Journal of Mathematical Research with Applications
Sept., 2024, Vol. 44, No. 5, pp. 583–595
DOI:10.3770/j.issn:2095-2651.2024.05.002
Http://jmre.dlut.edu.cn

Generalized Laws of Importation and Cross-Migrativity Based on (D, N)-Implications

Nana MA
School of Mathematics, Xi'an University of Finance and Economics, Shaanxi 710100*, P. R. China*

Abstract In the present article we study the generalized laws of importation and cross-migrativity based on (D, N)-implications. On the one hand, we consider the generalized law of importation $I(T(x, \alpha), y) = I(x, J(\alpha, y))$, when T is concretized as a fuzzy conjunction C, I and J are concretized as (D, N)-implications generated by a fuzzy disjunction D and a fuzzy negation N. On the other hand, we discuss the cross-migrativity of fuzzy disjunctions over fuzzy implications, that is, $I(x, D(y, z)) = D(y, I(x, z))$, where I is a fuzzy implication, D is a fuzzy disjunction. In addition, we study the relationship between the (α-)cross-migrative of (I, D) and the (α-)cross-law of importation $(I, I_{D,N}, C)$.

Keywords fuzzy implication; generalized laws of importation; cross-migrativity

MR(2020) Subject Classification 03B52; 94D05

1. Introduction

Fuzzy implications [1] are one of the main operations in fuzzy logic. They generalize the classical implication, which takes values in $\{0, 1\}$, to fuzzy logic, where the truth values belong to the unit interval $[0, 1]$. Fuzzy implications play an indispensable role in representing imprecise knowledge and performing logical inferences in many branches of computational intelligence [2,3], particularly in fuzzy logic systems [4, 5], decision theory [6], fuzzy control [7, 8] and expert systems [9]. As a result, constructing and characterizing new models of fuzzy implications have received a lot of attention in the community of fuzzy implications in the last decades, and a large number of new subclasses of fuzzy implications and their various further generalizations have appeared in the literature [10]. Important construction methods of fuzzy implications are derived from fuzzy negations, conjunctive and disjunctive aggregation functions, such as t-norms, t-conorms, uninorms, overlap and grouping functions, and more generally, fuzzy disjunction functions [10, 11]. The resulting implications can be divided into (T, N)-implications, (S, N)-implications, (U, N)-implications, (O, N)-implications, (G, N)-implications, (D, N)-implications, directional monotonic fuzzy implications, $\mathrm{R}_{\mathcal{O}}$-implications induced from C_L-overlap functions, the new contrapositivisation technique for fuzzy implications based on quasi-overlap and quasi-grouping functions, and their further generalizations [12–20].

Received August 31, 2023; Accepted June 27, 2024
Supported by the National Natural Science Foundation of China (Grant Nos. 12471438; 12001413), the Natural Science Basic Research Plan Project of Shaanxi Province (Grant No. 2022JM-048) and the Shaanxi Fundamental Science Research Project for Mathematics and Physics (Grant No. 23JSQ043).
E-mail address: manana@xaufe.edu.cn

The migrativity of the fuzzy implication about the t-norm is defined as $I(T(x,\alpha),y) = I(x,I(\alpha,y))$, $x,y,\alpha\in[0,1]$, where I is a fuzzy implication and T is a t-norm, known as the law of importation [1]. In addition, since the migrativity captured the invariance of the aggregation effect of the migrative t-norm induced by scalar multiplication, irrespective of which independent variables are multiplied by the same weights, the migrativity was very useful in image processing. The migrativity has stimulated widespread interest among researchers due to its wide range of applications. The generalized forms of the migrativity of aggregation functions have been studied in the literature by many scholars [21–23]. An important way is by replacing the t-norm T with other specific functions, for example, the migrativity of uninorms, nullnorms [24–27] and overlap functions [28–33] have been extensively studied and fruitful results have been achieved. It is well-known that t-norms can be regarded as fuzzy conjunctions. Recently, Baczyński et al. [34] have studied the generalized law of importation $I(C(x,\alpha),y) = I(x,J(\alpha,y))$, where I,J are fuzzy implications and C is a generalized conjunction. It is well known that the fuzzy disjunction and the fuzzy conjunction are dual concepts, inspired by [34], the first work in this paper is to consider the generalized law of importation when I and J are concretized as (D,N)-implications generated by a fuzzy disjunction D and a fuzzy negation N.

The migrativity of the fuzzy implication about the t-conorm cannot be satisfied [21]. Therefore, a meaningful way to define the migrativity t-conorm has become an urgent problem. [34, Section 4], Baczyński et al. proposed the α-migrative of a t-conorm S as their further work, that is, if it satisfies $S(1-\alpha+\alpha x,y) = S(x,1-\alpha+\alpha y)$ for all $x,y\in[0,1]$ and for a given $\alpha\in[0,1]$. Notice that the first term $1-\alpha+\alpha x$ can be replaced by the R-implication $I_{RC}(\alpha,x) = 1-\alpha+\alpha x$. Then, Pan, Zhou and Yan [35] continued the work of Baczyński et al., they considered the more general case, that is, I was general fuzzy implication, S was continuous t-conorm satisfying the equation $S(I(\alpha,x),y) = S(x,I(\alpha,y))$ for all $x,y\in[0,1]$ and $\alpha\in[0,1]$. On their basis, Fang studied the α-cross migrativity involving t-conorms and fuzzy implications, that is, $I(x,S(\alpha,y)) = S(y,I(x,\alpha))$ for all $x,y\in[0,1]$ (see [36]). In the fuzzy case, a t-conorm S can be regarded as a fuzzy disjunction D, and we found that the associativity of t-conorm was not involved in studying the migrativity in some situations. Therefore, the second work of this paper is to consider degenerating t-conorm into the simplest fuzzy disjunction, studying the cross-migrativity of fuzzy disjunction about fuzzy implication, and investigating the equivalent characterization for the cross-migrativity of (I,D).

