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Bi-Jordan n-Derivations on Triangular Rings: Maximal Quotient Rings and Faithful Module

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Abstract In this paper, we mainly study the structure of bi-Jordan n-derivations in triangular rings under conditions of maximal quotient rings and faithful bimodules, respectively. It is shown that every bi-Jordan n-derivation can be decomposed into the sum of an inner biderivation and an extremal biderivation in two different conditions. As by-products, the structures of bi-Jordan n-derivation over upper triangular matrix rings and nest algebras are characterized, respectively, and generalize the known results.

Keywords triangular rings; maximal quotient rings; faithful bimodules; nest algebras; extremal biderivation

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1. Introduction

In this paper, we focus on the structure of bi-Jordan n-derivations over triangular rings from the point of view of bimodules and maximal quotient rings, respectively. To this end, we first introduce some basics related to bi-Jordan n-derivations.

For a unital ring $\mathcal{A}$ and its center $\mathbf{C}(\mathcal{A})$. A ring $\mathcal{A}$ is said to be t-torsion free if $tx = 0$ implies $x = 0$ for some positive integer $t \in \mathfrak{N}$ and arbitrary $x \in \mathcal{A}$. Using equation $Q_2(x_1, x_2) = x_1 \circ x_2 = x_1x_2 + x_2x_1$ and mathematical induction, we define a polynomial $Q_n(x_1, x_2, \ldots, x_n)$ with respect to variables $x_1, x_2, \ldots, x_n$ in the following manner:

$$Q_n(x_1, x_2, \ldots, x_n) = Q_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ x_n$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{A}$. Specifically, $Q_1(x_1) = x_1$. A biadditive mapping $\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is known as bi-Jordan n-derivation if each component is Jordan n-derivation, i.e.,

$$\varphi(Q_n(x_1, x_2, \ldots, x_n), y) = \sum_{i=1}^{n} Q_n(x_1, x_2, \ldots, \varphi(x_i, y), \ldots, x_n) \text{ and}$$

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$$\varphi(x, Q_n(y_1, y_2, \ldots, y_n)) = \sum_{i=1}^{n} Q_n(y_1, y_2, \ldots, \varphi(x, y_i), \ldots, y_n)$$

for $x, y, x_i, y_i \in \mathcal{A}, i \in \{1, 2, \ldots, n\}$. For $n = 2$, it will evolve into Jordan biderivation. That is, the mapping $\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfies the condition:

$$\varphi(x \circ z, y) = \varphi(x, y) \circ z + x \circ \varphi(z, y) \text{ and } \varphi(x, y \circ z) = \varphi(x, y) \circ z + y \circ \varphi(x, z)$$

for all $x, y, z \in \mathcal{A}$. Clearly, every Jordan biderivation is a biderivation defined in the following way:

$$\varphi(xz, y) = \varphi(x, y)z + x\varphi(z, y) \text{ and } \varphi(x, yz) = \varphi(x, y)z + y\varphi(x, z)$$

for all $x, y, z \in \mathcal{A}$. If $\mathcal{A}$ is a noncommutative ring, inner biderivation $\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ can be defined as follows:

$$\varphi(x, y) = \chi[x, y]$$

for all $x, y \in \mathcal{A}$. It is clear that every inner biderivation is a Jordan biderivation. In addition, we also need to introduce a bilinear mapping concept extremal biderivation. A biadditive mapping $\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is said to be an extremal biderivation if it is of the form $\varphi(x, y) = [x, [y, a]]$ for all $x, y \in \mathcal{A}$ and some $a \notin \mathcal{Z}(\mathcal{A})$ such that $[[\mathcal{A}, \mathcal{A}], a] = 0$. And with the help of the respective definition formulas, it is clear that every biderivation, inner biderivation, Jordan biderivation and extremal biderivation is a bi-Jordan n-derivation.

The study of the structure of the biderivations of different algebras or rings originating from Maksa [1, 2] has attracted many scholars' attention [3–6]. This problem belongs to the Herstein Lie mapping framework proposed by Herstein [7] in his one-hour presentation to the American Mathematical Society in 1961. In recent years, the structure of some maps on triangular rings has been studied by many scholars, and they are described from the perspective of quotient rings or bimodules. This article presents the structure of bi-Jordan n-derivation on a triangular ring from two different perspectives: faithful bimodule and maximal quotient ring. Many mappings related to bi-Jordan n-derivation under the faithful bimodule condition have been studied by scholars. Biderivations of upper triangular matrix rings are studied by Ghosseiri [8], Biderivations of triangular algebras are worked by Benkovič [9]. After that, the first author and his coauthor studied the structure of Jordan biderivation on triangular algebra under the condition of Benkovič [9] and proved that it can be decomposed into the sum of inner biderivation and extremal biderivation, and generalized the derivation of Benkovič [9]. Inspired by papers [9, 10], the first main result of this paper investigates the structure of bi-Jordan n-derivations under faithful bimodule conditions and shows that every bi-Jordan n-derivation can be decomposed into the sum of inner biderivations and extremal biderivations.

At the same time, many researchers used the structure of the maximal quotient ring of the ring $\mathcal{R}$ introduced by Utumi [11] to study the structure of the map associated with the bi-Jordan n-derivation. Functional identities of degree 2 in triangular rings have been studied from different angles in the paper [12, 13], and it should be noted that this mapping has also been studied by Eremita [14] from a bimodule perspective. The biderivation and Jordan biderivation on a triangular ring were characterized by Wang [15] and Liu and his coauthor [16] by virtue of the properties of the maximal quotient ring respectively, and it is proved that every biderivation on the ring can be decomposed into the sum of inner biderivation and extremal biderivation, and every Jordan biderivation is biderivation. Inspired by the article [15–17] we study the structure of bi-Jordan n-derivation with the help of the structure of the quotient ring, and also prove that every bi-Jordan n-derivation can also be decomposed into the sum of inner biderivation and extremal biderivation. At the same time, we extend Wang's result [15].

In this paper, we describe the structure of the bi-Jordan n-derivations on triangular rings from the perspective of bimodule and quotient ring respectively. In the third part, we use the quotient ring structure to study the structure of the bi-Jordan n-derivation on the triangular ring. In this process, we use a more ingenious method to construct the form of the extremal biderivation, which makes the proof process more simple. In the fourth part, we describe the structure of the bi-Jordan n-derivations on the triangular ring from the structure of the faithful bimodules, and we illustrate the proof process in a concise but easy to understand way. In recent years, under the framework of Herstain Lie-type mapping, the methods to study the mapping structure on triangular rings are all under the condition of faithful bimodule and quotient ring. In this paper, we present both of them, in order to have a more comprehensive understanding of the structure of bi-Jordan n-derivations on triangular rings.

2. Triangular rings

In this part, we mainly introduce some basic theories of triangular rings.

For the associative ring $\mathcal{T}$ containing the identity element I and the idempotent elements e and f satisfying the equation $e+f=I$, if the associative ring $\mathcal{T}$ has the following decomposition form

$$\mathcal{T}=e\mathcal{T}e\oplus e\mathcal{T}f\oplus f\mathcal{T}f,$$

and satisfies the following conditions: $f\mathcal{T}e=0$ and $e\mathcal{T}f$ is an $(e\mathcal{T}e,f\mathcal{T}f)$-bimodule. The name of the triangular ring comes from the decomposition form of the associative ring $\mathcal{T}$, which can be written as a matrix form of order 2:

$$\mathcal{T}=\begin{bmatrix} e\mathcal{T}e & e\mathcal{T}f \\ 0 & f\mathcal{T}f \end{bmatrix}=\left\{ x=\begin{bmatrix} exe & exf \\ 0 & fxf \end{bmatrix} \middle| x\in\mathcal{T} \right\}.$$

It is obvious that each element $x\in\mathcal{T}$ has a decomposition form $x=exe+exf+fxf$. By virtue of the structure of triangular algebras, (block) upper triangular matrix algebras and nest algebras are classical examples of triangular algebras [10, 14].

According to the purpose of this article, the basic knowledge we researched is divided into two situations, respectively.

Case 1. $e\mathcal{T}f$ is a faithful $(e\mathcal{T}e,f\mathcal{T}f)$-bimodule.

Let us define two natural $\mathcal{T}$-linear projections $\pi_{e\mathcal{T}e}:\mathcal{T}\to e\mathcal{T}e$ and $\pi_{f\mathcal{T}f}:\mathcal{T}\to f\mathcal{T}f$ by

$$\pi_{f\mathcal{T}f}(exe+exf+fxf)=exe \text{ and } \pi_{f\mathcal{T}f}(exe+exf+fxf)=fxf,$$

respectively. It is easy to see that $\pi_{e\mathcal{T}e}(\mathcal{Z}(\mathcal{T}))$ is a subalgebra of $\mathcal{Z}(e\mathcal{T}e)$ and that $\pi_{f\mathcal{T}f}(\mathcal{Z}(\mathcal{T}))$ is a subalgebra of $\mathcal{Z}(f\mathcal{T}f)$. Furthermore, there exists a unique algebraic isomorphism $\tau\colon \pi_{e\mathcal{T}e}(\mathcal{Z}(\mathcal{T})) \longrightarrow \pi_{f\mathcal{T}f}(\mathcal{Z}(\mathcal{T}))$ such that $am = m\tau(a)$ for all $a \in \pi_{e\mathcal{T}e}(\mathcal{Z}(\mathcal{T}))$ and for all $m \in M$. By the instrumentality of [9], the center of $\mathcal{T}$ is

$$\mathcal{Z}(\mathcal{T}) = \{a + b \in e\mathcal{T}e \oplus f\mathcal{T}f | aexf = exfb,\ \forall\, x \in \mathcal{T}\}.$$

Case 2. The maximal left ring of quotients.

