C_0([0,1])$. Because $C_0([0,1])$ is a closed subset of $L_\infty([0,1])$, the same LDP holds in $L_\infty([0,1])$ by Lemma 3.7 in the opposite direction. Finally, by Lemma 3.2, LDP of $\{c_n\}$ in the metric space $L_\infty([0,1])$ follows from that of $\{\tilde{c}_n\}$ by Theorem 3.9. This completes the proof. □

Proof of Lemma 3.2 With Markov's inequality under sub-linear expectations in place of Markov's inequality, the proof here is similar to that of [29, Lemma 5.1.4], and so is omitted. This completes the proof. □

The proof of Lemma 3.5 relies on the following finite dimensional LDP.

Lemma 4.1 *Let $\mathcal{J}$ represent the collection of all ordered finite subsets of $[0,1]$. For any $j = \{0 < t_1 < t_2 < \cdots < t_{|j|} \leq 1\} \in \mathcal{J}$ and any $f : [0,1] \mapsto \mathbb{R}^d$, let $p_j(f)$ denote the vector $(f(t_1), f(t_2), \ldots, f(t_{|j|})) \in (\mathbb{R}^d)^{|j|}$. Then the sequence of $\{c_n \circ p_j^{-1}\}$ satisfies the LDP in $(\mathbb{R}^d)^{|j|}$ with the good rate function*

$$I_j(\mathbf{z}) = \sum_{\ell=1}^{|j|} (t_\ell - t_{\ell-1}) \bar{\Lambda}^*(\frac{z_\ell - z_{\ell-1}}{t_\ell - t_{\ell-1}}), \tag{4.1}$$

where $\mathbf{z} = (z_1, \ldots, z_{|j|})$ and $t_0 = 0$, $z_0 = 0$.

Proof By [6, Lemma 1.1], with Lemma 3.11 or [6, Theorem 2.1] in place of [29, Theorem 2.3.6], the proof here is similar to that of [29, Lemma 5.1.8], and so is omitted. This completes the proof. □

Remark 4.2 By the proof above, in Lemma 4.1, if the condition that $\bar{\Lambda}(\lambda) = \log \bar{\mathbb{E}}(\mathrm{e}^{\langle \lambda, \mathbf{X}_1 \rangle}) < \infty$ for all $\lambda \in \mathbb{R}^d$ is replaced by that

$$\bar{\Lambda}(\lambda) = \log \bar{\mathbb{E}}(\mathrm{e}^{\langle \lambda, \mathbf{X}_1 \rangle}) < \infty \text{ for all } |\lambda| < \eta,$$

where $\eta > 0$, and the other assumptions remain unchanged, then the sequence of $\{c_n \circ p_j^{-1}\}$ satisfies the LDP in $(\mathbb{R}^d)^{|j|}$ with the good rate function

$$\bar{I}_j(\mathbf{z}) = \sum_{\ell=1}^{|j|} (t_\ell - t_{\ell-1}) \bar{I}(\frac{z_\ell - z_{\ell-1}}{t_\ell - t_{\ell-1}}), \tag{4.2}$$

where $\bar{I}(\mathbf{x}) := -\lim_{\delta \to} \lim_{N \to \infty} \log c\{\frac{1}{N} \sum_{i=1}^N \mathbf{X}_i \in B(\mathbf{x}, \delta)\}$, $\mathbf{z} = (z_1, \ldots, z_{|j|})$ (see [6]), $B(\mathbf{x}, \delta) := \{\mathbf{y}; |\mathbf{y} - \mathbf{x}| < \delta\}$, and $t_0 = 0$, $z_0 = 0$.

From Lemma 3.2 follows that the capacities $\{c_n \circ p_j^{-1}\}$, $\{\tilde{c}_n \circ p_j^{-1}\}$ are exponentially equivalent in $(\mathbb{R}^d)^{|j|}$. Hence, the following is a result of Lemma 4.1.