This paper is organized as follows. In Section 2, we present several basic notions and results about fuzzy implications, (D,N)-implications, which will be used in our article. In Section 3, we introduce the notion of the generalized law of importation of (D,N)-implications. Moreover, we investigate the necessary and sufficient conditions of the generalized law of importation of (D,N)-implications. In Section 4, we discuss the fuzzy disjunction about fuzzy implication. By defining mutual exchangeability, we obtain an equivalent characterization for cross-migrativity of (I,D), that is, (I,D) is cross-migrative iff I and $I_{D,N}$ generated by the same fuzzy disjunction and strong fuzzy negations satisfy mutual exchangeability. Furthermore, by defining cross-law of importation, we obtain (I,D) is cross-migrative iff I, $I_{D,N}$ generated by the same fuzzy disjunction and strong fuzzy negations and C where $C(x, y) = N(D(N(x), N(y)))$ for all $x, y \in [0, 1]$, satisfy cross-law of importation. Concluding remarks and an outline of future works are contained in Section 5.

2. Preliminaries

In this section, we recall some basic concepts and conclusions about fuzzy disjunctions, fuzzy conjunctions, fuzzy implications, (D, N)-implications.

Definition 2.1 ([37,38]) *Let $D : [0, 1]^2 \to [0, 1]$ be a binary operation. If D satisfies*

(i) $D(1, 0) = 1 = D(0, 1)$;

(ii) $D(0, 0) = 0$;

(iii) D is non-decreasing in each of the variables,

then we call D a fuzzy disjunction.

Remark 2.2 ([37,38]) If D is a fuzzy disjunction, then $D(1, 1) = 1$ and $D(1, x) = D(x, 1) = 1$ for all $x \in [0, 1]$.

Example 2.3 ([37]) Define $D_1 : [0, 1]^2 \to [0, 1]$, $D_2 : [0, 1]^2 \to [0, 1]$, respectively, by $\forall\, x \in [0, 1]$,

$$D_1(x) = \begin{cases} 1, & x = 1 \text{ or } y = 1, \\ 0, & \text{otherwises;} \end{cases}$$

$$D_2(x) = \begin{cases} 0, & x = y = 0, \\ 1, & \text{otherwises.} \end{cases}$$

Then D_1 and D_2 are fuzzy disjunctions.

Definition 2.4 ([37,38]) *Let $C : [0, 1]^2 \to [0, 1]$ be a binary operation. If C satisfies*

(i) $C(1, 0) = 0 = C(0, 1)$;

(ii) $C(1, 1) = 1$;

(iii) C is non-decreasing in each of the variables,

then we call C a fuzzy conjunction.

Remark 2.5 If C is a fuzzy conjunction, then $D(x, y) = 1 - C(1 - x, 1 - y)$ for all $x, y \in [0, 1]$ is a fuzzy disjunction.

Definition 2.6 ([1]) *A decreasing function $N : [0, 1] \to [0, 1]$ is called a fuzzy negation if $N(0) = 1$, $N(1) = 0$.*

Definition 2.7 ([1]) *(i) A fuzzy negation N is called strong if it is an involution, i.e., $N(N(x)) = x$, $\forall\, x \in [0, 1]$.*

(ii) A fuzzy negation N is called strict if N is strictly decreasing, and N is continuous.

Example 2.8 ([1]) (i) Define $N_C(x) = 1 - x$, $\forall\, x \in [0, 1]$, then N_C is the classical (standard) fuzzy negation.

(ii) Define $N_1 : [0,1] \to [0,1]$, $N_2 : [0,1] \to [0,1]$, respectively, by $\forall\, x \in [0,1]$,

$$N_1(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwises;} \end{cases}$$

$$N_2(x) = \begin{cases} 0, & x = 1, \\ 1, & \text{otherwises.} \end{cases}$$

Then N_1 and N_2 are fuzzy negations.

Proposition 2.9 ([1]) *If N is a strong fuzzy negation, then N is a strict fuzzy negation.*

Proposition 2.10 ([1]) *If N is a continuous fuzzy negation, then the function $\mathcal{R}_N : [0,1] \to [0,1]$ defined by*

$$\mathcal{R}_N(x) = \begin{cases} N^{-1}(x), & x \in (0,1], \\ 1, & x = 0, \end{cases}$$

is a strictly decreasing fuzzy negation. Moreover, $\mathcal{R}_N^{-1} = N$ and $N \circ \mathcal{R}_N = \mathrm{id}_{[0,1]}$.

Definition 2.11 ([1]) *A function $I : [0,1]^2 \to [0,1]$ is a fuzzy implication if it fulfills (I1)–(I3):*

(I1) I is non-increasing in its first variable;

(I2) I is non-decreasing in its second variable;

(I3) $I(0,0) = I(1,1) = 1$, and $I(1,0) = 0$.

Note that, from the definition 2.11, it follows that $I(0,x) = I(x,1) = 1$ for all $x \in [0,1]$.

Definition 2.12 ([1]) *Let I be a fuzzy implication and N be a fuzzy negation. I is said to satisfy the law of contraposition (or in other words, the contrapositive symmetry) with respect to N, if $I(x,y) = I(N(y),N(x))$ for all $x,y \in [0,1]$.*

Definition 2.13 ([1]) *Let I be a fuzzy implication. The function N_I defined by $N_I(x) = I(x,0)$ for all $x \in [0,1]$, is called the natural negation of I.*

Definition 2.14 ([17]) *A binary function $I : [0,1]^2 \to [0,1]$ is said to be a (D,N)-implication if there exists a fuzzy disjunction D and a fuzzy negation N such that*

$$I(x,y) = D(N(x),y), \quad \forall\, x,y \in [0,1].$$

If I is a (D,N)-implication generated by D and N, then I is denoted by $I_{D,N}$.

Example 2.15 ([17]) I_{D_1,N_C} and I_{D_2,N_C} are (D,N)-implications.