Through the instrumentality of Utumi left quotient rings [11] (also called maximal left ring of quotients), we introduce another characterization of the triangular ring. For the unital ring, we introduced the concept of the maximal left ring of the quotient ring, which is defined as $\mathcal{Q}_{ml}(A)$. The center of $\mathcal{Q}_{ml}(A)$, also called the extended centroid of A, is defined using the symbol $\mathcal{C}(A)$.

By feat of [12, 13], the center $\mathcal{C}(\mathcal{T})$ of $\mathcal{Q}_{ml}(\mathcal{T})$ is

$$\mathcal{C}(\mathcal{T}) = \{k = c + d \in e\mathcal{Q}_{ml}(\mathcal{T})e \oplus f\mathcal{Q}_{ml}(\mathcal{T})f | keyf = eyfk,\ \forall\, y \in \mathcal{T}\}.$$

It is easy to verify that the map $\eta : \mathcal{C}(\mathcal{T})e \to \mathcal{C}(\mathcal{T})f$ is a ring isomorphism such that

$$\lambda e \cdot exf = exf \cdot \eta(\lambda e)$$

for all $x \in \mathcal{T}$ and $\lambda \in \mathcal{C}(\mathcal{T})$.

Let K, L be subsets of $\mathcal{Q}_{ml}(\mathcal{T})$. Set

$$\mathcal{C}(K, L) = \{q \in K | qx = xq,\ \forall\, x \in L\},$$

on behalf of [12, Proposition 2.5], we have $\mathcal{C}(\mathcal{T}) = \mathcal{C}(\mathcal{Q}_{ml}(\mathcal{T}), \mathcal{T})$.

By dint of [12, 13, 15, 18] and above notations, we have listed many important conclusions that are needed in the text. Since these conclusions have been proved, we have only listed them without giving their proofs.

Proposition 2.1 ([12, 13, 15, 18]) *Let $\mathcal{T}$ be a unital ring. The maximal left ring of quotients $\mathcal{Q}_{ml}(\mathcal{T})$ satisfies the following properties*

(1) $\mathcal{T}$ is a subring of the Utumi left quotient ring $\mathcal{Q}_{ml}(\mathcal{T})$ with the same I;

(2) For any dense left ideal $\mathcal{U}$ of $\mathcal{T}$ and a left $\mathcal{T}$-module homomorphism $\varrho : \mathcal{U} \to \mathcal{T}$, there exists $q \in \mathcal{Q}_{ml}(\mathcal{T})$ such that ϱ is of the form $\varrho(x) = xq$ for $x \in \mathcal{U}$;

(3) $\mathcal{Z}(\mathcal{T}) \subseteq \mathcal{C}(\mathcal{T})$. Furthermore, $\mathcal{Z}(\mathcal{T})e \subseteq \mathcal{C}(\mathcal{T})e$ and $\mathcal{Z}(\mathcal{T})f \subseteq \mathcal{C}(\mathcal{T})f$.

Proposition 2.2 ([12, Proposition 2.6]) *Let $\mathcal{T}$ be a triangular ring. Then $e\mathcal{T}$ is a dense left ideal of $\mathcal{T}$ and for each $q \in \mathcal{Q}_{ml}(\mathcal{T})$ the following hold:*

(1) $e\mathcal{T}fq = 0$ implies $fq = 0$;

(2) $qe\mathcal{T}f = 0$ implies $qe = 0$.

3. Bi-Jordan n-derivations: Utumi left ring of quotients

In this part, we study the structure of bi-Jordan n-derivation on triangular rings from the perspective of maximal left rings of quotient rings.

Theorem 3.1 *Let $\mathcal{T} = e\mathcal{T}e + e\mathcal{T}f + f\mathcal{T}f$ be a t-torsion free triangular ring, where $t \in \{2, n-1, 2^{n-1}-1\}$, and let $\varphi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Jordan n-derivation. If the following conditions hold:*

(1) $\mathcal{C}(f\mathcal{Q}_{ml}(\mathcal{R})f) = \mathcal{C}(\mathcal{R})f$;

(2) either $e\mathcal{R}\mathcal{C}(\mathcal{R})e$ or $f\mathcal{R}\mathcal{C}(\mathcal{R})f$ does not contain nonzero central ideals,

then every bi-Jordan n-derivation on $\mathcal{T}$ is a sum of an extremal biderivation and an inner biderivation.

For arbitrary elements $x = exe + exf + fxf \in \mathcal{T}$ and $y = eye + eyf + fyf \in \mathcal{T}$, according to the bilinear property of mapping φ, we have

$$\begin{aligned}\varphi(x,y) =&\varphi(exe+exf+fxf, eye+eyf+fyf)\\ =&\varphi(exe,eye)+\varphi(exe,eyf)+\varphi(exe,fyf)+\\ &\varphi(exf,eye)+\varphi(exf,eyf)+\varphi(exf,fyf)+\\ &\varphi(fxf,eye)+\varphi(fxf,eyf)+\varphi(fxf,fyf)\end{aligned} \tag{3.1}$$

for all $x, y \in \mathcal{T}$.

Let us begin this part of the proof with the following lemmas on extremal biderivation.

Lemma 3.2 *Let $\mathcal{T}$ be an associative algebra over a commutative ring $\mathcal{T}$ and $\varphi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Jordan n-derivation on $\mathcal{T}$. Then φ has the following properties*

(1) $\varphi(1, x) = \varphi(x, 1) = 0$ for all $x \in \mathcal{T}$;

(2) $\varphi(0, x) = \varphi(x, 0) = 0$ for all $x \in \mathcal{T}$;

(3) $\varphi(e, e) = -\varphi(e, f) = -\varphi(f, e) = \varphi(f, f)$, where $f = 1 - e$.

Proof In the process of proof, we mainly use the definition of bi-Jordan n-derivations, so we can obtain the following proof procedure.

(1) Corresponding to the definition of Q_n, we arrive at

$$\begin{aligned}2^{n-1}\varphi(1,x) &= \varphi(Q_n(1,1,\ldots,1),x) = \sum_{i=1}^{n} Q_n(1,\ldots 1,\varphi(1,x),1,\ldots,1)\\ &= 2^{n-1}n\varphi(1,x).\end{aligned}$$

Since $\mathcal{T}$ is 2 and $(n-1)$-torsion free, we have $\varphi(1, x) = 0$ for all $x \in \mathcal{T}$. Likewise, $\varphi(x, 1) = 0$ for all $x \in \mathcal{T}$.

(2) In relation to φ, the conclusion $\varphi(0, x) = \varphi(x, 0) = 0$ is clearly valid.

(3) It follows from (1) and equation $e + f = 1$ that

$$\varphi(e,e) = \varphi(e,1-f) = \varphi(e,1) - \varphi(e,f) = -\varphi(e,f).$$

In the same manner, we obtain $\varphi(f, e) = -\varphi(e, e)$ and $\varphi(f, f) = \varphi(e, e)$. And then it can be obtained that

$$\varphi(e,e) = -\varphi(e,f) = -\varphi(f,e) = \varphi(f,f). \quad \square$$

Lemma 3.3 ([9, Remark 4.5]) *A triangular ring $\mathcal{T} = e\mathcal{T}e + e\mathcal{T}f + f\mathcal{T}f$ has a nonzero extremal* *biderivation if and only if there exists* $0 \neq m_0 \in e\mathcal{T}f$ *such that*

$$[e\mathcal{T}e, e\mathcal{T}e]m_0 = 0 = m_0[f\mathcal{T}f, f\mathcal{T}f].$$

Lemma 3.4 *Let* $\varphi : \mathcal{T}\times\mathcal{T} \to \mathcal{T}$ *be a bi-Jordan* n*-derivation on* $\mathcal{T}$*. If* $\varphi(e, e) \neq 0$*, then* $\varphi = \kappa+\psi$*, where* $\kappa(x, y) = [x, [y, \varphi(e, e)]]$ *is an extremal biderivation and* ψ *is a bi-Jordan* n*-derivation and satisfies the relation* $e\psi(e, e)f = 0$*.*

Proof Since φ is a Jordan n-derivation of $\mathcal{T}$ for the second component, we arrive at

$$\begin{aligned}\varphi(e, exf) &= \varphi(e, Q_n(exf, f, \dots, f))\\ &= Q_n(\varphi(e, exf), f, \dots, f) + \sum_{i=2}^{n} Q_n(exf, f, \dots, f, \varphi(e, f), f, \dots, f)\\ &= 2^{n-1}f\varphi(e, exf)f + e\varphi(e, exf)f + (n-1)(exf\varphi(e, f) + \varphi(e, f)exf).\end{aligned}$$

Multiplying by e from left and f from right, and since $\mathcal{T}$ is $(n-1)$-torsion free, we get

$$e\varphi(e, f)exf + exf\varphi(e, f)f = 0 \text{ for all } x \in \mathcal{T}.$$

Replacing element exf with equation $exf = Q_n(e, exf, f, \dots, f)$ in $\varphi(e, exf)$, we can get

$$\begin{aligned}\varphi(e, exf) =&\varphi(e, Q_n(e, exf, f, \dots, f))\\ =&Q_n(\varphi(e, e), exf, f, \dots, f) + Q_n(e, \varphi(e, exf), f, \dots, f)+\\ &\sum_{i=3}^{n} Q_n(e, exf, f, \dots, f, \varphi(e, f), f, \dots, f)\\ =&\varphi(e, e)exf + exf\varphi(e, e) + e\varphi(e, exf)f.\end{aligned}$$

It follows from above equation that $e\varphi(e, e)exf + exf\varphi(e, e)f = 0$ for all $x \in \mathcal{T}$.

