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Ding Projective Modules and Dimensions over Formal Triangular Matrix Rings

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Abstract Let T be a formal triangular matrix ring. We prove that, if for each $1 \le j < i \le n, U_{ij}$ is flat on both sides, then a left T-module $\begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M}$ is Ding projective if and only if M_1 is a Ding projective left A_1-module and for each $1 \le k \le n-1$ the mapping $\varphi_{k+1,k} : U_{k+1,k} \otimes_{A_k} M_k \to M_{k+1}$ is injective with cokernel Ding projective over A_{k+1}. As a consequence, we describe Ding projective dimension of a left T-module.

Keywords formal triangular matrix ring; Ding projective module; Ding projective dimension

MR(2020) Subject Classification 16D90; 16E10; 18G05

1. Introduction

The origin of Gorenstein homological algebra may date back to 1960s when Auslander and Bridger introduced the concept of G-dimension for finitely generated modules over a two-sided Noetherian ring [1]. In 1990s, Enochs and Jenda extended the ideas of Auslander and Bridger and introduced the concepts of Gorenstein projective and Gorenstein injective modules over arbitrary rings [2]. In [3], Ding, Li and Mao considered a special case of the Gorenstein projective modules, which they called strongly Gorenstein flat modules. These modules over coherent rings possess many nice properties analogous to Gorenstein projective modules over noetherian rings. So Gillespie later renamed strongly Gorenstein flat as Ding projective modules (see [4, 5] for details).

Let $n \ge 2$, $T = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ U_{21} & A_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ U_{(n-1)1} & \cdots & U_{(n-1)(n-2)} & A_{n-1} & 0 \\ U_{n1} & \cdots & U_{n(n-2)} & U_{n(n-1)} & A_n \end{pmatrix}$ always means a formal triangular matrix ring where $A_1, \dots, A_n$ are rings and for each $1 \le j < i \le n$, U_{ij} is an (A_i, A_j)-bimodule. Formal triangular matrix rings play an important role in ring theory and the rep-

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resentation theory of algebra [6]. This kind of rings are often used to construct examples and counter examples, which make the theory of rings and modules more abundant and concrete. So the properties of formal triangular matrix rings and modules over them have deserved more and more interests. For example, Zhang explicitly described the Gorenstein projective modules over a triangular matrix Artin algebra [7]. Enochs and other authors presented general properties of formal triangular matrix rings and described projective, injective and flat modules over such rings [8]. After, they characterized when a left module over a formal triangular matrix ring of order 2 is Gorenstein projective or Gorenstein injective under the "Gorenstein regular" condition [9]. In [10], Mao investigated Ding modules and dimensions over triangular matrix ring of order 2 under the "U_{21} as left A_2-module and as right A_1-module have finite flat dimensions" condition.

The present paper is devoted to further study Ding projective modules and dimensions over formal triangular matrix rings of order n. Let U_{ij} be flat on both sides where $1 \le j < i \le n$. We prove that a left T-module $\begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M}$ is Ding projective if and only if M_1 is a Ding projective left A_1-module and for each $1 \le k \le n-1$ the mapping $\varphi_{k+1,k} : U_{k+1,k} \otimes_{A_k} M_k \to M_{k+1}$ is injective with cokernel Ding projective over A_{k+1}. In addition, we give an estimate of Ding projective dimension of a left T-module.

Throughout this paper, all rings are nonzero associative rings with identity and all modules are unitary. For a ring R, we write R-Mod for the category of left R-modules, ${}_RM$ denotes a left R-module, $\mathbb{Z}$ denotes set of integers.

Next we recall some notions and facts needed in the sequel.

Definition 1.1 ([8]) *By an $n \times n$ formal matrix ring we mean that we are given n rings A_i, that for each i and j we have an (A_i, A_j)-bimodule U_{ij} (or we sometimes write $U_{i,j}$) where $U_{ii} = A_i$ for each i and where we have (A_i, A_k)-linear maps $U_{ij} \otimes_{A_j} U_{jk} \to U_{ik}$ for each i, j, k that satisfy the obvious associativity condition on $(U_{ij} \otimes_{A_j} U_{jk}) \otimes_{A_k} U_{kl} = U_{ij} \otimes_{A_j} (U_{jk} \otimes_{A_k} U_{kl})$ and which are just scalar multiplication if $i = j$ or $j = k$.*

Given the data above, we can form a ring that we denote by

$$\mathbf{M} = \begin{pmatrix} A_1 & U_{12} & U_{13} & \cdots & U_{1n} \\ U_{21} & A_2 & U_{23} & \cdots & U_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ U_{(n-1)1} & \cdots & U_{(n-1)(n-2)} & A_{n-1} & U_{(n-1)n} \\ U_{n1} & \cdots & U_{n(n-2)} & U_{n(n-1)} & A_n \end{pmatrix},$$

where the addition is componentwise and where the multiplication comes from the given data. If $U_{ij} = 0$ when $i < j$, then we say that we have a formal (lower) triangular matrix ring and call it T, that is

$$T=\begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ U_{21} & A_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ U_{(n-1)1} & \cdots & U_{(n-1)(n-2)} & A_{n-1} & 0 \\ U_{n1} & \cdots & U_{n(n-2)} & U_{n(n-1)} & A_n \end{pmatrix}.$$

By [8], we see that every left T-module can be written as $\begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M}$ where M_i is a left A_i-module and where there are linear maps $\varphi_{ij}: U_{ij}\otimes_{A_j} M_j \to M_i$ for each i and j satisfying the obvious associativity condition. Since we have $\mathrm{Hom}_{A_i}(U_{ij}\otimes_{A_j} M_j, M_i)\cong \mathrm{Hom}_{A_j}(M_j, \mathrm{Hom}_{A_i}(U_{ij}, M_i))$ there is an alternate way of giving $\mathbf{M}$-modules in terms of maps $\widetilde{\varphi_{ij}}: M_j \to \mathrm{Hom}_{A_i}(U_{ij}, M_i)$ satisfying the conditions corresponding to the associativity.

Linear maps are $\begin{pmatrix} N_1 \\ \vdots \\ N_n \end{pmatrix}_{\varphi^N} \to \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M}$ are given by A_i-linear maps $N_i \to M_i$ such that the diagrams

$$\begin{array}{ccc} U_{ij}\otimes_{A_j} N_j & \longrightarrow & U_{ij}\otimes_{A_j} M_j \\ \downarrow & & \downarrow \\ N_i & \longrightarrow & M_i \end{array}.$$

Diagram 1 A_i-linear maps $N_i \to M_i$

are commutative.