Corollary 4.3 *For $j \in \mathcal{J}$, $\{\tilde{c}_n \circ p_j^{-1}\}$ satisfies the LDP in $(\mathbb{R}^d)^{|j|}$ with the good rate function I_j of (4.1).*

Proof of Lemma 3.3 With Dawson-Gärtner Theorem for capacities corresponding to [29, Theorem 4.6.1] and Corollary 4.3 in place of [29, Theorem 4.6.1, Corollary 5.1.10], the proof here is similar to that of [29, Lemma 5.1.6], and hence is omitted. This completes the proof. □

The proof of Lemma 3.5 depends on the following one-dimensional result.

Lemma 4.4 *Let X be a real valued random variable with $\bar{\mathbb{E}}(X) = -\bar{\mathbb{E}}(-X)$ while $\bar{\mathbb{E}}(X)$ exists, $\bar{\Lambda}_X(\lambda) = \log \bar{\mathbb{E}}(\mathrm{e}^{\lambda X})$. Then $\bar{\mathbb{E}}[\mathrm{e}^{\delta \bar{\Lambda}_X^*(X)}] < \infty$ for all $0 < \delta < 1$.*

Proof If $\bar{\Lambda}_X(\lambda) = \infty$ for all $\lambda \neq 0$, then $\bar{\Lambda}_X^*(\lambda)$ is identically zero and the lemma holds trivially. Assume otherwise and recall that then $\bar{x} = \bar{\mathbb{E}}(X) = -\bar{\mathbb{E}}(-X)$ exists, possible as an extended real number. Note that for any $x \in \mathbb{R}$,

$$\mathrm{e}^{\bar{\Lambda}_X(\lambda)} \geq \begin{cases} \mathrm{e}^{\lambda x} c(X \in [x, \infty)), & \text{if } \lambda > 0, \\ \mathrm{e}^{\lambda x} c(X \in (-\infty, x]), & \text{if } \lambda \leq 0. \end{cases}$$

Therefore, by similar formulas of [29, (2.2.6), (2.2.7)], we obtain

$$\bar{\Lambda}_X^*(x) = \begin{cases} -\log c(X \in [x, \infty)), & \text{if } x \geq \bar{\mathbb{E}}(X), \\ -\log c(X \in (-\infty, x]), & \text{if } x \leq \bar{\mathbb{E}}(X) = -\bar{\mathbb{E}}(-X). \end{cases}$$

Let $0 < \delta < 1$. Then

$$\begin{aligned}
&\bar{\mathbb{E}}(\mathrm{e}^{\delta \bar{\Lambda}_X^*(X)}) \leq \bar{\mathbb{E}}(\mathrm{e}^{\delta \bar{\Lambda}_X^*(X)} I(X \leq \bar{x})) + \bar{\mathbb{E}}(\mathrm{e}^{\delta \bar{\Lambda}_X^*(X)} I(X > \bar{x})) \\
&\leq \sup_{P \in \mathscr{P}} \mathbf{E}_P(\mathrm{e}^{\delta \bar{\Lambda}_X^*(X)} I(X \leq \bar{x})) + \sup_{P \in \mathscr{P}} \mathbf{E}_P(\mathrm{e}^{\delta \bar{\Lambda}_X^*(X)} I(X > \bar{x})) \\
&\leq \sup_{P \in \mathscr{P}} \mathbf{E}_P \int_{-\infty}^{\bar{x}} \frac{1}{c(X \in (-\infty, x])^{\delta}} \mathrm{d}P(X \leq x) + \sup_{P \in \mathscr{P}} \mathbf{E}_P \int_{\bar{x}}^{\infty} \frac{1}{c(X \in [x, \infty))^{\delta}} \mathrm{d}P(X \leq x) \\
&\leq \sup_{P \in \mathscr{P}} \mathbf{E}_P \int_{-\infty}^{\bar{x}} \frac{1}{P(X \in (-\infty, x])^{\delta}} \mathrm{d}P(X \leq x) + \sup_{P \in \mathscr{P}} \mathbf{E}_P \int_{\bar{x}}^{\infty} \frac{1}{P(X \in [x, \infty))^{\delta}} \mathrm{d}P(X \leq x) \\
&\leq \sup_{P \in \mathscr{P}} \lim_{M \to -\infty} \mathbf{E}_P \int_{M}^{\bar{x}} \frac{1}{P(X \in (-\infty, x])^{\delta}} \mathrm{d}P(X \leq x) + \\
&\quad \sup_{P \in \mathscr{P}} \lim_{M \to \infty} \mathbf{E}_P \int_{\bar{x}}^{M} \frac{1}{P(X \in [x, \infty))^{\delta}} \mathrm{d}P(X \leq x) \\
&\leq \sup_{P \in \mathscr{P}} \lim_{M \to -\infty} \frac{1}{1-\delta} \{P((-\infty, \bar{x}])^{1-\delta} - P((-\infty, M])^{1-\delta}\} + \\
&\quad \sup_{P \in \mathscr{P}} \lim_{M \to \infty} \frac{1}{1-\delta} \{P([\bar{x}, \infty))^{1-\delta} - P([M, \infty))^{1-\delta}\} \\
&\leq 2/(1-\delta),
\end{aligned}$$