Assume that $\varphi(e, e) \neq 0$ and let us prove the mapping $\kappa(x, y) = [x, [y, \varphi(e, e)]]$ is an extremal biderivation of $\mathcal{T}$. Note that

$$\begin{aligned}\kappa(e, e) &= [e, [e, \phi(e, e)]] = [e, [e, e\phi(e, e)e + e\phi(e, e)f + f\phi(e, e)f]]\\ &= e\phi(e, e)f,\end{aligned}$$

then $\varphi(e, e) - \kappa(e, e) = e\varphi(e, e)e + f\varphi(e, e)f$.

In the following part, we prove that the element $e\varphi(e, e)f$ satisfies conditions

$$e\varphi(e, e)f[fxf, fyf] = 0 \text{ and } [exe, eye]e\varphi(e, e)f = 0.$$

In fact, for any $x, y \in \mathcal{T}$ we have

$$\begin{aligned}&\varphi(Q_n(e, exe, f, \dots, f), Q_n(e, eye, f, \dots, f))\\ &= Q_n(\varphi(Q_n(e, exe, f, \dots, f), e), eye, f, \dots, f)+\\ &\quad Q_n(e, \varphi(Q_n(e, exe, f, \dots, f), eye), f, \dots, f)+\\ &\quad \sum_{i=3}^{n} Q_n(e, eye, f, \dots, f, \varphi(Q_n(e, exe, f, \dots, f), f), f, \dots, f)\\ &= eye\varphi(Q_n(e, exe, f, \dots, f), e)f + e\varphi(Q_n(e, exe, f, \dots, f), eye)f+\end{aligned}$$

$$\begin{aligned}&2eye\varphi(Q_n(e,exe,f,\dots,f),f)f\\&=eyexe\varphi(e,e)f-eye\varphi(exe,e)f-exe\varphi(e,eye)f+e\varphi(exe,eye)f.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\varphi(Q_n(e,exe,f,\dots,f),Q_n(e,eye,f,\dots,f))\\&=Q_n(\varphi(e,Q_n(e,eye,f,\dots,f)),exe,f,\dots,f)+\\&\quad Q_n(e,\varphi(exe,Q_n(e,eye,f,\dots,f)),f,\dots,f)+\\&\quad Q_n(e,exe,\varphi(exe,Q_n(e,eye,f,\dots,f)),f,\dots,f)\\&=exe\varphi(e,Q_n(e,eye,f,\dots,f))f+e\varphi(exe,Q_n(e,eye,f,\dots,f))f+\\&\quad 2exe\varphi(f,Q_n(e,eye,f,\dots,f))f\\&=exeye\varphi(e,e)f-a\varphi(e,eye)f-eye\varphi(exe,e)f+e\varphi(exe,eye)f.\end{aligned}$$

Considering above two equations, we get that

$$[exe,eye]e\varphi(e,e)f=0.$$

Through similar calculation process, it can be obtained that

$$e\varphi(e,e)f[fxf,fyf]=0$$

for all $x,y\in\mathcal{T}$. Therefore, according to Lemma 3.2, we obtain that the biadditive mapping $\kappa(x,y)=[x,[y,\varphi(e,e)]]$ is an extremal biderivation of $\mathcal{T}$ and also is a biderivation. And then $\kappa(x,y)=[x,[y,\phi(e,e)]]$ is a Jordan biderivation. We know that every Jordan derivation is a Jordan n-derivation of triangular rings, so we know that $\kappa(x,y)=[x,[y,\varphi(e,e)]]$ is a bi-Jordan n-derivation of triangular rings. Set $\varphi-\kappa=\psi$. it is easy to check that ψ is a bi-Jordan n-derivation. The proof of the lemma is now completed. □

Before proceeding further study, let us remake a note on the rationality of this method.

Remark 3.5 Owing to Lemma 3.3, we may always subtract an extremal biderivation $\kappa(x,y)=[x,[y,\varphi(e,e)]]$ from bi-Jordan n-derivation φ on $\mathcal{T}$ in Theorem 3.1. Therefore, we consider only those bi-Jordan n-derivations ψ which satisfies $e\psi(e,e)f=0$. Further in view of Lemma 3.2, we see that

$$e\psi(e,e)f=-e\psi(f,e)f=-e\psi(e,f)f=e\psi(f,f)f=0.$$

Lemma 3.6 *Let $\psi:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ be a bi-Jordan n-derivation on $\mathcal{T}$. Then ψ satisfies*

(1) $\psi(exf,y)\in e\mathcal{T}f$;

(2) $\psi(x,eyf)\in e\mathcal{T}f$

for all $x,y\in\mathcal{T}$.

Proof For any $x,y\in\mathcal{T}$, we have

$$\begin{aligned}\psi(exf,y)&=\psi(Q_n(exf,f,\dots,f),y)\\&=Q_n(\psi(exf,y),f,\dots,f)+\sum_{i=2}^{n}Q_n(exf,f,\dots,f,\psi(f,y),f,\dots,f)\end{aligned}$$

$$= 2^{n-1} f\psi(exf, y)f + e\psi(exf, y)f + (n-1)(exf\psi(f, y) + \psi(f, y)exf). \tag{3.2}$$

According to above relation (3.2), multiplying e and f on both sides of the above identity respectively, we find that

$$e\psi(exf, y)e = 0 = f\psi(exf, y)f.$$

Hence we have $\psi(exf, y) \in e\mathcal{T}f$. Using similar methods, we can show that $\psi(x, eyf) \in e\mathcal{T}f$. □

Lemma 3.7 *Let $\psi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Jordan n-derivation on $\mathcal{T}$. Then ψ satisfies*

(1) $\psi(exe, eyf) = -\psi(eyf, exe) = \lambda exeyf$;

(2) $\psi(exf, fyf) = -\psi(fyf, exf) = \lambda exfyf$

for all $x, y \in \mathcal{T}$.

Proof For any $x, y \in \mathcal{T}$, thanks to the equation $eyf = Q_n(eyf, f, \ldots, f)$, we have

$$\begin{aligned}\psi(exe, eyf) &= \psi(exe, Q_n(eyf, f, \ldots, f)) \\ &= Q_n(\psi(exe, eyf), f, \ldots, f) + \sum_{i=2}^{n} Q_n(eyf, f, \ldots, f, \psi(exe, f), f, \ldots, f) \\ &= e\psi(exe, eyf)f + (n-1)(eyf\psi(exe, f) + \psi(exe, f)eyf).\end{aligned} \tag{3.3}$$

Multiplying by e from left and f from right, and since $\mathcal{T}$ is $(n-1)$-torsion free, we obtain

$$eyf\psi(exe, f)e + \psi(exe, f)eyf = 0, \tag{3.4}$$

thus we arrive that

$$\psi(exe, eyf) = e\psi(exe, eyf)f \in e\mathcal{T}f.$$

On the other hand, it follows from (3.4) that

$$\begin{aligned}\psi(exe, eyf) =& \psi(exe, Q_n(e, eyf, f, \ldots, f)) \\ =& Q_n(\psi(exe, e), eyf, f, \ldots, f) + Q_n(e, \psi(exe, eyf), f, \ldots, f) + \\ & \sum_{i=3}^{n} Q_n(e, eyf, f, \ldots, f, \psi(exe, f), f, \ldots, f) \\ =& \psi(exe, e)eyf + eyf\psi(exe, e) + e\psi(exe, eyf)f.\end{aligned}$$