Note that a sequence of T-modules

$$0\to \begin{pmatrix} M_1' \\ M_2' \\ \vdots \\ M_n' \end{pmatrix}_{\varphi^{M'}} \to \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} M_1'' \\ M_2'' \\ \vdots \\ M_n'' \end{pmatrix}_{\varphi^{M''}} \to 0$$

is exact if and only if for each i the sequence $0\to M_i' \to M_i \to M_i'' \to 0$ of A_i-modules is exact.

We now assume that our ring T is formal triangular, that it is $n\times n$ for some $n\geq 2$ and that the maps $\psi_{ijt}: U_{ij}\otimes_{A_j} U_{jt}\to U_{it}$ are isomorphic where $i>j>t$. The functors $q_i\begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix} = M_i$ from the category of T-modules to that of A_i-modules have left and right adjoints. These are given by

$$p_i(M_i) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ M_i \\ U_{i+1,i} \otimes_{A_i} M_i \\ \vdots \\ U_{n,i} \otimes_{A_i} M_i \end{pmatrix}_{\varphi} \quad \text{and} \quad h_i(M_i) = \begin{pmatrix} \mathrm{Hom}_{A_i}(U_{i1}, M_i) \\ \mathrm{Hom}_{A_i}(U_{i2}, M_i) \\ \vdots \\ M_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{\phi},$$

where the corresponding morphisms are $\varphi_{jl} = \psi_{jli} \otimes \mathrm{Id}_{M_i}$ for $l > i, \varphi_{ji} : U_{ji} \otimes_{A_i} M_i \to U_{ji} \otimes_{A_i} M_i$ is the canonical isomorphism, and $\varphi_{jl} = 0$ for $l < i$; $\phi_{jl} : U_{jl} \otimes_{A_l} \mathrm{Hom}_{A_i}(U_{il}, M_i) \to \mathrm{Hom}_{Ai}(U_{ij}, M_i)$ are given by $\phi_{jl}(u_{jl} \otimes f)(u_{ij}) = f(u_{ij} \otimes u_{jl})$ for $j < i$ where $f \in \mathrm{Hom}_{A_i}(U_{il}, M_i)$, $u_{ij} \in U_{ij}, u_{jl} \in U_{jl}$, $\phi_{il} : U_{il} \otimes_{A_l} \mathrm{Hom}_{A_i}(U_{il}, M_i) \to M_i$ is the evaluation map, and $\phi_{jl} = 0$ for $j > i$.

So in the first case we get a projective T-module if M_i is projective and get an injective T-module if M_i is injective in the second case.

In the rest of the article, unless specified, module will mean a left module and T will be as above.

2. Ding projective modules over formal triangular matrix rings

Recall that a left R-module M is Ding projective if there is an exact sequence $\cdots \to P^1 \to P^0 \to P^{-1} \to \cdots$ of projective left R-modules with $M \cong \ker(P^0 \to P^{-1})$, which remains exact after applying $\mathrm{Hom}_R(-, F)$ for any flat left R-module F.

Lemma 2.1 ([8]) *Let* $M = \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M}$ *be a left T-module. Then*

(1) M is projective if and only if M_1 is a projective left A_1-module and for each $1 \le i \le n-1$, the mapping $\varphi^M_{i+1,i} : U_{i+1,i} \otimes_{A_i} M_i \to M_{i+1}$ is injective with cokernel projective over A_{i+1}.

(2) M is flat if and only if M_1 is a flat left A_1-module and for each $1 \le i \le n-1$, the mapping $\varphi^M_{i+1,i} : U_{i+1,i} \otimes_{A_i} M_i \to M_{i+1}$ is an injection with a flat cokernel.

Now we describe explicitly the structure of a Ding projective left T-module.

Proposition 2.2 *Let U_{ij} as left A_i-module be flat and as right A_j-module have finite flat dimension. If M_t is a Ding projective left A_t-module, then $p_t(M_t)$ is a Ding projective left T-module where $1 \le j < i \le n$, $1 \le t \le n$.*

Proof Suppose $t = 1$, M_1 is a Ding projective left A_1-module. There is an exact sequence

$$\mathbb{P}_1 : \cdots \longrightarrow P_1^1 \xrightarrow{\partial^1_{\mathbb{P}_1}} P_1^0 \xrightarrow{\partial^0_{\mathbb{P}_1}} P_1^{-1} \longrightarrow \cdots$$

of projective left A_1-modules with $M_1 \cong \ker\partial^0_{\mathbb{P}_1}$, which remains exact after applying $\mathrm{Hom}_{A_1}(-, H_1)$ for each flat left A_1-module H_1. Since U_{i1} has finite flat dimension, $U_{i1} \otimes_{A_1} \mathbb{P}_1$ is exact in A_i-Mod by [9, Lemma 2.3], which implies the exact sequence of projective left T-modules

$$p_1(\mathbb{P}_1): \cdots \longrightarrow \begin{pmatrix} P_1^1 \\ U_{21} \otimes_{A_1} P_1^1 \\ \vdots \\ U_{n1} \otimes_{A_1} P_1^1 \end{pmatrix} \xrightarrow{\partial^1_{p_1(\mathbb{P}_1)}} \begin{pmatrix} P_1^0 \\ U_{21} \otimes_{A_1} P_1^0 \\ \vdots \\ U_{n1} \otimes_{A_1} P_1^0 \end{pmatrix} \xrightarrow{\partial^0_{p_1(\mathbb{P}_1)}} \begin{pmatrix} P_1^{-1} \\ U_{21} \otimes_{A_1} P_1^{-1} \\ \vdots \\ U_{n1} \otimes_{A_1} P_1^{-1} \end{pmatrix} \longrightarrow \cdots$$

with $p_1(M_1) = \ker\partial^0_{p_1(\mathbb{P}_1)}$. For any flat left T-module $F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}_{\varphi^F}$, F_1 is a flat left A_1-module by Lemma 2.1. Then by adjointness of functors p_1 and q_1, $\mathrm{Hom}_T(p_1(\mathbb{P}_1), F) \cong \mathrm{Hom}_{A_1}(\mathbb{P}_1, F_1)$ is exact. So $p_1(M_1)$ is a Ding projective left T-module.