3. The generalized law of importation of (D,N)-implications

In [34], Baczyński et al. discussed the generalized law of importation, that is, $I(C(x,\alpha),y) = I(x,J(\alpha,y))$, where I,J are fuzzy implications and C is a generalized conjunction. It is well known that fuzzy disjunction and fuzzy conjunction are dual concepts, inspired by [34], in this section we further consider the generalized law of importation when I and J are concretized as

(D, N)-implications generated by fuzzy disjunction D and fuzzy negation N.

Definition 3.1 *Suppose C is a fuzzy conjunction and $I_{D,N}$ is a (D, N)-implication. If $\forall\, \alpha, x, y \in [0, 1]$,*

$$I_{D,N}(C(x, \alpha), y) = I_{D,N}(x, I_{D,N}(\alpha, y)), \tag{3.1}$$

then we call the tuple $(C, I_{D,N})$ satisfying the generalized law of importation.

Example 3.2 (i) If $C : [0, 1]^2 \to [0, 1]$ is an operation such that

$$C(x, y) = \begin{cases} 0, & x = 0 \text{ or } y = 0, \\ x, & y > x \neq 0, \\ y, & x \geq y \neq 0, \end{cases}$$

and $D(x, y)$ =max$\{x, y\}$, $N(x) = 1 - x$, then the tuple $(C, I_{D,N})$ satisfies (3.1).

According to Definitions 2.1 and 2.4, it is easy to prove that D is a fuzzy disjunction and C is a fuzzy conjunction. $\forall\, \alpha, x, y \in [0, 1]$, if $x = 0$ or $\alpha = 0$, we have

$$\begin{aligned} I_{D,N}(C(x, \alpha), y) &= \max(1 - C(x, \alpha), y) = \max(1, y) = 1 \\ &= \max(1 - x, 1 - \alpha, y) = \max(1 - x, \max(1 - \alpha, y)) \\ &= I_{D,N}(x, I_{D,N}(\alpha, y)). \end{aligned}$$

If $x \neq 0$, $\alpha \neq 0$ and $x \geq \alpha$, then

$$\begin{aligned} I_{D,N}(C(x, \alpha), y) &= \max(1 - C(x, \alpha), y) = \max(1 - \alpha, y) \\ &= \max(1 - x, 1 - \alpha, y) = \max(1 - x, \max(1 - \alpha, y)) \\ &= I_{D,N}(x, I_{D,N}(\alpha, y)). \end{aligned}$$

If $x \neq 0$, $\alpha \neq 0$ and $x < \alpha$, then

$$\begin{aligned} I_{D,N}(C(x, \alpha), y) &= \max(1 - C(x, \alpha), y) = \max(1 - x, y) \\ &= \max(1 - x, 1 - \alpha, y) = \max(1 - x, \max(1 - \alpha, y)) \\ &= I_{D,N}(x, I_{D,N}(\alpha, y)). \end{aligned}$$

Then the tuple $(C, I_{D,N})$ satisfies (3.1).

(ii) If C is an operation such that

$$C(x, y) = \begin{cases} 0, & y = 0, \\ x, & \text{otherwise}, \end{cases}$$

and $D(x, y) = \max\{x, y\}$, $N(x) = 1 - x$. Let $x = 0.5$, $y = 0.6$, $\alpha = 0.3$. Then it is easy to verify that the tuple $(C, I_{D,N})$ does not satisfy (3.1). □

By Example 3.2, we know that there are fuzzy conjunctions and (D, N)-implications satisfying (3.1), at the same time there exist fuzzy conjunctions and (D, N)-implications that do not satisfy (3.1), the first natural question to answer is the following:

Under what conditions, the fuzzy conjunction and (D, N)-implication satisfy (3.1)?

Proposition 3.3 *Let C be a fuzzy conjunction, D be an associative fuzzy disjunction and N be a strong negation. If D is the N-dual of C, that is, $D(N(x), y) = N(C(x, N(y)))$ for all $x, y \in [0,1]$, then the tuple $(C, I_{D,N})$ satisfies (3.1).*

Proof $\forall\, \alpha, x, y \in [0,1]$, we have

$$\begin{aligned} I_{D,N}(C(x,\alpha), y) &= D(N(C(x,\alpha)), y) = D(N(C(x, NN(\alpha))), y) \\ &= D(D(N(x), N(\alpha)), y) = D(N(x), D(N(\alpha), y)) \\ &= I_{D,N}(x, I_{D,N}(\alpha, y)). \ \square \end{aligned}$$

Remark 3.4 Proposition 3.3 is only a sufficient condition for $(C, I_{D,N})$ to satisfy Eq. (3.1), it is not a necessary condition. For example, Example 3.2 (i), $(C, I_{D,N})$ satisfies (3.1). However, let $x = \frac{3}{4}$ and $y = \frac{1}{3}$. Then

$$\begin{aligned} D(N(x), y) &= D(N(\frac{3}{4}), \frac{1}{3}) = \max\{1 - \frac{3}{4}, \frac{1}{3}\} \\ &= \frac{1}{3} \neq \frac{3}{4} = 1 - C(\frac{1}{4}, 1 - \frac{1}{3}) \\ &= N(C(\frac{1}{4}, N(\frac{1}{3}))) = N(C(x, N(y))). \end{aligned}$$

Example 3.5 If $C = \min$, $D = \max$ and $N(x) = 1 - x$ for all $x \in [0,1]$, it can be easily verified that $\max(1 - x, y) = N(\min(x, 1 - y))$, by Proposition 3.3, $(C, I_{D,N})$ satisfies (3.1).

In order to explore what conclusions will be obtained when $(C, I_{D,N})$ satisfies (3.1), we define the functions: $\mathcal{F} = \{f : [0,1] \to [0,1] | f \text{ is non-decreasing with } f(0) = 0,\ f(1) = 1\}$.