It is easy to conclude $\psi(exe, e)eyf + eyf\psi(exe, e) = 0$ and $\psi(exe, eyf) = e\psi(exe, eyf)f$. Through similar calculations, we can see

$$\psi(eyf, exe) = e\psi(eyf, exe)f \in e\mathcal{T}f$$

and

$$\psi(e, exe)eyf + eyf\psi(e, exe) = 0.$$

Defining a map $\mathfrak{f} : e\mathcal{T} \to \mathcal{T}$ by $\mathfrak{f}(x) = \psi(e, exf)$ for all $x \in e\mathcal{T}$, then we get that

$$\begin{aligned}\mathfrak{f}(rx) =& \psi(e, Q_n(ere, exf, f, \ldots, f)) \\ =& Q_n(\psi(e, ere), exf, f, \ldots, f) + Q_n(ere, \psi(e, exf), f, \ldots, f) +\end{aligned}$$

$$\sum_{i=3}^{n} Q_n(ere, exf, f, \ldots, f, \psi(e,f), f, \ldots, f)$$
$$= ere\psi(e, exf) = r\mathfrak{f}(x)$$

for all $x \in e\mathcal{T}$, $r \in \mathcal{T}$. This implies that $\mathfrak{f}$ is a left $\mathcal{T}$-module homomorphism. On account of conclusion (3) in Proposition 2.1, there exists $q \in \mathcal{Q}_{ml}(\mathcal{T})$ such that $\mathfrak{f}(x) = xq$ for all $x \in e\mathcal{T}$. In particular, $\mathfrak{f}(e) = eq = 0$. This implies that $q = fq$. Thus, $\mathfrak{f}(x) = xfqf$ for all $x \in e\mathcal{T}$. For any $r \in f\mathcal{T}f$, we have

$$\begin{aligned}\mathfrak{f}(xr) =& \psi(e, Q_n(exf, frf, f, \ldots, f)) \\ =& Q_n(\psi(e, exf), frf, f, \ldots, f) + Q_n(exf, \psi(e, frf), f, \ldots, f) + \\ & \sum_{i=3}^{n} Q_n(exf, frf, f, \ldots, f, \psi(e,f), f, \ldots, f) \\ =& \psi(e, exf)frf = \mathfrak{f}(x)r,\end{aligned}$$

which leads to $xfrfqf = xfqfrf$ for all $x \in e\mathcal{T}$. Then we get $e\mathcal{T}(frfqf - fqfrf) = 0$ for all $r \in \mathcal{T}$. Using Proposition 2.1 (3), we see that $frfqf = fqfrf$ for all $r \in f\mathcal{T}f$. Then $fqf \in \mathcal{C}(f\mathcal{Q}_{ml}(\mathcal{T})f, f\mathcal{T}f)$ and hence $fqf \in \mathcal{C}(\mathcal{T})f$. Setting $\lambda = \tau^{-1}(fqf)$ and owing to the conclusion (3) in Proposition 2.1, we have $\lambda exf = xfqf$ for all $x \in e\mathcal{T}$. Thus $\psi(e, exf) = \lambda exf$ for all $x \in e\mathcal{T}$. Therefore,

$$\begin{aligned}0 &= \psi(Q_n(e, exe, f, \ldots, f), eyf) \\ &= Q_n(\psi(e, eyf), exe, f, \ldots, f) + Q_n(e, \psi(exe, eyf), f, \ldots, f) + Q_n(e, exe, \psi(f, eyf), f, \ldots, f) \\ &= -exe\psi(e, eyf)f + e\psi(exe, eyf)f\end{aligned}$$

for all $x, y \in \mathcal{T}$. It follows from above equation that $\psi(exe, eyf) = exe\psi(e, eyf)f = \lambda exeyf$. Similarly, there exists $\mu \in \mathcal{C}(\mathcal{T})e$, such that $\psi(exf, e) = \mu exf$ for all $x \in \mathcal{T}$.

Next we will prove that $\psi(e, exf) = \lambda exf = -\psi(exf, e)$ for all $x \in \mathcal{T}$. It is sufficient to prove that $\lambda + \mu = 0$. For any $x, y, z \in \mathcal{T}$, we have

$$\begin{aligned}&\psi(Q_n(eze, e, f, \ldots, f), Q_n(eye, exf, f, \ldots, f)) \\ &= Q_n(\psi(Q_n(eze, e, f, \ldots, f), eye), exf, f, \ldots, f) + \\ &\quad Q_n(eye, \psi(Q_n(eze, e, f, \ldots, f), exf), f, \ldots, f) + \\ &\quad \sum_{i=3}^{n} Q_n(eye, exf, f, \ldots, f, \psi(Q_n(eze, e, f, \ldots, f), f), f, \ldots, f) \\ &= -eyeze\psi(e, exf)f + eye\psi(eze, exf)f.\end{aligned} \tag{3.5}$$

On the other hand,

$$\begin{aligned}&\psi(Q_n(eze, e, f, \ldots, f), Q_n(eye, exf, f, \ldots, f)) \\ &= Q_n(\psi(eze, Q_n(eye, exf, f, \ldots, f)), e, f, \ldots, f) + \\ &\quad Q_n(eze, \psi(e, Q_n(eye, exf, f, \ldots, f)), f, \ldots, f) + \\ &\quad Q_n(eze, e, \psi(f, Q_n(eye, exf, f, \ldots, f)), f, \ldots, f)\end{aligned}$$

$$= exf\psi(eze, eye) + \psi(eze, eye)exf + eye\psi(eze, exf)f - ezeye\psi(e, exf)f. \quad (3.6)$$

Considering (3.5) and (3.6) together, we get that

$$-[eye, eze]\psi(e, exf) = exf\psi(eze, eye) + \psi(eze, eye)exf \quad (3.7)$$

for all $x, y, z \in \mathcal{T}$. Similarly, we obtain

$$\begin{aligned}&\psi(Q_n(eze, exf, f, \dots, f), Q_n(eye, e, f, \dots, f))\\ &\quad = exf\psi(eze, eye) + \psi(eze, eye)exf + eze\psi(exf, eye)f - eyeze\psi(exf, e)f\\ &\quad = -ezeye\psi(exf, e)f + eze\psi(exf, eye)f.\end{aligned}$$

And then we get that

$$[eye, eze]\psi(exf, e) = exf\psi(eze, eye) + \psi(eze, eye)exf. \quad (3.8)$$

With the help of the preceding two equations (3.7) and (3.8), we get

$$[eye, eze](\psi(e, exf) + \psi(exf, e)) = 0, \text{ i.e., } (\lambda + \mu)[eye, eze]exf = 0$$

for all $x, y, z \in \mathcal{T}$. With the aid of Proposition 2.2, we obtain that

$$0 = (\lambda + \mu)[eye, eze] = [(\lambda + \mu)eye, eze].$$

It follows from above equation that

$$[(\lambda + \mu)e\mathcal{RC}(\mathcal{R})e, e\mathcal{RC}(\mathcal{R})e] = 0.$$

Thus $(\lambda + \mu)e\mathcal{RC}(\mathcal{R})e$ is a central ideal of $e\mathcal{RC}(\mathcal{R})e$. And then $\lambda + \mu = 0$ via condition (2) in Theorem 3.1 as desired.

In the same way we can prove the other equation was established. □

Lemma 3.8 *Let $\psi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Jordan n-derivation on $\mathcal{T}$. Then ψ satisfies*

(1) $\psi(exe, fyf) = 0$;

(2) $\psi(fxf, eye) = 0$

for all $x, y \in \mathcal{T}$.

Proof Let us first prove the conclusion:

$$\psi(exe, fyf) = \psi(fxf, eye) = exe\psi(e, f)fyf$$

for all $x, y \in \mathcal{T}$.

Because mapping ψ is a Jordan n-derivation for both components, we obtain

$$\begin{aligned}2^{n-1}\psi(exe, fyf) =&\psi(Q_n(exe, e, \dots e), fyf)\\ =&Q_n(\psi(exe, fyf), e, \dots e) + \sum_{i=2}^{n} Q_n(exe, e, \dots, e, \psi(e, fyf), e, \dots e)\\ =&2^{n-1}e\psi(exe, fyf)e + e\psi(exe, fyf)f +\\ &2^{n-2}(n-1)[exe\psi(e, fyf)e + e\psi(e, fyf)exe] +\\ &(2^{n-1} - 1)exe\psi(e, fyf)f\end{aligned} \quad (3.9)$$

for all $x, y \in \mathcal{T}$. Multiplying the above identity (3.9) by f from left and right, since $\mathcal{T}$ is 2-torsion free we conclude that

$$f\psi(exe, fyf)f = 0 \tag{3.10}$$

for all $x, y \in \mathcal{T}$.