Next we prove that for each $2 \leq t \leq n$, $p_t(M_t)$ is a Ding projective left T-module. There exists an exact sequence of projective left A_t-modules

$$\mathbb{P}_t: \cdots \longrightarrow P_t^1 \xrightarrow{\partial^1_{\mathbb{P}_t}} P_t^0 \xrightarrow{\partial^0_{\mathbb{P}_t}} P_t^{-1} \longrightarrow \cdots$$

with $M_t \cong \ker\partial^0_{\mathbb{P}_t}$. Since U_{ij} as right A_j-module have finite flat dimension, $U_{it} \otimes_{A_t} \mathbb{P}_t$ is exact in A_i-Mod by [9, Lemma 2.3], which implies the exact sequence of projective left T-modules

$$p_t(\mathbb{P}_t): \cdots \longrightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ P_t^1 \\ U_{t+1,t} \otimes_{A_t} P_t^1 \\ \vdots \\ U_{nt} \otimes_{A_t} P_t^1 \end{pmatrix} \xrightarrow{\partial^1_{p_t(\mathbb{P}_t)}} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ P_t^0 \\ U_{t+1,t} \otimes_{A_t} P_t^0 \\ \vdots \\ U_{nt} \otimes_{A_t} P_t^0 \end{pmatrix} \xrightarrow{\partial^0_{p_t(\mathbb{P}_t)}} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ P_t^{-1} \\ U_{t+1,t} \otimes_{A_t} P_t^{-1} \\ \vdots \\ U_{nt} \otimes_{A_t} P_t^{-1} \end{pmatrix} \longrightarrow \cdots$$

with $p_t(M_t) = \ker\partial^0_{p_t(\mathbb{P}_t)}$. Let $F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}_{\varphi^F}$ be a flat left T-module. By Lemma 2.1, there exists the exact sequence in A_2-Mod

$$0 \longrightarrow U_{21} \otimes_{A_1} F_1 \xrightarrow{\varphi^F_{21}} F_2 \longrightarrow \mathrm{coker}\varphi^F_{21} \longrightarrow 0$$

with $\mathrm{coker}\varphi^F_{21}$ being flat. By [11, Theorem 9.48], $U_{21} \otimes_{A_1} F_1$ is flat and so F_2 is flat. Continuing this way, we have $F_3, \ldots, F_n$ is flat. Hence $\mathrm{Hom}_T(p_t(\mathbb{P}_t), F) \cong \mathrm{Hom}_{A_t}(\mathbb{P}_t, F_t)$ is exact. Thus $p_t(M_t)$ is a Ding projective left T-module. □

Theorem 2.3 *Let U_{ij} be flat on both sides where $1 \leq j < i \leq n$, and $M = \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M}$ be a left T-module. The following conditions are equivalent:*

(1) M is a Ding projective left T-module;

(2) M_1 is a Ding projective left A_1-module, and for each $1 \leq k \leq n-1$ the mapping $\varphi^M_{k+1,k} : U_{k+1,k} \otimes_{A_k} M_k \to M_{k+1}$ is injective with cokernel Ding projective over A_{k+1}.

In this case, $U_{k+1,k} \otimes_{A_k} M_k$ is a Ding projective left A_{k+1}-module if and only if M_{k+1} is a Ding projective left A_{k+1}-module.

Proof (1)⇒(2). There is an exact sequence of projective left T-modules

$$\Delta : \cdots \longrightarrow \begin{pmatrix} P_1^1 \\ \vdots \\ P_n^1 \end{pmatrix}_{\varphi^1} \xrightarrow{\begin{pmatrix} \partial_1^1 \\ \vdots \\ \partial_n^1 \end{pmatrix}} \begin{pmatrix} P_1^0 \\ \vdots \\ P_n^0 \end{pmatrix}_{\varphi^0} \xrightarrow{\begin{pmatrix} \partial_1^0 \\ \vdots \\ \partial_n^0 \end{pmatrix}} \begin{pmatrix} P_1^{-1} \\ \vdots \\ P_n^{-1} \end{pmatrix}_{\varphi^{-1}} \longrightarrow \cdots$$

with $M \cong \ker \begin{pmatrix} \partial_1^0 \\ \vdots \\ \partial_n^0 \end{pmatrix}$, which remains exact after applying $\mathrm{Hom}_T(-, H)$ for each flat left T-module H.

By Lemma 2.1, we get the exact sequence of projective left A_1-modules

$$\Delta_1 : \cdots \longrightarrow P_1^1 \xrightarrow{\partial_1^1} P_1^0 \xrightarrow{\partial_1^0} P_1^{-1} \longrightarrow \cdots$$

with $M_1 \cong \ker \partial_1^0$. Let F_1 be a flat left A_1-module, there exists the short exact sequence in T-Mod

$$0 \to \begin{pmatrix} 0 \\ U_{21} \otimes_{A_1} F_1 \\ \vdots \\ U_{n1} \otimes_{A_1} F_1 \end{pmatrix} \to \begin{pmatrix} F_1 \\ U_{21} \otimes_{A_1} F_1 \\ \vdots \\ U_{n1} \otimes_{A_1} F_1 \end{pmatrix} \to \begin{pmatrix} F_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to 0,$$

which induces the exact sequence of complexes

$$0 \to \mathrm{Hom}_T(\Delta, \begin{pmatrix} 0 \\ U_{21} \otimes_{A_1} F_1 \\ \vdots \\ U_{n1} \otimes_{A_1} F_1 \end{pmatrix}) \to \mathrm{Hom}_T(\Delta, \begin{pmatrix} F_1 \\ U_{21} \otimes_{A_1} F_1 \\ \vdots \\ U_{n1} \otimes_{A_1} F_1 \end{pmatrix}) \to \mathrm{Hom}_T(\Delta, \begin{pmatrix} F_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}) \to 0.$$

By Lemma 2.1, $\begin{pmatrix} F_1 \\ U_{21} \otimes_{A_1} F_1 \\ \vdots \\ U_{n1} \otimes_{A_1} F_1 \end{pmatrix}$ is a flat left T-module, so the complex $\mathrm{Hom}_T(\Delta, \begin{pmatrix} F_1 \\ U_{21} \otimes_{A_1} F_1 \\ \vdots \\ U_{n1} \otimes_{A_1} F_1 \end{pmatrix})$ is exact.

Since $\psi_{ijt} : U_{ij} \otimes_{A_j} U_{jt} \to U_{it}$ are isomorphic where $i > j > t$, $\begin{pmatrix} 0 \\ U_{21} \otimes_{A_1} F_1 \\ \vdots \\ U_{n1} \otimes_{A_1} F_1 \end{pmatrix}$ is flat by Lemma 2.1, and so $\mathrm{Hom}_T(\Delta, \begin{pmatrix} 0 \\ U_{21} \otimes_{A_1} F_1 \\ \vdots \\ U_{n1} \otimes_{A_1} F_1 \end{pmatrix})$ is exact. It follows that $\mathrm{Hom}_T(\Delta, \begin{pmatrix} F_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix})$ is exact by [11, Theorem 6.3]. By adjointness of functors p_1 and q_1, we have $\mathrm{Hom}_{A_1}(\Delta_1, F_1) \cong \mathrm{Hom}_T(\Delta, \begin{pmatrix} F_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix})$ is exact. Thus M_1 is a Ding projective left A_1-module.