Similar to [34, Lemma 3.8], we have the following Corollary 3.6.

Corollary 3.6 *Let D be a fuzzy disjunction, C be a fuzzy conjunction and N be a fuzzy negation. If the tuple $(C, I_{D,N})$ satisfies (3.1), and $e \in [0,1]$ is the left-neutral element of C, then there exists an $f \in \mathcal{F}$ such that $I_{D,N}(x, y) = f(I_{D,N}(x, y))$.*

Corollary 3.7 *Let D be a fuzzy disjunction, C be a fuzzy conjunction and N be a fuzzy negation. If the tuple $(C, I_{D,N})$ satisfies (3.1), $e \in [0,1]$ is the left-neutral element of C, then f is the identity mapping if and only if $N(e)$ is the left-neutral element of D.*

Proof On the one hand, if f is the identity mapping, then $D(N(e), x) = I_{D,N}(e, x) = f(x) = x$, that is, $N(e)$ is the left-neutral element of D. On the other hand, if $N(e)$ is the left-neutral element of D, by Corollary 3.6, $\forall\, x \in [0,1]$, $f(x) = I_{D,N}(e, x) = D(N(e), x) = x$. This shows that f is the identity mapping. $\square$

Proposition 3.8 *Let D be a fuzzy disjunction, C be a fuzzy conjunction and N be a fuzzy negation. If the tuple $(C, I_{D,N})$ satisfies (3.1), and $e \in [0,1]$ is the left-neutral element of C, then $N_{I_{D,N}}(x) = 0$ for all $x \in [e, 1]$.*

Proof Since $I_{D,N}$ is the non-increasingness in the first variable, we have $\forall\, x \in [e,1]$, $N_{I_{D,N}}(x) = I_{D,N}(x, 0) \leq I_{D,N}(e, 0) = f(0) = 0$. $\square$

The following proposition gives a necessary and sufficient condition for $(C, I_{D,N})$ to satisfy Eq. (3.1).

Proposition 3.9 *Let C be a fuzzy conjunction, $I_{D,N}$ be a (D, N)-implication, $e \in [0, 1]$ be the left-neutral element of C, and $N(e)$ be not the left-neutral element of D. Then $(C, I_{D,N})$ satisfies (3.1) if and only if there exists a non identity mapping $f \in \mathcal{F}$ such that $I_{D,N}(x, y) = f(I_{D,N}(x, y))$ and $f(I_{D,N}(C(x, \alpha), y)) = f(I_{D,N}(x, I_{D,N}(\alpha, y)))$.*

Proof If $(C, I_{D,N})$ satisfies (3.1), by Corollaries 3.6 and 3.7, there exists a non identity mapping $f \in \mathcal{F}$ such that $I_{D,N}(x, y) = f(I_{D,N}(x, y))$, and since $(C, I_{D,N})$ satisfies (3.1), then $f(I_{D,N}(C(x, \alpha), y)) = f(I_{D,N}(x, I_{D,N}(\alpha, y)))$. Conversely, if there exists a non identity mapping $f \in \mathcal{F}$ such that $I_{D,N}(x, y) = f(I_{D,N}(x, y))$, then $I_{D,N}(C(x, \alpha), y) = f(I_{D,N}(C(x, \alpha), y)) = f(I_{D,N}(x, I_{D,N}(\alpha, y))) = I_{D,N}(x, I_{D,N}(\alpha, y))$. □

Example 3.10 Suppose $C(x, y) = \min\{x, y\}$. If $\forall\, \alpha, x, y \in [0, 1]$,

$$D(x, y) = \begin{cases} x, & y = 0, \\ 1, & y > 0; \end{cases}$$

$$N(x) = \begin{cases} 0, & x = 1, \\ 1, & x < 1. \end{cases}$$

Obviously, 1 is the left-neutral element of C, but 0 is not the left-neutral element of D. We can verify that $(C, I_{D,N})$ satisfies (3.1). There exists a non identity mapping $f \in \mathcal{F}$: $f(x) = I_{D,N}(1, x) = D(0, x)$ such that $I_{D,N}(x, y) = f(I_{D,N}(x, y))$ and $f(I_{D,N}(C(x, \alpha), y)) = f(I_{D,N}(x, I_{D,N}(\alpha, y)))$.

4. The cross-migrativity based on (D, N)-implications

In [36], Fang discussed the α-cross-migrativity of t-conorms over fuzzy implications, that is, $I(x, S(\alpha, y)) = S(y, I(x, \alpha))$ for all $\alpha, x, y \in [0, 1]$. However, we found that the associativity of t-conorm was not involved in studying the migrativity in some situations. Therefore, in this section, we consider degenerating t-conorms into the simplest fuzzy disjunctions, studying the cross-migrativity of fuzzy disjunctions about fuzzy implications. On the basis of Section 3, we studied the cross-migrativity based on (D, N)-implications. We begin with the definition of cross-migrative about fuzzy implications.

Definition 4.1 *Let $D : [0, 1] \to [0, 1]$ be a fuzzy disjunction and $I : [0, 1]^2 \to [0, 1]$ be a fuzzy implication. If $\forall\, x, y \in [0, 1]$,*

$$I(x, D(y, z)) = D(y, I(x, z)), \tag{4.1}$$

then (I, D) is said to be cross-migrative. Moreover, if $z = \alpha \in [0, 1]$, then (I, D) is α-cross-migrative, that is,

$$I(x, D(y, \alpha)) = D(y, I(x, \alpha)). \tag{4.2}$$

If (I,D) is α-cross-migrative, then (I,D) is also known as satisfying α-cross-migrativity.

Remark 4.2 In Definition 4.1, we define the α-cross-migrativity of (I,D) to be different from the α-cross-migrativity of the t-conorms, due to the fact that the t-conorms are commutative, whereas D is non-commutative.

In the following, we give an example.