where the second to last inequality above comes from integration by parts. □

Proof of Lemma 3.5 With Lemma 4.4 and Markov's inequality under sub-linear expectations in place of [29, Lemma 5.1.14] and Markov's inequality, the proof here is similar to that of [29, Lemma 5.1.7] and so is omitted. This completes the proof. □

Theorem 3.1 can be generalized to the capacities $\hat{c}_\epsilon$ induced by

$$Y_\epsilon(t) = \epsilon \sum_{i=1}^{[t/\epsilon]} \mathbf{X}_i, \quad 0 \leq t \leq 1, \tag{4.3}$$

where $\tilde{c}_n$ (and $Z_n(t)$) corresponds to the special case of $\epsilon = n^{-1}$. The precise result is given in the following.

Theorem 4.5 *The capacities $\hat{c}_\epsilon$ induced on $L_\infty([0,1])$ by $Y_\epsilon(t)$ satisfy the LDP with the good rate function $I(\cdot)$ of (3.1).*

Proof For any sequence $\epsilon_m \to 0$ fulfilling that ϵ_m^{-1} are integers, Theorem 4.5 is a conclusion of Theorem 3.1. Consider now an arbitrary sequence $\epsilon_m \to 0$ and write $n_m \equiv [\epsilon_m^{-1}]$. By Theorem 3.1, $\{\tilde{c}_{n_m}\}_{m=1}^{\infty}$ satisfies an LDP with the rate function $I(\cdot)$ of (3.1) and rate $\frac{1}{n_m}$. Since $n_m \epsilon_m \to 1$, the proof of the theorem is finished by applying Theorem 3.9, provided that for any $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n_m} \log c(\|Y_{\epsilon_m} - Z_{n_m}\| > \delta) = -\infty. \tag{4.4}$$

To this end, note that $\epsilon_m n_m \in [1-\epsilon_m, 1]$ and $[\frac{t}{\epsilon_m}] \in \{[n_m t], [n_m t]+1\}$. Therefore, by (2.2) and (4.3),

$$\begin{aligned} |Y_{\epsilon_m}(t) - Z_{n_m}(t)| \le& (1-\epsilon_m n_m)|Z_{n_m}(t)| + \epsilon_m |\mathbf{X}_{[\frac{t}{\epsilon_m}]}| \\ \le& 2\epsilon_m \max_{i=1,\dots,n_m} |\mathbf{X}_i|. \end{aligned}$$

Now, by the union of events bound,

$$\frac{1}{n_m} \log c(\|Y_{\epsilon_m} - Z_{n_m}\| > \delta) \le \frac{1}{n_m} \log n_m + \frac{1}{n_m} \log c(|\mathbf{X}_1| > \frac{\delta}{2\epsilon_m}),$$

and (4.4) holds, since

$$\begin{aligned} \lim_{\epsilon\to 0} \epsilon \log c(|\mathbf{X}_1| > \frac{1}{\epsilon}) \le& \lim_{\epsilon\to 0} \epsilon \log c\Big(\bigcup_{j=1}^d (|X_1^j| > \frac{1}{\epsilon})\Big) \\ \le& \lim_{\lambda\to\infty} \Big[\lim_{\epsilon\to 0} \max_{j=1,\dots,d} \log \bar{\mathbb{E}}(\mathrm{e}^{\lambda|X_1^j|}) - \lambda\Big] = -\infty. \end{aligned}$$

The proof is completed. □

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