Let $\lambda_k : M_k \to P_k^0$ and $\lambda_n : M_n \to P_n^0$ be the inclusions. Consider the exact sequence of projective left A_k-modules

$$\Delta_k : \cdots \longrightarrow P_k^1 \xrightarrow{\partial_k^1} P_k^0 \xrightarrow{\partial_k^0} P_k^{-1} \longrightarrow \cdots$$

with $M_k \cong \ker \partial_k^0$, and the following commutative diagram in A_{k+1}-Mod:

$$\begin{array}{ccc} U_{k+1,k} \otimes_{A_k} M_k & \xrightarrow{1\otimes\lambda_k} & U_{k+1,k} \otimes_{A_k} P_k^0 \\ \downarrow{\scriptstyle \varphi_{k+1,k}^M} & & \downarrow{\scriptstyle \varphi_{k+1,k}^0} \\ M_{k+1} & \xrightarrow{\lambda_{k+1}} & P_{k+1}^0. \end{array}$$

Diagram 2 Commutative diagram in A_{k+1}-Mod

Since U_{ij} as right A_j-module is flat, $U_{k+1,k} \otimes_{A_k} \Delta_k$ is exact. Thus $1 \otimes \lambda_k$ is injective. Also $\varphi_{k+1,k}^0$ is injective by Lemma 2.1, so $\varphi_{k+1,k}^M$ is injective.

For any $m \in \mathbb{Z}$, there exists $\overline{\partial_{k+1}^m} : \mathrm{coker}(\varphi_{k+1,k}^m) \to \mathrm{coker}(\varphi_{k+1,k}^{m-1})$ such that the following diagram with exact rows is commutative.

Since the first column and the second column are exact, we get the exact sequence

$$\Xi_{k+1} : \cdots \longrightarrow \mathrm{coker}(\varphi_{k+1,k}^1) \xrightarrow{\overline{\partial_{k+1}^1}} \mathrm{coker}(\varphi_{k+1,k}^0) \xrightarrow{\overline{\partial_{k+1}^0}} \mathrm{coker}(\varphi_{k+1,k}^{-1}) \longrightarrow \cdots$$

of projective left A_{k+1}-modules by [11, Theorem 6.3] with $\operatorname{coker}(\varphi^M_{k+1,k}) \cong \ker(\overline{\partial^0_{k+1}})$.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_{k+1,k} \otimes_{A_k} P^1_k & \xrightarrow{\varphi^1_{k+1,k}} & P^1_{k+1} & \longrightarrow & \operatorname{coker}(\varphi^1_{k+1,k}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\partial^1_k} & & \downarrow{\scriptstyle \partial^1_{k+1}} & & \downarrow{\scriptstyle \overline{\partial^1_{k+1}}} & & \\
0 & \longrightarrow & U_{k+1,k} \otimes_{A_k} P^0_k & \xrightarrow{\varphi^0_{k+1,k}} & P^0_{k+1} & \longrightarrow & \operatorname{coker}(\varphi^0_{k+1,k}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\partial^0_k} & & \downarrow{\scriptstyle \partial^0_{k+1}} & & \downarrow{\scriptstyle \overline{\partial^0_{k+1}}} & & \\
0 & \longrightarrow & U_{k+1,k} \otimes_{A_k} P^{-1}_k & \xrightarrow{\varphi^{-1}_{k+1,k}} & P^{-1}_{k+1} & \longrightarrow & \operatorname{coker}(\varphi^{-1}_{k+1,k}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Diagram 3 Commutative diagram with exact rows in A_{k+1}-Mod

Let F_2 be a flat left A_2-module. There exists the short exact sequence in T-Mod

$$0 \to \begin{pmatrix} 0 \\ 0 \\ U_{32} \otimes_{A_2} F_2 \\ \vdots \\ U_{n2} \otimes_{A_2} F_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ F_2 \\ U_{32} \otimes_{A_2} F_2 \\ \vdots \\ U_{n2} \otimes_{A_2} F_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ F_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to 0,$$

which induces the exact sequence of complexes

$$0 \to \operatorname{Hom}_T(\Delta, \begin{pmatrix} 0 \\ 0 \\ U_{32} \otimes_{A_2} F_2 \\ \vdots \\ U_{n2} \otimes_{A_2} F_2 \end{pmatrix}) \to \operatorname{Hom}_T(\Delta, \begin{pmatrix} 0 \\ F_2 \\ U_{32} \otimes_{A_2} F_2 \\ \vdots \\ U_{n2} \otimes_{A_2} F_2 \end{pmatrix}) \to \operatorname{Hom}_T(\Delta, \begin{pmatrix} 0 \\ F_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}) \to 0.$$

By Lemma 2.1, $\begin{pmatrix} 0 \\ F_2 \\ U_{32} \otimes_{A_2} F_2 \\ \vdots \\ U_{n2} \otimes_{A_2} F_2 \end{pmatrix}$ is flat, hence $\operatorname{Hom}_T(\Delta, \begin{pmatrix} 0 \\ F_2 \\ U_{32} \otimes_{A_2} F_2 \\ \vdots \\ U_{n2} \otimes_{A_2} F_2 \end{pmatrix})$ is exact. Since ψ_{ijt} :