In a parallel way, we can prove that

$$\begin{aligned} 2^{n-1}\psi(exe, fxf) =& \psi(exe, Q_n(fxf, f, \dots f)) \\ =& Q_n(\psi(exe, fxf), f, \dots, f) + \\ & \sum_{i=2}^{n} Q_n(fxf, f, \dots, f, \underbrace{\psi(exe, f)}_{i\text{-th component}}, f, \dots f) \\ =& e\psi(exe, fxf)f + (2^{n-1} - 1)\psi(exe, f)fxf. \end{aligned} \tag{3.11}$$

In accordance with above relation (3.11), multiplying the above identity by e from left and right, it follows from 2-torsion free commutative ring $\mathcal{T}$ that

$$e\psi(exe, fyf)e = 0. \tag{3.12}$$

It follows from $(2^{n-1} - 1)$-torsion free of $\mathcal{T}$ and Eqs. (3.9)–(3.12) that

$$\psi(exe, fyf) = e\psi(exe, fyf)f = exe\psi(e, fyf)f = e\psi(exe, f)fyf$$

for all $x, y \in \mathcal{T}$. Taking $y = f$ into the relation $e\psi(exe, fyf)f = exe\psi(e, fyf)f$, we pick up $e\psi(exe, f)f = exe\psi(e, f)f$. By invoking the fact $e\psi(exe, fyf)f = e\psi(exe, f)fyf$, we get hold of $e\psi(exe, fyf)f = exe\psi(e, f)fyf$ for all $x, y \in \mathcal{T}$. Therefore, we have

$$\psi(exe, fyf) = exe\psi(e, f)fyf.$$

Using a similar approach, we can acquire that

$$\psi(fyf, exe) = exe\psi(f, e)fyf$$

for all $x, y \in \mathcal{T}$. □

Lemma 3.9 *Let $\psi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Jordan n-derivation on $\mathcal{T}$. Then ψ satisfies*

(1) $\psi(exe, eye) = \lambda[exe, eye]$;

(2) $\psi(fxf, fyf) = \eta(\lambda)[fxf, fyf]$

for all $x, y \in \mathcal{T}$.

Proof For any $x, y \in \mathcal{T}$, by Lemma 3.7 we find that

$$\begin{aligned} 0 =& \psi(exe, Q_n(eye, f, \dots, f)) \\ =& Q_n(\psi(exe, eye), f, \dots, f) + Q_n(eye, \psi(exe, f), f, \dots, f) \\ =& 2^{n-1}f\psi(exe, eye)f + e\psi(exe, eye)f, \end{aligned}$$

hence $e\psi(exe, eye)f = 0$, $f\psi(exe, eye)f = 0$, and then $\psi(exe, eye) = e\psi(exe, eye)e$. At the same time, using Lemma 3.7, for all $x, y, z \in \mathcal{T}$, we see that

$$\psi(Q_n(exe, e, f, \dots, f), Q_n(eye, ezf, f, \dots, f))$$

$$
\begin{aligned}
&= Q_n(\psi(Q_n(exe, e, f, \ldots, f), eye), ezf, f, \ldots, f)+\\
&\quad Q_n(eye, \psi(Q_n(exe, e, f, \ldots, f), ezf), f, \ldots, f)+\\
&\quad \sum_{i=3}^{n} Q_n(eye, ezf, f, \ldots, f, \psi(Q_n(exe, e, f, \ldots, f), f), f, \ldots, f)\\
&= eye\psi(exe, ezf)f - eyexe\psi(e, ezf)f. \qquad (3.13)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\psi(Q_n(exe, e, f, \ldots, f), Q_n(eye, ezf, f, \ldots, f))\\
&\quad = Q_n(\psi(exe, Q_n(eye, ezf, f, \ldots, f)), e, f, \ldots, f)+\\
&\qquad Q_n(exe, \psi(e, Q_n(eye, ezf, f, \ldots, f)), f, \ldots, f)+\\
&\qquad Q_n(exe, e, \psi(f, Q_n(eye, ezf, f, \ldots, f)), f, \ldots, f)\\
&\quad = \psi(exe, eye)ezf + eye\psi(exe, ezf)f - exeye\psi(e, ezf)f. \qquad (3.14)
\end{aligned}
$$

Taking advantage of (3.13) and (3.14), and from Lemma 3.6 we arrive at

$$e\psi(exe, eye)[e, ezf] = [exe, eye]\psi(e, ezf) = \lambda[exe, eye]ezf$$

for all $x, y, z \in \mathcal{T}$. Thus, we arrive that for all $x, y \in \mathcal{T}$,

$$(\psi(exe, eye) - \lambda[exe, eye])e\mathcal{T}f = 0.$$

And then $\psi(exe, eye) = \lambda[exe, eye] \in e\mathcal{T}e$ for all $x, y \in \mathcal{T}$. Similarly, we can prove that $\psi(fxf, fyf) \in f\mathcal{T}f$ and

$$[e, ezf]\psi(fxf, fyf) = \psi(e, ezf)[fxf, fyf] = \lambda ezf[fxf, fyf]$$

for all $x, y \in \mathcal{T}$. Hence, $e\mathcal{T}f(\psi(fxf, fyf) - \eta(\lambda)[fxf, fyf]) = 0$. And then

$$\psi(fxf, fyf) = \eta(\lambda)[fxf, fyf]$$

for all $x, y \in \mathcal{T}$. □

Lemma 3.10 *Let $\psi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Jordan n-derivation on $\mathcal{T}$. Then*

$$\psi(exf, eyf) = 0$$

for all $x, y \in \mathcal{T}$.

Proof Let us fix element $y \in \mathcal{T}$. We define a map $\mathfrak{g} : e\mathcal{T} \to \mathcal{T}$ by $\mathfrak{g}_y(x) = \psi(exf, eyf)$ for all $x \in e\mathcal{T}$. Then by Lemma 3.6, we get

$$
\begin{aligned}
\mathfrak{g}_y(rx) =& \psi(Q_n(ere, exf, f, \ldots, f), eyf)\\
=& Q_n(\psi(ere, eyf), exf, f, \ldots, f) + Q_n(ere, \psi(exf, eyf), f, \ldots, f)+\\
& \sum_{i=3}^{n} Q_n(ere, exf, f, \ldots, f, \psi(f, eyf), f, \ldots, f)\\
=& ere\psi(exf, eyf)f = r\mathfrak{g}_y(x)
\end{aligned}
$$

for all $x \in e\mathcal{T}$, $r \in \mathcal{T}$ and hence $\mathfrak{g}_y$ is a left $\mathcal{T}$-module homomorphism. By Proposition 2.1 (2) and (3), there exists $q_y \in \mathcal{Q}_{ml}(\mathcal{T})$ such that $\mathfrak{g}_y(x) = xq_y$ for all $x \in e\mathcal{T}$. Clearly, $eq_y = \mathfrak{g}_y(e) = 0$. So $q_y = fq_yf$ implies that $\mathfrak{g}_y(x) = xfq_yf$ for all $x \in e\mathcal{T}$. And then we also have

$$\begin{aligned}\mathfrak{g}_y(xr) =&\psi(Q_n(exf, frf, f, \dots, f), eyf)\\ =&Q_n(\psi(exf, eyf), frf, f, \dots, f) + Q_n(exf, \psi(frf, eyf), f, \dots, f)+\\ &\sum_{i=3}^{n} Q_n(exf, frf, f, \dots, f, \psi(f, eyf), f, \dots, f)\\ =&e\psi(exf, eyf)frf = \mathfrak{g}_y(x)r,\end{aligned}$$

and hence $xfrfq_yf = xfq_yfrf$ for all $x \in e\mathcal{T}$, $r \in \mathcal{T}$. Then $e\mathcal{T}(frfq_yf - fq_yfrf) = 0$ for all $r \in \mathcal{T}$. In view of Proposition 2.1 (3), we get $frfq_yf = fq_yfrf$ for all $r \in \mathcal{T}$. Consequently, by the assumption of Theorem 3.1, we have $fq_yf \in \mathcal{C}(\mathcal{T})f$. Now, for any $x, y, x', y' \in \mathcal{T}$, by Lemma 3.6, we have

$$\begin{aligned}&\psi(Q_n(ex'f, exe, f, \dots, f), Q_n(ey'f, eye, f, \dots, f))\\ &= Q_n(\psi(Q_n(ex'f, exe, f, \dots, f), ey'f), eye, f, \dots, f)+\\ &\quad Q_n(ey'f, \psi(Q_n(ex'f, exe, f, \dots, f), eye), f, \dots, f)+\\ &\quad \sum_{i=3}^{n} Q_n(ey'f, eye, f, \dots, f, \psi(Q_n(ex'f, exe, f, \dots, f), f), f, \dots, f)\\ &= eyexe\psi(ex'f, ey'f). \end{aligned} \tag{3.15}$$

On the other hand,

$$\begin{aligned}&\psi(Q_n(ex'f, exe, f, \dots, f), Q_n(ey'f, eye, f, \dots, f))\\ &= Q_n(\psi(ex'f, Q_n(ey'f, eye, f, \dots, f)), exe, f, \dots, f)+\\ &\quad Q_n(ex'f, \psi(exe, Q_n(ey'f, eye, f, \dots, f)), f, \dots, f)+\\ &\quad \sum_{i=3}^{n} Q_n(ex'f, exe, f, \dots, f, \psi(f, Q_n(ey'f, eye, f, \dots, f), f, \dots, f))\\ &= exeye\psi(ex'f, ey'f)f. \end{aligned} \tag{3.16}$$

Taking advantage of (3.15) and (3.16) together, we get that

$$0 = [\psi(ex'f, ey'f), [exe, eye]] = [exe, eye]ex'fq_{y'}f = \eta^{-1}(fq_{y'}f)[exe, eye]ex'f$$

for all $x, y, x', y' \in \mathcal{T}$. By Proposition 2.1 (3), we have $\eta^{-1}(fq_{y'}f)[e\mathcal{T}e, e\mathcal{T}e] = 0$, this implies that

$$[\eta^{-1}(fq_{y'}f)e\mathcal{T}\mathcal{C}(\mathcal{T})e, e\mathcal{T}\mathcal{C}(\mathcal{T})e] = 0.$$

It follows from above equation that $\eta^{-1}(fq_{y'}f)e\mathcal{T}\mathcal{C}(\mathcal{T})e$ is a central ideal of $e\mathcal{T}\mathcal{C}(\mathcal{T})e$. Assume without loss of generality that $e\mathcal{T}\mathcal{C}(\mathcal{T})e$ does not contain nonzero central ideals. Then $\eta^{-1}(fq_{y'}f) = 0$, which leads to $fq_{y'}f = 0$ for all $y' \in \mathcal{T}$. So we conclude that

$$\psi(exf, eyf) = exfq_yf = 0$$

for all $x, y \in \mathcal{T}$. This Lemma is proved. □

Proof of Theorem 3.1 As maintained by Lemmas 3.4, 3.6–3.10 and Eq. (3.1), we assert that Theorem 3.1 holds. □

When $n = 2$, we have the following corollary.