Example 4.3 Suppose

$$D(x,y)=\begin{cases}0, & x=y=0,\\ 1, & \text{otherwises},\end{cases}$$

and

$$I(x,y)=\begin{cases}1, & x<1,\\ y, & x=1.\end{cases}$$

Take $\alpha=0.5$, on the one hand, $I(x,D(y,0.5))=I(x,1)=1$. On the other hand,

$$D(y,I(x,0.5))=\begin{cases}D(y,1), & x<1,\\ D(y,0.5), & x=1,\end{cases}=\begin{cases}1, & x<1,\\ 1, & x=1.\end{cases}$$

Therefore, $D(y,I(x,0.5))=1$, (I,D) is 0.5-cross-migrative. In fact, when $\alpha\in(0,1)$, (I,D) is α-cross-migrative.

Take $\alpha=1$, on the one hand, $I(x,D(y,1))=I(x,1)=1$. On the other hand, $D(y,I(x,1))=D(y,1)=1$. Therefore, (I,D) is 1-cross-migrative.

Take $\alpha=0$, when $y=0,x=1$, then $I(x,D(y,0))=I(x,0)=0$, when $x<1$, then $I(x,D(y,0))=1$. In addition,

$$D(y,I(x,0))=\begin{cases}0, & x=1\text{ and }y=0,\\ 1, & \text{otherwises}.\end{cases}$$

Therefore, (I,D) is 0-cross-migrative. □

Through the above example, we found that there exist fuzzy implications and fuzzy conjunctions satisfy (4.1) and (4.2). In the following, it is necessary to study the equivalent characterization of cross-migrativity. In order to investigate the cross-migrativity, we provide the following definition.

Definition 4.4 ([39]) *Suppose I and J are fuzzy implications. If for all $x,y,z\in[0,1]$,*

$$I(x,J(y,z))=J(y,I(x,z)),\tag{4.3}$$

then (I,J) satisfies mutual exchangeability. Especially, if $z=\alpha\in[0,1]$, then (I,J) satisfies α-mutual exchangeability, that is,

$$I(x,J(y,\alpha))=J(y,I(x,\alpha)).\tag{4.4}$$

Proposition 4.5 *Suppose D is a fuzzy disjunction, I is a fuzzy implication, N is a strong fuzzy negation and $\alpha\in[0,1]$. Then (I,D) satifies (4.2) iff $(I,I_{D,N})$ satisfies (4.4), that is, $\forall\,x,y\in[0,1]$, $I(x,I_{D,N}(y,\alpha))=I_{D,N}(y,I(x,\alpha))$.*

Proof Necessity. Suppose (I, D) satifies (4.2). Then

$$I(x, I_{D,N}(y, \alpha)) = I(x, D(N(y), \alpha)) = D(N(y), I(x, \alpha)) = I_{D,N}(y, I(x, \alpha)).$$

Sufficiency. Suppose $(I, I_{D,N})$ satisfies (4.4). Then

$$\begin{aligned} I(x, D(y, \alpha)) &= I(x, D(NN(y), \alpha)) = I(x, I_{D,N}(N(y), \alpha)) = I_{D,N}(N(y), I(x, \alpha)) \\ &= D(NN(y), I(x, \alpha)) = D(y, I(x, \alpha)). \quad \square \end{aligned}$$

Similar to Proposition 4.5, we have the following corollary.

Corollary 4.6 *Suppose D is a fuzzy disjunction, I is a fuzzy implication and N is a strong fuzzy negation. Then (I, D) satifies (4.1) iff $(I, I_{D,N})$ satisfies (4.3).*

Proposition 4.7 *Suppose D is a fuzzy disjunction, I is a fuzzy implication, N is a continuous fuzzy negation and $\alpha \in [0, 1]$. If $(I, I_{D,N})$ satisfies (4.4), then (I, D) satisfies (4.2).*

Proof $\forall\, x, y \in [0, 1]$, by Proposition 2.10, we have

$$\begin{aligned} I(x, D(y, \alpha)) &= I(x, D(N \circ \mathcal{R}_N(y), \alpha)) = I(x, I_{D,N}(\mathcal{R}_N(y), \alpha)) \\ &= I_{D,N}(\mathcal{R}_N(y), I(x, \alpha)) = D(N \circ \mathcal{R}_N(y), I(x, \alpha)) \\ &= D(y, I(x, \alpha)). \quad \square \end{aligned}$$

Similar to Proposition 4.7, we have the following Corollary 4.8.

Corollary 4.8 *Suppose D is a fuzzy disjunction, I is a fuzzy implication and N is a continuous fuzzy negation. If $(I, I_{D,N})$ satisfies (4.3), then (I, D) satisfies (4.1).*

In the following, we investigate the relationship between cross-migrativity and cross-law of importation, starting with the definition of cross-law of importation.

Definition 4.9 ([34]) *Suppose I and J are fuzzy implications, C is a fuzzy conjunction. If for all $x, y, z \in [0, 1]$,*

$$I(C(x, z), y) = J(x, I(z, y)), \tag{4.5}$$

then (I, J, C) satisfies cross-law of importation. Especially, if $z = \alpha \in [0, 1]$, then (I, J, C) satisfies α-cross-law of importation, that is,

$$I(C(x, \alpha), y) = J(x, I(\alpha, y)). \tag{4.6}$$

Theorem 4.10 *Suppose D is a fuzzy disjunction, N is a strong fuzzy negation, I is a fuzzy implication satisfying the law of contraposition and $\alpha \in [0, 1]$. Then (I, D) satisfies (4.2) iff $(I, I_{D,N}, C)$ satisfies (4.6), that is, $I(C(x, \alpha), y) = I_{D,N}(x, I(\alpha, y))$ for all $x, y \in [0, 1]$, where $C(x, y) = N(D(N(x), N(y)))$ for all $x \in [0, 1]$.*