$U_{ij}\otimes_{A_j}U_{jt}\to U_{it}$ are isomorphic where $i>j>t$, $\begin{pmatrix}0\\0\\U_{32}\otimes_{A_2}F_2\\\vdots\\U_{n2}\otimes_{A_2}F_2\end{pmatrix}$ is flat by Lemma 2.1, so $\mathrm{Hom}_T(\Delta,\begin{pmatrix}0\\0\\U_{32}\otimes_{A_2}F_2\\\vdots\\U_{n2}\otimes_{A_2}F_2\end{pmatrix})$ is exact. By [11, Theorem 6.3], $\mathrm{Hom}_T(\Delta,\begin{pmatrix}0\\F_2\\0\\\vdots\\0\end{pmatrix})$ is exact. For each $m\in\mathbb{Z}$, $\begin{pmatrix}P_1^m\\\vdots\\P_n^m\end{pmatrix}_{\varphi^m}=\bigoplus_{s=1}^{n}(p_s(C_s^m))$, where $C_s^m=\begin{cases}P_1^m, & s=1\\ \mathrm{coker}(\varphi_{s,s-1}^m), & 2\le s\le n\end{cases}$ by [8, Corollary 2.3]. By adjointness of functors p_s and q_s, we have $\mathrm{Hom}_T(\bigoplus_{s=1}^{n}(p_s(C_s^m)),\begin{pmatrix}0\\F_2\\0\\\vdots\\0\end{pmatrix})\cong(\mathrm{Hom}_{A_1}(P_1^m,0),$ $\mathrm{Hom}_{A_2}(\mathrm{coker}(\varphi_{21}^m),F_2),\mathrm{Hom}_{A_3}(\mathrm{coker}(\varphi_{32}^m),0),\dots,\mathrm{Hom}_{A_n}(\mathrm{coker}(\varphi_{n,n-1}^m),0))\cong\mathrm{Hom}_{A_2}(\mathrm{coker}(\varphi_{21}^m),$ $F_2)$, so $\mathrm{Hom}_{A_2}(\Xi_2,F_2)\cong\mathrm{Hom}_T(\Delta,\begin{pmatrix}0\\F_2\\0\\\vdots\\0\end{pmatrix})$ is exact. Thus $\mathrm{coker}\varphi_{21}^M$ is a Ding projective left A_2-module. Continuing this way, we can get that $\mathrm{coker}\varphi_{32}^M,\dots,\mathrm{coker}\varphi_{n-1,n-2}^M$ are Ding projective.

Let F_n be a flat left A_n-module. Then $\begin{pmatrix}0\\\vdots\\0\\F_n\end{pmatrix}$ is a flat left T-module, and so $\mathrm{Hom}_T(\Delta,\begin{pmatrix}0\\\vdots\\0\\F_n\end{pmatrix})$ is exact. By adjointness of functors p_n and q_n, we have $\mathrm{Hom}_{A_n}(\Xi_n,F_n)\cong\mathrm{Hom}_T(\Delta,\begin{pmatrix}0\\\vdots\\0\\F_n\end{pmatrix})$ is exact. Thus $\mathrm{coker}\varphi_{n,n-1}^M$ is a Ding projective left A_n-module.

(2)⇒(1). Since U_{32} as right A_2-module is flat and $\psi_{ijt}:U_{ij}\otimes_{A_j}U_{jt}\to U_{it}$ are isomorphic where $i>j>t$, so consider the following commutative diagram.

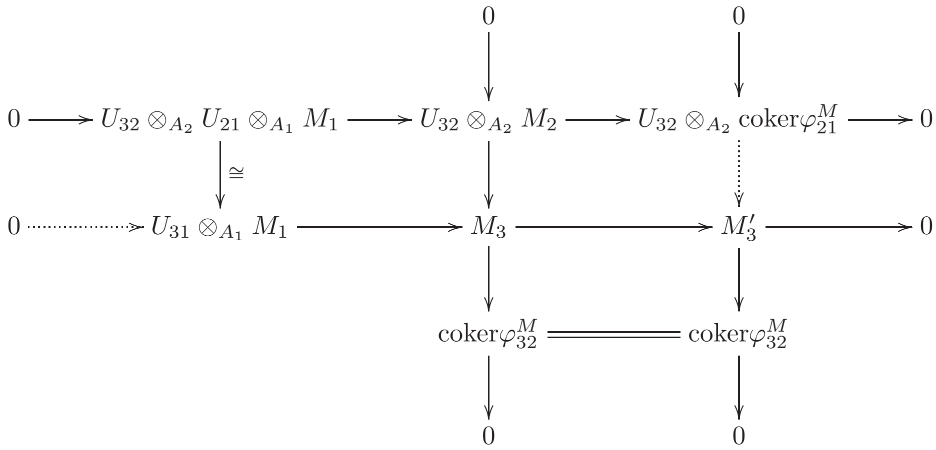

Diagram 4 Commutative diagram in A_3-Mod

Similarly since U_{43} as right A_3-module is flat, so consider the following commutative diagram.

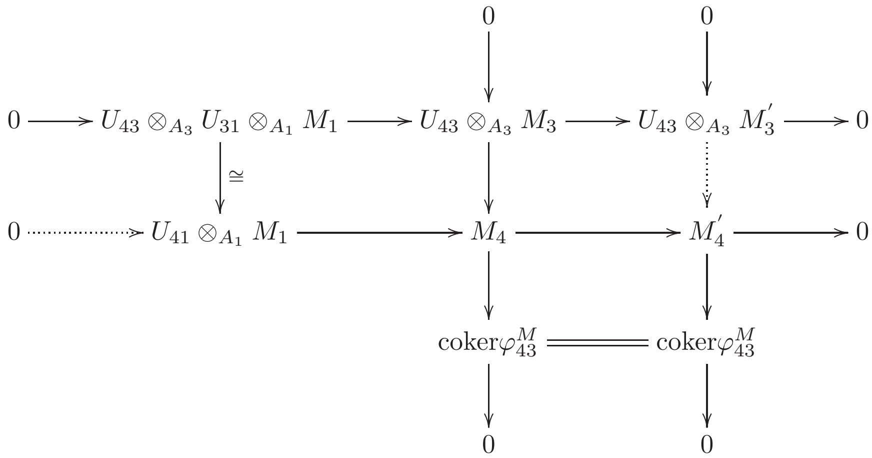

Diagram 5 Commutative diagram in A_4-Mod

Repeating this way, we can get an exact sequence in T-Mod

$$0 \to (p_1(M_1) =) \begin{pmatrix} M_1 \\ U_{21} \otimes_{A_1} M_1 \\ U_{31} \otimes_{A_1} M_1 \\ \vdots \\ U_{n1} \otimes_{A_1} M_1 \end{pmatrix} \to (M =) \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_n \end{pmatrix} \to (M' =) \begin{pmatrix} 0 \\ \mathrm{coker}\varphi_{21}^M \\ M_3' \\ \vdots \\ M_n' \end{pmatrix} \to 0.$$

Clearly, the quotient M' satisfies our condition. Also we see that if M' is Ding projective, then so is M. Repeating the procedure with M', consider the following commutative diagram:

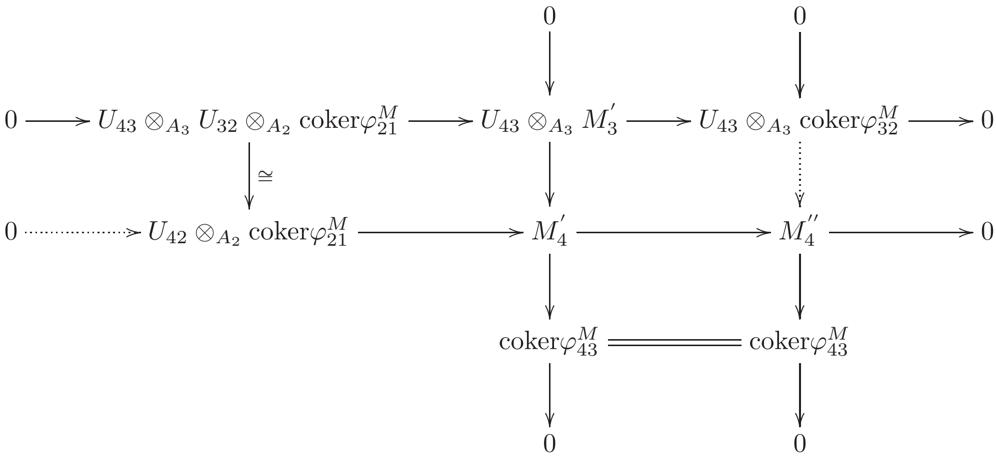

Diagram 6 Commutative diagram

Repeating this way, we can get an exact sequence in T-Mod

$$0 \to (p_2(\mathrm{coker}\varphi_{21}^M) =) \begin{pmatrix} 0 \\ \mathrm{coker}\varphi_{21}^M \\ U_{32} \otimes_{A_2} \mathrm{coker}\varphi_{21}^M \\ U_{42} \otimes_{A_2} \mathrm{coker}\varphi_{21}^M \\ \vdots \\ U_{n2} \otimes_{A_2} \mathrm{coker}\varphi_{21}^M \end{pmatrix} \to (M' =) \begin{pmatrix} 0 \\ \mathrm{coker}\varphi_{21}^M \\ M_3' \\ M_4' \\ \vdots \\ M_n' \end{pmatrix} \to (M'' =) \begin{pmatrix} 0 \\ 0 \\ \mathrm{coker}\varphi_{32}^M \\ M_4'' \\ \vdots \\ M_n'' \end{pmatrix} \to 0.$$

Clearly, the quotient M'' satisfies our condition. Also we see that if M'' is Ding projective, then so is M'. But repeating the procedure with M'', and then continuing we see that we need note that if $N = p_n(\mathrm{coker}\varphi_{n,n-1}^M) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{coker}\varphi_{n,n-1}^M \end{pmatrix}$ satisfies our conditions, then N is Ding projective. But this is obvious.

Finally, if the equivalent conditions above hold, then there exists the exact sequence

$$0 \longrightarrow U_{k+1,k} \otimes_{A_k} M_k \xrightarrow{\varphi_{k+1,k}^M} M_{k+1} \longrightarrow \mathrm{coker}\varphi_{k+1,k}^M \longrightarrow 0,$$

where $1 \leq k \leq n-1$. Since $\mathrm{coker}\varphi_{k+1,k}^M$ is a Ding projective left A_{k+1}-module, $U_{k+1,k} \otimes_{A_k} M_k$ is Ding projective if and only if M_{k+1} is Ding projective by [5, Theorem 2.6]. □

3. Ding projective dimensions of modules over formal triangular matrix rings

Next we investigate Ding projective dimensions of modules over formal triangular matrix ring.

Given a left R-module X, let $\mathrm{Dpd}_R(X)$ denote inf $\{n$: there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to X \to 0$ of left R-modules, where each G_i is Ding projective$\}$, and call $\mathrm{Dpd}_R(X)$ the Ding projective dimension of X. If no such n exists, set $\mathrm{Dpd}_R(X) = \infty$ (see [3]).

Proposition 3.1 *Let m be a non-negative integer, U_{ij} be flat on both sides where $1 \le j < i \le n$, and $M = \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M}$ be a left T-module. If $\mathrm{Dpd}_{A_1}(M_1) \le m$, and for each $1 \le k \le n-1$ the mapping $\varphi^M_{k+1,k}$ is injective with cokernel Ding projective over A_{k+1}. Then $\mathrm{Dpd}_T(M) \le m$.*

Proof There is an exact sequence of left A_1-modules

$$\Lambda_1 : 0 \to K_1 \to P^1_{m-1} \to \cdots \to P^1_0 \to M_1 \to 0$$

with $P^1_0, P^1_1, \ldots, P^1_{m-1}$ being projective, and K_1 being Ding projective by [12, Theorem 2.4]. Since $\mathrm{coker}\varphi^M_{21}$ is a Ding projective left A_2-modules, there is an exact sequence of left A_2-modules

$$0 \to K^2_m \to P^2_{m-1} \to \cdots \to P^2_0 \to \mathrm{coker}\varphi^M_{21} \to 0$$

with each $P^2_0, P^2_1, \ldots, P^2_{m-1}$ being projective, and K^2_m Ding projective. Hence, we can get a commutative diagram with exact rows.

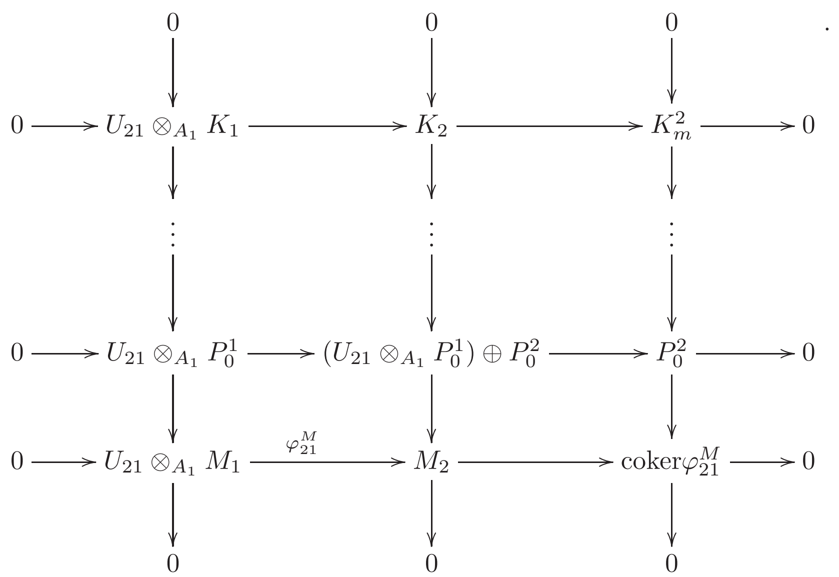

Diagram 7 Commutative diagram with exact rows

Since the first column and the third column are exact, the sequence

$$\Lambda_2 : 0 \to K_2 \to (U_{21} \otimes_{A_1} P^1_{m-1}) \oplus P^2_{m-1} \to \cdots \to (U_{21} \otimes_{A_1} P^1_0) \oplus P^2_0 \to M_2 \to 0$$

is exact by [11, Theorem 6.3].