Corollary 3.11 ([16]) *For a triangular ring $\mathcal{T} = e\mathcal{T}e + e\mathcal{T}f + f\mathcal{T}f$, if the following conditions hold:*

(1) $\mathcal{C}(f\mathcal{Q}_{ml}(\mathcal{R})f) = \mathcal{C}(\mathcal{R})f$;

(2) either $e\mathcal{R}\mathcal{C}(\mathcal{R})e$ or $f\mathcal{R}\mathcal{C}(\mathcal{R})f$ does not contain nonzero central ideals,

then every Jordan biderivation on $\mathcal{T}$ is a sum of an extreme bi-derivation and an inner bi-derivation.

4. Bi-Jordan n-derivations:faithful bimodule

In this part, we mainly use the faithful bimodules to study the structure of bi-Jordan n-derivations on triangular rings. In the third part we study the structure of the bi-Jordan n-derivations from the point of view of the quotient ring, and in the process of proving it, we introduce Lemma 3.4. The key of this lemma is to construct a extreme bi-derivation, which simplifies the whole proof process. However, the construction of extreme bi-derivations is very clever and has a large amount of computation. Therefore, in this part, we take a more direct and concise method to study the bi-Jordan n-derivations of triangular rings from the point of view of bimodule. This is one of the highlights of this section.

Let $e\mathcal{T}f$ be a unital $(e\mathcal{T}e, f\mathcal{T}f)$-bimodule. An additive mapping $\mho : e\mathcal{T}f \to e\mathcal{T}f$ satisfying the condition $\mho(exexf) = exe\mho(exf)$ and $\mho(exfxf) = \mho(exf)fxf$ for all $x \in \mathcal{T}$ is called a bimodule homomorphism. A bimodule homomorphism $\mho : e\mathcal{T}f \to e\mathcal{T}f$ is of the standard form if there exist $a_0 \in \mathcal{Z}(e\mathcal{T}e), b_0 \in \mathcal{Z}(f\mathcal{T}f)$ such that $\mho(exf) = a_0exf + exfb_0$ for all $x \in \mathcal{T}$.

Theorem 4.1 *Let $\mathcal{T} = e\mathcal{T}e+e\mathcal{T}f+f\mathcal{T}f$ be a t-torsion free triangular ring with faithful bimodule $e\mathcal{T}f$, where $t \in \{2, n-1, 2^{n-1}-1\}$, and let $\varphi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Jordan n-derivation. If the following conditions hold:*

(1) $\pi_{e\mathcal{T}e}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(e\mathcal{T}e)$ and $\pi_{f\mathcal{T}f}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(f\mathcal{T}f)$;

(2) at least one of the algebras $e\mathcal{T}e$ and $f\mathcal{T}f$ is noncommutative;

(3) each bimodule homomorphism $f : e\mathcal{T}f \to e\mathcal{T}f$ is of the standard form;

(4) for any $\alpha \in \mathcal{Z}(\mathcal{T})$ and $0 \neq a \in \mathcal{T}$, if $\alpha a = 0$, then $\alpha = 0$,

then each bi-Jordan n-derivation is of a form

$$\varphi(x, y) = \lambda_0[x, y] + [x, [y, \varphi(e, e)]]$$

for all $x, y \in \mathcal{T}$ and some $\lambda_0 \in \mathcal{Z}(\mathcal{T})$.

Remark 4.2 After a more concise test, Lemma 3.2 is still valid for the study of bi-Jordan n-derivation from the perspective of bimodule. So I omit it here.

Proof For arbitrary elements $x = exe+exf+fxf \in \mathcal{T}$ and $y = eye+eyf+fyf \in \mathcal{T}$. According to the bilinear property of mapping φ, we have

$$\begin{aligned}\varphi(x, y) =&\varphi(exe + exf + fxf, eye + eyf + fyf)\\ =&\varphi(exe, eye) + \varphi(exe, eyf) + \varphi(exe, fyf)+\\ &\varphi(exf, eye) + \varphi(exf, eyf) + \varphi(exf, fyf)+\\ &\varphi(fxf, eye) + \varphi(fxf, eyf) + \varphi(fxf, fyf)\end{aligned} \tag{4.1}$$

for all $x, y \in \mathcal{T}$. □

Let us identify the 6 parts of Eq. (4.1), which are divided into several steps.

By fine-tuning Lemmas 3.8 and 3.9, we find that Steps 1 and 2 hold.

Step 1. Let us first prove the conclusion:

$$\varphi(exe, fyf) = \varphi(fxf, eye) = exe\varphi(e, f)fyf.$$

Step 2. There exist two bi-Jordan n-derivations $\delta : e\mathcal{T}e \times e\mathcal{T}e \to e\mathcal{T}e$, $\gamma : f\mathcal{T}f \times f\mathcal{T}f \to f\mathcal{T}f$ such that

$$\begin{aligned}\varphi(exe, eye) =&\delta(exe, eye) + exeye\varphi(e, e)f = \delta(exe, eye) + eyexe\varphi(e, e)f,\\ \varphi(fxf, fyf) =&\gamma(fxf, fyf) + e\varphi(f, f)fxfyf = \gamma(fxf, fyf) + e\varphi(f, f)fyfxf\end{aligned}$$

for all $x, y \in \mathcal{T}$.

Step 3. And in front of the symbols, we have

$$\varphi(exe, exf) = \alpha exexf = -\varphi(exf, exe) \text{ and} \varphi(exf, fxf) = exfxf\tau^{-1}(\alpha) = -\varphi(fxf, exf)$$

for all $x \in \mathcal{T}$. τ is an algebra isomorphism $\tau : e\mathcal{Z}(\mathcal{T})e \to f\mathcal{Z}(\mathcal{T})f$ such that $exexf = exf\tau(exe)$ for all $x \in \mathcal{T}$.

Actually, by Step 1 we know that

$$\begin{aligned}\varphi(exe, exf) =&\varphi(exe, Q_n(exf, f, \ldots, f))\\ =&Q_n(\varphi(exe, exf), f, \ldots, f) + \sum_{i=2}^{n} Q_n(exf, f, \ldots, f, \underbrace{\varphi(exe, f)}_{i\text{-th component}}, f, \ldots, f)\\ =&2^{n-1} f\varphi(exe, exf)f + e\varphi(exe, exf)f\end{aligned}$$

for all $x \in \mathcal{T}$, which is due to the fact $\varphi(exe, f) = e\varphi(exe, f)f \in e\mathcal{T}f$. It should be remarked that $e\varphi(exe, exf)e = 0$. Multiplying the above identity by f from left and right, since $\mathcal{T}$ is $(2^{n-1} - 1)$-torsion free, we obtain $f\varphi(exe, exf)f = 0$. Thus we arrive that

$$\varphi(exe, exf) = e\varphi(exe, exf)f \in e\mathcal{T}f. \tag{4.2}$$

On the other hand, it follows from (4.2) that

$$\begin{aligned}2^{n-1}\varphi(exe, exf) =&\varphi(Q_n(exe, e, \ldots, e), exf)\\ =&Q_n(\varphi(exe, exf), e, \ldots, e) + \sum_{i=2}^{n} Q_n(exe, e, \ldots, e, \underbrace{\varphi(e, exf)}_{i\text{-th component}}, e, \ldots, e)\end{aligned}$$

$$=e\varphi(exe,exf)f+(2^{n-1}-1)exe\varphi(e,exf)f$$

for all $x,y\in\mathcal{T}$. Since $\mathcal{T}$ is $(2^{n-1}-1)$-torsion free, we immediately see that $\varphi(exe,exf)=exe\varphi(e,exf)f$.

Similarly, we get $\varphi(exf,exe)=exe\varphi(exf,e)f$ and

$$e\varphi(e,exe)e=0 \tag{4.3}$$

for all $x,y\in\mathcal{T}$.