Proof Necessity. Suppose that (I, D) satisfies (4.2). Then

$$\begin{aligned} I(C(x, \alpha), y) &= I(N(D(N(x), N(\alpha))), y) = I(N(y), NN(D(N(x), N(\alpha)))) \\ &= I(N(y), D(N(x), N(\alpha))) = D(N(x), I(N(y), N(\alpha))) \end{aligned}$$

$$= I_{D,N}(x, I(N(y), N(\alpha))) = I_{D,N}(x, I(\alpha, y)).$$

Sufficiency. Suppose that $(I, I_{D,N}, C)$ satisfies (4.6). Then

$$\begin{aligned} I(x, D(y,\alpha)) &= I(x, N(C(N(y), N(\alpha)))) = I(NN(C(N(y), N(\alpha))), N(x)) \\ &= I(C(N(y), N(\alpha)), N(x)) = I_{D,N}(N(y), I(N(\alpha), N(x))) \\ &= D(NN(y), I(N(\alpha), N(x))) = D(y, I(x,\alpha)). \quad \square \end{aligned}$$

Corollary 4.11 *Suppose D is a fuzzy disjunction, I is a fuzzy implication satisfying the law of contraposition and N is a strong fuzzy negation. Then (I, D) satisfies (4.1) iff $(I, I_{D,N}, C)$ satisfies (4.5), where $C(x, y) = N(D(N(x), N(y)))$ for all $x \in [0, 1]$.*

Theorem 4.12 *Suppose C is a fuzzy conjunction, I is a fuzzy implication satisfying the law of contraposition, N is a continuous fuzzy negation and $\alpha \in [0, 1]$. D is given as follows: $\forall\, x, y \in [0, 1]$, $D(x, y) = \mathcal{R}_N(C(N(x), N(y)))$. If $(I, I_{D,N}, C)$ satisfies (4.6). Then (I, D) satisfies (4.2).*

Proof Clearly, D is a fuzzy disjunction. $\forall\, x, y \in [0, 1]$, $\alpha \in [0, 1]$

$$\begin{aligned} I(x, D(y,\alpha)) &= I(x, \mathcal{R}_N \circ C(N(y), N(\alpha))) = I(N\mathcal{R}_N \circ C(N(y), N(\alpha)), N(x)) \\ &= I(C(N(y), N(\alpha)), N(x)) = I_{D,N}(N(y), I(N(\alpha), N(x))) \\ &= D(NN(y), I(N(\alpha), N(x))) = D(y, I(x,\alpha)). \quad \square \end{aligned}$$

Corollary 4.13 *Suppose C is a fuzzy conjunction, I is a fuzzy implication satisfying the law of contraposition, N is a continuous fuzzy negation and $\alpha \in [0, 1]$. D is given as Theorem 4.12. If $(I, I_{D,N}, C)$ satisfies (4.5), then (I, D) is (4.1).*

Theorem 4.14 *Suppose D_1, D_2 are two fuzzy disjunctions, N is a strong fuzzy negation, I is a (D_1, N)-implication and $\alpha \in [0, 1]$, and C_1, C_2 are N-dual to D_1, D_2, that is, $C_1(x, y) = ND_1(N(x), N(y))$, $C_2 = ND_2(N(x), N(y))$. Then (I, D_2) is $N(\alpha)$-cross-migrative iff (C_1, C_2) satisfies $C_1(x, C_2(y, \alpha)) = C_2(y, C_1(x, \alpha))$.*

Proof Necessity. Suppose that (I, D_2) is $N(\alpha)$-cross-migrative. Then

$$\begin{aligned} C_1(x, C_2(y,\alpha)) &= ND_1(N(x), NC_2(y,\alpha)) = ND_1(N(x), NND_2(N(y), N(\alpha))) \\ &= ND_1(N(x), D_2(N(y), N(\alpha))) = NI_{D_1,N}(x, D_2(N(y), N(\alpha))) \\ &= ND_2(N(y), I_{D_1,N}(x, N(\alpha))) = ND_2(N(y), D_1(N(x), N(\alpha))) \\ &= ND_2(N(y), NND_1(N(x), N(\alpha))) = C_2(y, C_1(x,\alpha)). \end{aligned}$$

Sufficiency. Suppose that (C_1, C_2) satisfies $C_1(x, C_2(y, \alpha)) = C_2(y, C_1(x, \alpha))$. Then

$$\begin{aligned} I_{D_1,N}(x, D_2(y, N(\alpha))) &= D_1(N(x), D_2(y, N(\alpha))) = NC_1(x, ND_2(y, N(\alpha))) \\ &= NC_1(x, ND_2(NN(y), N(\alpha))) = NC_1(x, C_2(N(y), \alpha)) \\ &= NC_2(N(y), C_1(x,\alpha)) = NC_2(N(y), ND_1(N(x), N(\alpha))) \\ &= NC_2(N(y), NI_{D_1,N}(x, N(\alpha))) = D_2(y, I_{D_1,N}(x, N(\alpha))). \quad \square \end{aligned}$$

Proposition 4.15 *Suppose C_1, C_2 are fuzzy conjunctions, N is a strong fuzzy negation and $\alpha \in [0,1]$. If (C_1, C_2) is α-cross-migrative, that is, $C_2(x, C_1(y,\alpha)) = C_1(y, C_2(x,\alpha))$, $\forall\, x, y \in [0,1]$, define $D(x,y) = \mathcal{R}_N(C_1(N(x), N(y)))$, $I(x,y) = \mathcal{R}_N(C_2(x, N(y)))$, then (I, D) is $\mathcal{R}_N(\alpha)$-cross-migrative.*