Since $\mathrm{coker}\varphi^M_{32}$ is a Ding projective left A_3-modules, there is an exact sequence of left A_3-

modules

$$0 \to K_m^3 \to P_{m-1}^3 \to \cdots \to P_0^3 \to \mathrm{coker}\varphi_{32}^M \to 0$$

with $P_0^3, P_1^3, \ldots, P_{m-1}^3$ being projective, and K_m^3 Ding projective. Repeating this way, we get the exact sequence in A_3-Mod

$$\Lambda_3 : 0 \to K_3 \to \overset{2}{\underset{t=1}{\oplus}} (U_{3t} \otimes_{A_t} P_{m-1}^t) \oplus P_{m-1}^3 \to \cdots \to \overset{2}{\underset{t=1}{\oplus}} (U_{3t} \otimes_{A_t} P_0^t) \oplus P_0^3 \to M_3 \to 0.$$

And then continuing, we have the exact sequence in A_n-Mod

$$\Lambda_n : 0 \to K_n \to \overset{n-1}{\underset{t=1}{\oplus}} (U_{nt} \otimes_{A_t} P_{m-1}^t) \oplus P_{m-1}^n \to \cdots \to \overset{n-1}{\underset{t=1}{\oplus}} (U_{nt} \otimes_{A_t} P_0^t) \oplus P_0^n \to M_n \to 0.$$

Hence, we get a projective resolution of M

$$0 \to \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_n \end{pmatrix}_{\varphi^K} \to \begin{pmatrix} P_{m-1}^1 \\ (U_{21} \otimes_{A_1} P_{m-1}^1) \oplus P_{m-1}^2 \\ \overset{2}{\underset{t=1}{\oplus}} (U_{3t} \otimes_{A_t} P_{m-1}^t) \oplus P_{m-1}^3 \\ \vdots \\ \overset{n-1}{\underset{t=1}{\oplus}} (U_{nt} \otimes_{A_t} P_{m-1}^t) \oplus P_{m-1}^n \end{pmatrix} \to \cdots \to \begin{pmatrix} P_0^1 \\ (U_{21} \otimes_{A_1} P_0^1) \oplus P_0^2 \\ \overset{2}{\underset{t=1}{\oplus}} (U_{3t} \otimes_{A_t} P_0^t) \oplus P_0^3 \\ \vdots \\ \overset{n-1}{\underset{t=1}{\oplus}} (U_{nt} \otimes_{A_t} P_0^t) \oplus P_0^n \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M} \to 0,$$

where $\begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{pmatrix}_{\varphi^K}$ is a Ding projective left T-module by Theorem 2.3. Thus $\mathrm{Dpd}_T(M) \le m$. □

Proposition 3.2 *Let m be a non-negative integer, U_{ij} be flat on both sides where $1 \le j < i \le n$, and $M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M}$ be a left T-module. If for each $1 \le k \le n-1$, the mapping $\varphi_{k+1,k}^M$ is injective, then $\mathrm{Dpd}_T(M) \le m$ if and only if $\mathrm{Dpd}_{A_1}(M_1) \le m$, $\mathrm{Dpd}_{A_{k+1}}(\mathrm{coker}\varphi_{k+1,k}^M) \le m$.*

Proof (⇒). Since $\mathrm{Dpd}_T(M) = m$, there exists an exact sequence of left T-modules

$$\Delta : \ 0 \to \begin{pmatrix} N_1^m \\ N_2^m \\ \vdots \\ N_n^m \end{pmatrix}_{\varphi^m} \to \begin{pmatrix} N_1^{m-1} \\ N_2^{m-1} \\ \vdots \\ N_n^{m-1} \end{pmatrix}_{\varphi^{m-1}} \to \cdots \to \begin{pmatrix} N_1^0 \\ N_2^0 \\ \vdots \\ N_n^0 \end{pmatrix}_{\varphi^0} \to \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M} \to 0$$

with $\begin{pmatrix} N_1^0 \\ N_2^0 \\ \vdots \\ N_n^0 \end{pmatrix}_{\varphi^0}, \begin{pmatrix} N_1^1 \\ N_2^1 \\ \vdots \\ N_n^1 \end{pmatrix}_{\varphi^1}, \ldots, \begin{pmatrix} N_1^m \\ N_2^m \\ \vdots \\ N_n^m \end{pmatrix}_{\varphi^m}$ is Ding projective. Hence the sequence

$$\Lambda_1 : 0 \to N_1^m \to N_1^{m-1} \to \cdots \to N_1^0 \to M_1 \to 0$$

is exact. By Theorem 2.3, we have $N_1^0, N_1^1, \ldots, N_1^m$ are Ding projective, thus $\mathrm{Dpd}_{A_1}(M_1) \leq m$.

Consider the following commutative diagram with exact rows.

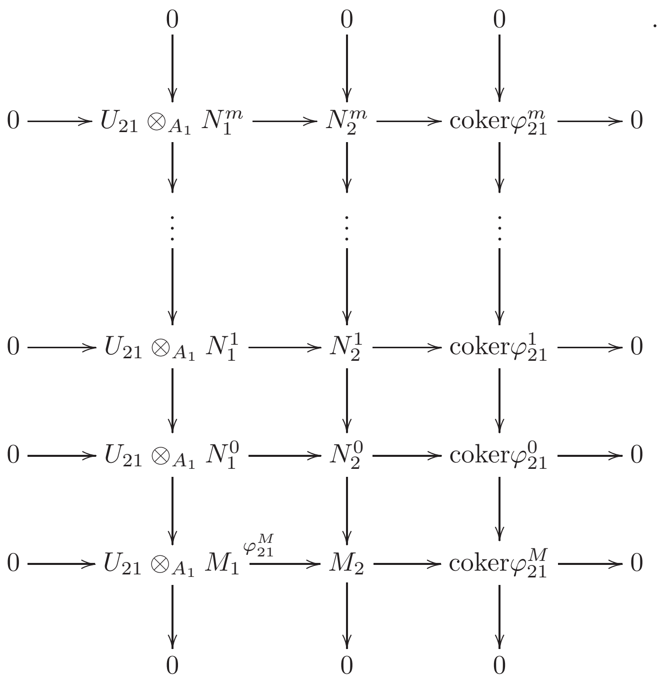

Diagram 8 Commutative diagram with exact rows in A_2-Mod

Since U_{21} as right A_1-module is flat, $U_{21} \otimes_{A_1} \Lambda_1$ is exact. Hence the sequence

$$0 \to \mathrm{coker}\varphi_{21}^m \to \mathrm{coker}\varphi_{21}^{m-1} \to \cdots \to \mathrm{coker}\varphi_{21}^0 \to \mathrm{coker}\varphi_{21}^M \to 0$$

is exact by [11, Theorem 6.3]. For each $0 \leq s \leq m$, $\mathrm{coker}\varphi_{21}^s$ is Ding projective. Thus $\mathrm{Dpd}_{A_2}(\mathrm{coker}\varphi_{21}^M) \leq m$.