Let us see a mapping $\mathfrak{h}:e\mathcal{T}f\to e\mathcal{T}f$ which is defined to be $\mathfrak{h}(exf)=e\varphi(e,exf)f$ for all $x\in\mathcal{T}$. Then $\mathfrak{h}$ is an $(e\mathcal{T}e,f\mathcal{T}f)$-bimodule homomorphism. Namely, for all $x\in\mathcal{T}$, according to (4.3), we have

$$\begin{aligned}\mathfrak{h}(exexf)=&e\varphi(e,exexf)f=e\varphi(e,Q_n(exe,exf,e,\ldots,e))f\\=&e[Q_n(\varphi(e,exe),exf,e,\ldots,e)+Q_n(exe,\varphi(e,exf),e,\ldots,e)+\\&\sum_{i=3}^{n}Q_n(exe,exf,e,\ldots,e,\underbrace{\varphi(e,e)}_{i\text{-th component}},e,\ldots,e)]f\\=&exe\varphi(e,exf)f=exe\mathfrak{h}(exf)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{h}(exfxf)=&e\varphi(e,exfxf)f=e\varphi(e,Q_n(exf,fxf,e,\ldots,e))f\\=&e[Q_n(\varphi(e,exf),fxf,e,\ldots,e)+Q_n(m,\varphi(e,fxf),e,\ldots,e)+\\&\sum_{i=3}^{n}Q_n(exf,fxf,e,\ldots,e,\underbrace{\varphi(e,e)}_{i\text{-th component}},e,\ldots,e)]f\\=&e\varphi(e,exf)b=\mathfrak{h}(exf)fxf.\end{aligned}$$

Applying condition (3) yields that the bimodule homomorphism $\mathfrak{h}$ is of the standard form

$$\mathfrak{h}(exf)=a_0exf+b_0exf$$

for some $a_0\in\mathcal{Z}(e\mathcal{T}e)$, $b_0\in\mathcal{Z}(f\mathcal{T}f)$.

In light of condition (1), we know that $a_0\in\pi_{e\mathcal{T}e}(\mathcal{Z}(e\mathcal{T}e))$ and $b_0\in\pi_{f\mathcal{T}f}(\mathcal{Z}(f\mathcal{T}f))$. We may write

$$e\varphi(e,exf)f=\mathfrak{h}(exf)=a_0exf+exfb_0=(a_0+\tau^{-1}(b_0))exf=\alpha_0exf,$$

where $\alpha_0=a_0+\tau^{-1}(b_0)\in\mathcal{Z}(e\mathcal{T}e)=\pi_{e\mathcal{T}e}(\mathcal{Z}(e\mathcal{T}e))$ for all $x\in\mathcal{T}$.

Likewise, one can define a mapping $\mathfrak{g}:e\mathcal{T}f\to e\mathcal{T}f$ by $\mathfrak{g}(exf)=e\varphi(exf,e)f$ for all $x\in\mathcal{T}$, which is an $(e\mathcal{T}e,f\mathcal{T}f)$-bimodule homomorphism. Therefore, there exists $\beta\in\pi_A(\mathcal{Z}(e\mathcal{T}e))$ such that $\mathfrak{g}(exf)=e\varphi(exf,e)f=\beta_0exf$ for all $x\in\mathcal{T}$. Let us next show that

$$\mathfrak{h}(exf)=\alpha_0exf=-\mathfrak{g}(exf),\ \text{ i.e., } e\varphi(e,exf)f=\alpha_0exf=-e\varphi(exf,e)f$$

for all $x\in\mathcal{T}$. It is sufficient for us to prove that $\alpha_0+\beta_0=0$.

According to condition (2), we may assume that $e\mathcal{T}e$ is noncommutative as a $\mathcal{T}$-algebra. Choose $exe, eye \in e\mathcal{T}e$ such that $[exe, eye] \neq 0$. Since $\mathfrak{h}(exf) = \alpha_0 exf$ and $\mathfrak{g}(exf) = \beta_0 exf$, we get

$$\begin{aligned}&\varphi(Q_n(exe, exf, e, \dots, e), Q_n(eye, e, e, \dots, e))\\ &\quad= 2^{n-1}e\varphi(exe, eye)exf + exe\varphi(exf, eye)f + (2^{n-1} - 1)exeye\varphi(exf, e)f. \end{aligned}\tag{4.4}$$

On the other hand,

$$\begin{aligned}&\varphi(Q_n(exe, exf, e, \dots, e), Q_n(eye, e, e, \dots, e))\\ &\quad= e\varphi(exe, eye)exf + exe\varphi(exf, eye)f + (2^{n-1} - 1)eyexe\varphi(exf, e)f\end{aligned}\tag{4.5}$$

for all $x, y \in \mathcal{T}$. Considering (4.4) and (4.5) together, since $\mathcal{T}$ is $(2^{n-1} - 1)$-torsion free, we see that

$$e\varphi(exe, eye)exf = [exe, eye]\varphi(exf, e)f = [exe, eye]\alpha_0 exf \tag{4.6}$$

for all $x, y \in \mathcal{T}$. Adopting similar methods, we obtain

$$\begin{aligned}&\varphi(Q_n(exe, e, e, \dots, e), Q_n(eye, exf, e, \dots, e))\\ &\quad= e\varphi(exe, eye)exf + eye\varphi(exe, exf)f + (2^{n-1} - 1)exeye\varphi(e, exf)f\\ &\quad= 2^{n-1}e\varphi(exe, eye)exf + eye\varphi(exe, exf)f + (2^{n-1} - 1)eyexe\varphi(e, exf)f.\end{aligned}$$

Then we have

$$e\varphi(exe, eye)exf = [exe, eye]\varphi(e, exf)f = [exe, eye]\beta_0 exf \tag{4.7}$$

for all $x, y \in \mathcal{T}$.

It follows from (4.6) and (4.7) that $(\alpha_0+\beta_0)[exe, eye]exf = 0$ for all $x, y \in \mathcal{T}$. The faithfulness of the left A-module now implies $(\alpha_0 + \beta_0)[exe, eye] = 0$. Since $[exe, eye] \neq 0$, we assert that $\alpha_0 + \beta_0 \neq 0$ by condition (3). Considering $\alpha_0 + \beta_0 \neq 0$ and $\varphi(f, exf) + \varphi(e, exf) = 0$ together with $\varphi(exf, e) + \varphi(exf, f) = 0$, we see that

$$\varphi(exf, f) = \alpha_0 exf = -\varphi(f, exf)$$

for all $x \in \mathcal{T}$.

Then we can write

$$\varphi(exe, exf) = exe\varphi(e, exf)f = \alpha_0 exexf$$

for all $x \in \mathcal{T}$. This proves the first equality. The other three equations can be proved in an analogous manner. □

Step 4. As the above notations, we have $\delta(exe, eye) = \alpha_0[exe, eye]$ and $\gamma(fxf, fyf) = \tau(\alpha_0)[fxf, fyf]$ for all $x, y \in \mathcal{T}$.

In fact, for any $x, y \in \mathcal{T}$, by Steps 1–3, we get

$$\begin{aligned}2^{n-1}\varphi(exe, eyexf) =&\varphi(Q_n(exe, e, \dots, e), eyexf)\\ =&Q_n(\varphi(exe, eyexf), e, \dots, e) + \sum_{i=2}^{n} Q_n(exe, e, \dots, e, \varphi(e, eyexf), e, \dots, e)\end{aligned}$$

$$=e\varphi(exe, eyexf)f + (2^{n-1}-1)exeye\varphi(e, exf)f.$$

Since $\mathcal{T}$ is $(2^{n-1}-1)$-torsion free, then we conclude that

$$\varphi(exe, eyexf) = exe\varphi(e, eyexf)f = exeye\varphi(e, exf)f \tag{4.8}$$

for all $x, y \in \mathcal{T}$.

On the other hand,

$$\begin{aligned}\varphi(exe, eyexf) =& \varphi(exe, Q_n(eyexf, e, \ldots, e))\\ =& Q_n(\varphi(exe, eye), exf, e, \ldots, e) + Q_n(eye, \varphi(exe, exf), e, \ldots, e) +\\ & \sum_{i=3}^{n} Q_n(eye, exf, e, \ldots, e, \underbrace{\varphi(exe, e)}_{i\text{-th component}}, e, \ldots, e)\\ =& \varphi(exe, eye)m + eyexe\varphi(e, exf)f \end{aligned} \tag{4.9}$$

for all $x, y \in \mathcal{T}$. Combining (4.8) with (4.9) gives

$$\delta(exe, eye)exf = [exe, eye]\varphi(e, exf)f$$

for all $x, y \in \mathcal{T}$. In light of Step 3, one can define an $(e\mathcal{T}e, f\mathcal{T}f)$-bimodule homomorphism

$$\begin{aligned} h : & e\mathcal{T}f \mapsto e\mathcal{T}f,\\ & exf \mapsto e\varphi(e, exf)f \end{aligned}$$

for all $x \in \mathcal{T}$. Taking into account condition (3), there exists an $\alpha_0 \in \mathcal{Z}(A)$ such that $e\varphi(e, exf)f = \alpha_0 exf$. Then we can say that

$$\delta(exe, eye)exf = [exe, eye]\alpha_0 exf$$

for all $x \in \mathcal{T}$. Since M is faithful as a left A-module and also as a right B-module, we obtain $\delta(exe, eye) = \alpha_0[exe, eye]$ for all $x, y \in \mathcal{T}$. Similarly, one can show that

$$\gamma(fxf, fyf) = \tau(\alpha_0)[fxf, fyf]$$

for all $x, y \in \mathcal{T}$. □

Step 5. As the above notations, we have $\varphi(exf, eyf) = 0$ for all $x, y \in \mathcal{T}$.