Proof Firstly, we verify D is a fuzzy conjunction. Clearly, D is nondecreasing and $D(0,0) = 0$, $D(0,1) = D(1,0) = 1$. By Definition 2.1, D is a fuzzy conjunction. Similarly, I is a fuzzy implication. $\forall\, x, y \in [0,1]$, we have

$$\begin{aligned} I(x, D(y, \mathcal{R}_N(\alpha))) &= \mathcal{R}_N C_2(x, ND(y, \mathcal{R}_N(\alpha))) \\ &= \mathcal{R}_N C_2(x, N\mathcal{R}_N C_1(N(y), N\mathcal{R}_N(\alpha))) \\ &= \mathcal{R}_N C_2(x, C_1(N(y), \alpha)) \\ &= \mathcal{R}_N C_1(N(y), C_2(x, \alpha)) \\ &= \mathcal{R}_N C_1(N(y), N\mathcal{R}_N(C_2(x, N\mathcal{R}_N(\alpha)))) \\ &= \mathcal{R}_N C_1(N(y), NI(x, \mathcal{R}_N(\alpha))) \\ &= D(y, I(x, \mathcal{R}_N(\alpha))). \quad \square \end{aligned}$$

Example 4.16 (i) Suppose $\forall\, x, y \in [0,1], C_1(x,y) = C_2(x,y) = \min\{x,y\}$, $N(x) = 1 - x$. We have $\mathcal{R}_N(x) = 1 - x$. Obviously, $C_2(x, C_1(y,\alpha)) = C_1(y, C_2(x,\alpha))$. If $\forall\, x, y \in [0,1]$, define $D(x,y) = 1 - \min(1-x, 1-y))$, $I(x,y) = 1 - \min(x, 1-y))$. Then $\forall\, \alpha \in [0,1]$, (I, D) is $(1-\alpha)$-cross-migrative.

(ii) Suppose $\forall\, x, y \in [0,1]$,

$$C_1(x,y) = \begin{cases} 0, & y = 0, \\ x, & \text{otherwises}; \end{cases}$$

$$C_2(x,y) = \begin{cases} y, & x > 0, \\ 0, & \text{otherwises}; \end{cases}$$

$$N(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwises}. \end{cases}$$

We can easily prove that C_1, C_2 are fuzzy conjunctions, N is a strong fuzzy negation. In addition, if $\alpha = 0$, $C_2(x, C_1(y,\alpha)) = C_2(x, C_1(y,0)) = 0 = C_1(y, C_2(x,0)) = C_1(y, C_2(x,\alpha))$. If $\alpha \neq 0$, on the one hand,

$$C_2(x, C_1(y,\alpha)) = C_2(x,y) = \begin{cases} y, & x > 0, \\ 0, & \text{otherwises}. \end{cases}$$

On the other hand,

$$\begin{aligned} C_1(y, C_2(x,\alpha)) &= \begin{cases} C_1(y,\alpha), & x > 0, \\ C_1(y,0), & \text{otherwises}, \end{cases} \\ &= \begin{cases} y, & x > 0, \\ 0, & \text{otherwises}. \end{cases} \end{aligned}$$

Therefore, $C_2(x, C_1(y, \alpha)) = C_1(y, C_2(x, \alpha))$. By Proposition 4.15, $\forall\, x, y \in [0,1]$, define $D(x,y) = \mathcal{R}_N(C_1(N(x), N(y)))$, $I(x,y) = \mathcal{R}_N(C_2(x, N(y)))$. Then $\forall\, \alpha \in [0,1]$, (I, D) is $(1-\alpha)$-cross-migrative. □

5. Conclusions and future works

In this paper we study the generalized laws of importation and cross-migrativity based on (D, N)-implications. First, we study the generalized law of importation of (D, N)-implications. In addition, we investigate the necessary and sufficient conditions of the generalized law of importation of (D, N)-implications. Secondly, we discuss the α-cross-migrativity of fuzzy disjunctions over fuzzy implications, we obtain that (I, D) is cross-migrative iff I and $I_{D,N}$ generated by the same fuzzy disjunction and strong fuzzy negations satisfy mutual exchangeability. Furthermore, by defining cross-law of importation, we obtain (I, D) is cross-migrative iff I, $I_{D,N}$ generated by the same fuzzy disjunction and strong fuzzy negations and C where $C(x,y) = N(D(N(x), N(y)))$ for all $x, y \in [0,1]$, satisfy cross-law of importation. For further studies, we would focus on the cross-migrativity of some specific fuzzy implications, such as R-implications,I_g, I_f, as well as other classes of fuzzy implications.

Acknowledgements We thank the referees for their time and comments.

References

[1] M. BACZYNSKI, B. JAYARAM. *Fuzzy Implications*. Springer, Berlin, Germany, 2008.

[2] M. MAS, M. MONSERRAT, J. TORRENS, et al. *A survey on fuzzy implication functions*. IEEE Trans. Fuzzy Syst., 2007, **15**: 1107–1121.

[3] J. KACPRZYK, W. PEDRYCZ. *Springer Handbook of Computational Intelligence*. Springer-Verlag, Heidelberg, Germany, 2015.

[4] I. IANCU. *Heart disease diagnosis based on mediative fuzzy logic*. Artif. Intell. Med., 2018, **89**: 51–60.

[5] K. SHIHABUDHEEN, G. PILLAI. *Recent advances in neuro-fuzzy system: a survey*. Knowl.-Based Syst., 2018, **52**: 136–162.

[6] F. KUMAR, D. CLAUDIO. *Implications of estimating confidence intervals on group fuzzy decision making scores*. Expert Syst. Appl., 2016, **65**: 152–163.

[7] Y. ABDOLRAHIMAHIM, P. OLIVIER, H. BERND. *Application of fuzzy logic control for the dough proofing process*. Food Bioprod. Process., 2019, **115**: 36–46.

[8] Xichen YE, Tong ZHANG, Huicui CHEN, et al. *Fuzzy control of hydrogen pressure in fuel cell system*. Int. J. Hydrog. Energy, 2019, **44**: 8460–8466.

[9] J. P. POLI, L. BOUDET. *A fuzzy expert system architecture for data and event stream processing*. Fuzzy Sets and Systems, 2018, **343**: 20–34.