Repeating this way, we can get $\mathrm{Dpd}_{A_3}(\mathrm{coker}\varphi_{32}^M) \leq m, \ldots, \mathrm{Dpd}_{A_n}(\mathrm{coker}\varphi_{n,n-1}^M) \leq m$.

($\Leftarrow$). There exists an exact sequence of left T-modules

$$\Delta:\ 0 \to \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{pmatrix}_{\varphi^K} \to \begin{pmatrix} P_1^{m-1} \\ P_2^{m-1} \\ \vdots \\ P_n^{m-1} \end{pmatrix}_{\varphi^{m-1}} \to \cdots \to \begin{pmatrix} P_1^0 \\ P_2^0 \\ \vdots \\ P_n^0 \end{pmatrix}_{\varphi^0} \to \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M} \to 0\,,$$

where $\begin{pmatrix} P_1^0 \\ P_2^0 \\ \vdots \\ P_n^0 \end{pmatrix}_{\varphi^0}, \ldots, \begin{pmatrix} P_1^{m-1} \\ P_2^{m-1} \\ \vdots \\ P_n^{m-1} \end{pmatrix}_{\varphi^{m-1}}$ are projective. The sequence above induces the following exact sequence

$$0 \to K_1 \to P_1^{m-1} \to \cdots \to P_1^0 \to M_1 \to 0,$$

where $P_1^0, \ldots, P_1^{m-1}$ are projective. Since $\mathrm{Dpd}_{A_1}(M_1) \leq m$, K_1 is Ding projective by [12, Theorem 2.4]. We can get the following exact sequence

$$0 \to \mathrm{coker}\varphi^K_{k+1,k} \to \mathrm{coker}\varphi^{m-1}_{k+1,k} \to \cdots \to \mathrm{coker}\varphi^0_{k+1,k} \to \mathrm{coker}\varphi^M_{k+1,k} \to 0,$$

where $\mathrm{coker}\varphi^0_{k+1,k}, \ldots, \mathrm{coker}\varphi^{m-1}_{k+1,k}$ are projective.

Since $\mathrm{Dpd}_{A_{k+1}}(\mathrm{coker}\varphi^M_{k+1,k}) \leq m$, $\mathrm{coker}\varphi^K_{k+1,k}$ is Ding projective, hence $\begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{pmatrix}_{\varphi^K}$ is Ding projective by Theorem 2.3. Thus $\mathrm{Dpd}_T(M) \leq m$. □

Let $\mathrm{lDPD}(R) = \sup\{\mathrm{Dpd}_R(X) : X$ be any left R-module$\}$ and call $\mathrm{lDPD}(R)$ the left global Ding projective dimension of R.

Lemma 3.3 *Let $\mathrm{lDPD}(A_i) < \infty$, U_{ij} as left A_i-module be projective and as right A_j-module have finite flat dimension. If M_j is a Ding projective left A_j-module, then $U_{ij} \otimes_{A_j} M_j$ is a Ding projective left A_i-module where $1 \leq j < i \leq n$.*

Proof It is obtained by the proof of [10, Lemma 3.7]. □

Proposition 3.4 *Let $\mathrm{lDPD}(A_i) < \infty$, U_{ij} as left A_i-module be projective and as right A_j-module be flat. If $M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M}$ is a left T-module, then*

$$\max\{\mathrm{Dpd}_{A_1}(M_1), \mathrm{Dpd}_{A_2}(M_2), \ldots, \mathrm{Dpd}_{A_n}(M_n)\} \leq \mathrm{Dpd}_T(M),$$

where $1 \leq j < i \leq n$.

Proof Assume $\mathrm{Dpd}_T(M) = \infty$, the inequality '$\leq$' is clear. Naturally we may assume that $\mathrm{Dpd}_T(M) = m < \infty$.

There is an exact sequence in T-Mod

$$\Delta:\ 0 \to \begin{pmatrix} N_1^m \\ N_2^m \\ \vdots \\ N_n^m \end{pmatrix}_{\varphi^m} \to \begin{pmatrix} N_1^{m-1} \\ N_2^{m-1} \\ \vdots \\ N_n^{m-1} \end{pmatrix}_{\varphi^{m-1}} \to \cdots \to \begin{pmatrix} N_1^0 \\ N_2^0 \\ \vdots \\ N_n^0 \end{pmatrix}_{\varphi^0} \to \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}_{\varphi^M} \to 0\,.$$

For each $0 \leq s \leq m$, $\begin{pmatrix} N_1^s \\ N_2^s \\ \vdots \\ N_n^s \end{pmatrix}_{\varphi^s}$ is Ding projective. By Theorem 2.3, all N_1^s and $\mathrm{coker}\varphi^s_{k+1,k}$ are

Ding projective where $1 \leq k \leq n-1$. Since there exists the exact sequence

$$0 \to N_1^m \to N_1^{m-1} \to \cdots \to N_1^0 \to M_1 \to 0,$$

$\mathrm{Dpd}_{A_1}(M_1) \leq m$. Consider the short exact sequence

$$0 \longrightarrow U_{21} \otimes_{A_1} N_1^s \xrightarrow{\varphi_{21}^s} N_2^s \longrightarrow \mathrm{coker}\varphi_{21}^s \longrightarrow 0\ ,$$

where $U_{21} \otimes_{A_1} N_1^s$ is Ding projective by Lemma 3.3, hence each N_2^s is Ding projective by Theorem 2.3. Since there exists the exact sequence

$$0 \to N_2^m \to N_2^{m-1} \to \cdots \to N_2^0 \to M_2 \to 0,$$

$\mathrm{Dpd}_{A_2}(M_2) \leq m$. Repeating the procedure with N_2^s and then continuing, we see that $\mathrm{Dpd}_{A_3}(M_3) \leq m, \ldots, \mathrm{Dpd}_{A_n}(M_n) \leq m$, as desired. □

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