Indeed, for any $x, y \in \mathcal{T}$, we see that

$$\begin{aligned}\varphi(exf, eyf) =& \varphi(Q_n(exf, f, \ldots, f), eyf)\\ =& Q_n(\varphi(exf, eyf), f, \ldots, f) + \sum_{i=2}^{n} Q_n(exf, f, \ldots, f, \varphi(f, eyf), f, \ldots, f)\\ =& 2^{n-1} f\varphi(exf, eyf)f + e\varphi(exf, eyf)f.\end{aligned}$$

And hence $e\varphi(exf, eyf)e = 0$ and $f\varphi(exf, eyf)f = 0$. Furthermore, we have $\varphi(exf, eyf) = e\varphi(exf, eyf)f$. Let us fix $x \in \mathcal{T}$, then the mapping $\mathfrak{t} : e\mathcal{T}f \to e\mathcal{T}f$ defined by $\mathfrak{t}_n(exf) = \varphi(exf, eyf) = e\varphi(exf, eyf)f$ for all $y \in \mathcal{T}$ is a bimodule homomorphism as a left A-module and also as a right B-module. By invoking Step 3 again, we conclude that

$$\varphi(Q_n(exe, exf, e, \ldots, e), Q_n(eye, eyf, e, \ldots, e))$$

$$
\begin{aligned}
&= Q_n(\varphi(exe, Q_n(eye, eyf, e, \dots, e)), exf, e, \dots, e)+\\
&\quad Q_n(exe, \varphi(exf, Q_n(eyf, eyf, e, \dots, e)), e, \dots, e)+\\
&\quad \sum_{i=3}^{n} Q_n(exe, exf, e, \dots, e, \varphi(e, Q_n(eye, eyf, e, \dots, e)), e, \dots, e)\\
&= exeye\varphi(exf, eyf)f \qquad (4.10)
\end{aligned}
$$

for all $x, y \in \mathcal{T}$.

On the other hand,

$$
\begin{aligned}
&\varphi(Q_n(exe, exf, e, \dots, e), Q_n(eye, eyf, e, \dots, e))\\
&= Q_n(\varphi(Q_n(exe, exf, e, \dots, e), eye), eyf, e, \dots, e)+\\
&\quad Q_n(eye, \varphi(Q_n(exe, exf, e, \dots, e), eyf), e, \dots, e)+\\
&\quad \sum_{i=3}^{n} Q_n(eye, eyf, e, \dots, e, \varphi(Q_n(exe, exf, e, \dots, e), e), e, \dots, e)\\
&= exeye\varphi(exf, eyf)f \qquad (4.11)
\end{aligned}
$$

for all $x, y \in \mathcal{T}$. Considering (4.10) and (4.11) together, we get that $[exe, eye]\varphi(exf, eyf) = 0$ for all $x, y \in \mathcal{T}$.

Without loss of generality, we might assume that $e\mathcal{T}e$ is a noncommutative algebra. Let $exe, eye \in e\mathcal{T}e$ be fixed elements such that $[exe, eye] \neq 0$. We therefore say that $\varphi(exf, eyf) = 0$ for all $x, y \in \mathcal{T}$.

At the end, to get the final result, we may obtain the relation

$$
-exe\varphi(e, e)fyf - eye\varphi(e, e)fxf + exeye\varphi(e, e) + \varphi(e, e)fyfxf = [x, [y, \varphi(e, e)]], \qquad (4.12)
$$

where $x = exe + exf + fxf \in \mathcal{T}$, $y = eye + eyf + fyf \in \mathcal{T}$ for all $x, y \in \mathcal{T}$.

In fact, considering the relation $\varphi(e, e) = \varphi(e, e)e + e\varphi(e, e)$, we have $e\varphi(e, e)e = 0$ and $f\varphi(e, e)f = 0$. And hence

$$
\varphi(e, e) = e\varphi(e, e)f.
$$

According to the above relation, complicated matrix calculations show that equation (4.12) can be proved.

Finally we prove the main result. Let $\lambda_0 = \alpha_0 + \eta(\alpha_0) \in \mathcal{Z}(\mathcal{T})$. Substituting Steps 1-5 leads to

$$
\begin{aligned}
\varphi(x, y) =&\alpha_0[exe, eye] + \alpha_0 exeyf - \alpha_0 exeyf + \alpha_0 exfyf - \alpha_0 exfyf + \eta(\alpha_0)[fxf, fyf]-\\
&exe\varphi(e, e)fyf - eye\varphi(e, e)fxf + exeye\varphi(e, e) + \varphi(e, e)fxfyf\\
=&\begin{bmatrix}\alpha_0 & 0\\ 0 & \eta(\alpha_0)\end{bmatrix}\left[\begin{bmatrix}exe & exf\\ 0 & fxf\end{bmatrix}, \begin{bmatrix}eye & eyf\\ 0 & fyf\end{bmatrix}\right] + \left[\begin{bmatrix}exe & exf\\ 0 & fxf\end{bmatrix}, \left[\begin{bmatrix}eye & eyf\\ 0 & fyf\end{bmatrix}, \varphi(e, e)\right]\right]\\
=&\lambda_0[x, y] + [x, [y, \varphi(e, e)]]
\end{aligned}
$$

for all $x, y \in \mathcal{T}$. Hence every bi-Jordan n-derivation φ is a sum of inner and extremal biderivation. □

By means of papers [9, 10] and Theorem 4.1, we can immediately deduce the following:

Corollary 4.3 ([10, Theorem 2.1]) *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra over a commutative ring $\mathcal{R}$ and let $\varphi : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ be a Jordan biderivation. If the following conditions hold:*

(1) $\pi_A(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(B)$;

(2) at least one of the algebras A and B is noncommutative;

(3) each bimodule homomorphism $\mathfrak{f}\colon M \longrightarrow M$ is of the standard form;

(4) for any $\alpha \in \mathcal{Z}(\mathcal{T})$ and $0 \neq a \in \mathcal{T}$, if $\alpha a = 0$, then $\alpha = 0$,

then each Jordan biderivation $\varphi : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ is of the form

$$\varphi(x, y) = \lambda_0[x, y] + [x, [y, \varphi(e, e)]]$$

for all $x, y \in \mathcal{T}$ and some $\lambda_0 \in \mathcal{Z}(\mathcal{T})$.

Corollary 4.4 ([9, Theorem 4.11]) *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra over a commutative ring $\mathcal{R}$ and let $\varphi : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ be a biderivation. If the following conditions hold:*

(1) $\pi_A(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(B)$;

(2) at least one of the algebras A and B is noncommutative;

(3) each bimodule homomorphism $\mathfrak{f}\colon M \longrightarrow M$ is of the standard form;

(4) for any $\alpha \in \mathcal{Z}(\mathcal{T})$ and $0 \neq a \in \mathcal{T}$, if $\alpha a = 0$, then $\alpha = 0$,

then each biderivation $\varphi : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ is of the form

$$\varphi(x, y) = \lambda_0[x, y] + [x, [y, \varphi(e, e)]]$$

for all $x, y \in \mathcal{T}$ and some $\lambda_0 \in \mathcal{Z}(\mathcal{T})$.

5. Applications

Since upper triangular matrix rings and nest algebras are classical examples of triangular rings [9, 10], in this part, we mainly give the applications of Theorems 3.1 and 4.1 to these algebras, and obtain the following inferences:

Corollary 5.1 *Let $T_m(R)$ be a upper triangular matrix ring with $m \geq 3$, where R is a unital ring. If a bi-additive mapping $\varphi : T_m(R) \times T_m(R) \rightarrow T_m(R)$ is a bi-Jordan n-derivation of $T_m(R)$, then it has the form $\phi = \zeta + \delta$, where $\delta : T_m(R) \times T_m(R) \rightarrow T_m(R)$ is an inner biderivation, and $\zeta : T_m(R) \times T_m(R) \rightarrow T_m(R)$ is an extremal biderivation.*

Corollary 5.2 *Let $\mathcal{N}$ be a nest of a Hilbert space H with* $\dim H \geq 3$ and $\mathcal{T}(\mathcal{N})$ *be the nest algebra associated with $\mathcal{N}$. Then each bi-Jordan n-derivation of $\mathcal{T}(\mathcal{N})$ is the sum of an extremal biderivation and an inner biderivation.*

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