[10] M. BACZYŃSKI, G. BELIAKOV, H. SOLA, et al. *Advances in Fuzzy Implication Functions*. Springer, NY, New York, 2013.

[11] Hongjun ZHOU. *Two general construction ways toward unified framework of ordinal sums of fuzzy implications*. IEEE Trans. Fuzzy Syst., 2021, **29**(4): 846–860.

[12] R. FERNANDEZ-PERALTA, S. MASSANET, A. MEAIAROVÁ-ZEMÁNKOVÁ, et al. *A general framework for the characterization of (S,N)-implications with a non-continuous negation based on completions of t-conorms*. Fuzzy Sets and Systems, 2022, **441**: 1–32.

[13] M. BACZYŃSKI, B. JAYARAM. *On the characterizations of (S,N)-implications*. Fuzzy Sets and Systems, 2007, **158**(15): 1713–1727.

[14] G. P. DIMURO, B. BEDREGAL, R. SANTIAGO. *On (G,N)-implications derived from grouping functions*. Inform. Sci., 2014, **279**: 1–17.

[15] J. PINHEIRO, B. BEDREGAL, R. SANTIAGO, et al. *A study of (T, N)-implications and its use to construct a new class of fuzzy subsethood measure*. Internat. J. Approx. Reason., 2018, **97**: 1–16.

[16] Meng CAO, Baoqing HU, Junsheng QIAO. *On interval (G, N)-implications and (O,G,N)-implications derived from interval overlap and grouping functions*. Internat. J. Approx. Reason., 2018, **100**: 135–160.

[17] Qin MA, Hongjun ZHOU. *(D, N)-implication and its characterization*. Fuzzy Systems and Mathematics, 2017, **31**(5): 1–12.

[18] Junsheng QIAO. *Directional monotonic fuzzy implication functions induced from directional increasing quasi-grouping functions*. Comput. Appl. Math., 2022, **41**(5): Paper No. 218, 16 pp.

[19] F. NERES, B. BEDREGAL, R. SANTIAGO. *On a new contrapositivisation technique for fuzzy implications constructed from quasi-overlap and quasi-grouping functions*. Internat. J. Approx. Reason., 2023, **162**: Paper No. 109012, 33 pp.

[20] Junsheng QIAO. *$R_{\mathcal{O}}$-implications induced from C_L-overlap functions on complete lattices*. Soft Computing, 2022, **26**: 8229–8243.

[21] J. FODOR, I. J. RUDAS. *On continuous triangular norms that are migrative*. Fuzzy Sets and Systems, 2007, **158**(15): 1692–1697.

[22] J. FODOR, I. J. RUDAS. *An extension of the migrative property for triangular norms*. Fuzzy Sets and Systems, 2011, **168**: 70–80.

[23] Yao OUYANG. *Generalizing the migrativity of continuous t-norms*. Fuzzy Sets and Systems, 2013, **211**: 73–83.

[24] E. AAICI. *On the α-migrativity of t-norms and t-conorms over nullnorms and uninorms*. New Trends Math. Sci., 2018, **6**(1): 153–158.

[25] Chuyao HUANG, Feng QIN. *Migrativity properties of uninorms over 2-uninorms*. Internat. J. Approx. Reason., 2021, **139**: 104–129.

[26] Wenhuang LI, Feng QIN. *Migrativity equation for uninorms with continuous underlying operators*. Fuzzy Sets and Systems, 2021, **414**: 115–134.

[27] M. MAS, M. MONSERRAT, D. RUIZ-AGUILERA, et al. *Migrative uninorms and nullnorms over t-norms and t-conorms*. Fuzzy Sets and Systems, 2015, **261**: 20–32.

[28] Junsheng QIAO, Baoqing HU. *On generalized migrativity property for overlap functions*. Fuzzy Sets and Systems, 2019, **357**: 91–116.

[29] Junsheng QIAO, Baoqing HU. *On the migrativity of uninorms and nullnorms over overlap and grouping functions*. Fuzzy Sets and Systems, 2018, **346**: 1–54.

[30] Junsheng QIAO, Bin ZHAO. *On α-cross-migrativity of overlap (0-overlap) functions*. IEEE Trans. Fuzzy Syst., 2022, **30**(2): 448–461.

[31] Hongjun ZHOU, Xinxin YAN. *Migrativity properties of overlap functions over uninorms*. Fuzzy Sets and Systems, 2021, **403**: 10–37.

[32] Kuanyun ZHU, Jingru WANG, Yongwei YANG. *A short note on the migrativity properties of overlap functions over uninorms*. Fuzzy Sets and Systems, 2021, **414**: 135–145.

[33] Kuanyun ZHU, Baoqing HU. *Addendum to on the migrativity of uninorms and nullnorms over overlap and grouping functions*. Fuzzy Sets and Systems, 2020, **386**: 48–59.

[34] M. BACZYŃAKI, B. JAYARAM, R. MESIAR. *Fuzzy implications: alpha migrativity and generalised laws of importation*. Inform. Sci., 2020, **531**: 87–96.

[35] Deng PAN, Hongjun ZHOU, Xinxin YAN. *Characterizations for the migrativity of continuous t-conorms over fuzzy implications*. Fuzzy Sets and Systems, 2023, **456**: 173–196.

[36] Bowen FANG. *On α-cross-migrativity of t-conorms over fuzzy implications*. Fuzzy Sets and Systems, 2023, **466**: Paper No. 108463, 30 pp.

[37] U. BENTKOWSKA, A. KRÓL. *Preservation of fuzzy relation properties based on fuzzy conjunctions and disjunctions during aggregation process*. Fuzzy Sets and Systems, 2016, **291**: 98–113.

[38] J. DREWNIAK, A. KRÓL. *A survey of weak connectives and the preservation of their properties by aggregations*. Fuzzy Sets and Systems, 2010, **161**: 202–215.

[39] N. RAO VEMURI. *Mutually exchangeable fuzzy implications*. Inform. Sci., 2015, **317**: 1–24.