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On Compatible Hom-Lie Triple Systems

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Abstract In this paper, we consider compatible Hom-Lie triple systems. More precisely, compatible Hom-Lie triple systems are characterized as Maurer-Cartan elements in a suitable bidifferential graded Lie algebra. We also define a cohomology theory for compatible Hom-Lie triple systems. As applications of cohomology, we study linear deformations and abelian extensions of compatible Hom-Lie triple systems.

Keywords compatible Hom-Lie triple system; cohomology; linear deformations; abelian extensions

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1. Introduction

The notion of Lie triple systems first appeared in Cartan's work [1] on Riemannian geometry. Since then, in [2, 3], Jacobson studied Lie triple systems from Jordan theory and quantum mechanics. After that, Lie triple systems have important applications in physics, such as quantum mechanics theory, elementary particle theory and numerical analysis of differential equations. The representation theory, deformation theory, cohomology and related properties of Lie triple systems can be found in [4–6]. As a Hom-type algebra [7] generalization of a Lie triple system, the notion of Hom-Lie triple systems was introduced by Yau in [8]. Later, Ma et al. considered cohomology and 1-parameter formal deformation of Hom-Lie triple systems in [9]. The authors have studied the relative Rota-Baxter operators on Hom-Lie triple systems in [10, 11]. The representation, cohomology and abelian extension of modified λ-differential Hom-Lie triple systems are investigated in [12].

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The purpose of this paper is to study the compatible Hom-Lie triple system, which is a pair of Hom-Lie triple systems, so that the linear combination of their algebraic structures is also a Hom-Lie triple system. Compatible algebraic structures have been widely used in various fields of mathematics and mathematical physics. For example, in the study of bi-Hamiltonian mechanics, the concept of compatibility of two Poisson structures on manifolds was introduced from a mathematical perspective [13, 14]. Recently, compatible algebraic structures have been widely studied, such as compatible Lie algebras [15], compatible L_∞-algebras [16], compatible pre-Lie algebras [17], compatible associative algebras [18], compatible Lie triple systems [19], compatible 3-Lie algebras [20], compatible Hom-Lie algebras [21] and compatible Hom-Leibniz algebras [22]. Motivated by these works, in this paper, we study the cohomology, linear deformations and abelian extensions of compatible Hom-Lie triple systems.

The paper is organized as follows. In Section 2, we introduce the notion of compatible Hom-Lie triple systems and give the bidifferential graded Lie algebra whose Maurer-Cartan elements are compatible Hom-Lie triple system structures. In Section 3, we introduce the cohomology of a compatible Hom-Lie triple system with coefficients in a representation. In Section 4, we study linear deformations of a compatible Hom-Lie triple system. We introduce Nijenhuis operators that generate trivial linear deformations. In Section 5, we study abelian extensions of compatible Hom-Lie triple systems and give a classification of equivalence classes of abelian extensions.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All the vector spaces and (multi)linear maps are taken over $\mathbb{K}$.

2. Maurer-Cartan characterizations of compatible Hom-Lie triple systems

In this section, we recall concepts of Hom-Lie triple systems from [8] and [9]. Then we introduce the notion of compatible Hom-Lie triple systems and give the bidifferential graded Lie algebra whose Maurer-Cartan elements are compatible Hom-Lie triple system structures. We also construct the bidifferential graded Lie algebra governing deformations of a compatible Hom-Lie triple system.

Definition 2.1 ([8]) *(i) A Hom-Lie triple system (Hom-Lts) is a vector space $\mathfrak{g}$ together with a trilinear operation $[-,-,-]_\mathfrak{g}$ on $\mathfrak{g}$ and a linear map $\alpha_\mathfrak{g}:\mathfrak{g}\to\mathfrak{g}$, satisfying $\alpha_\mathfrak{g}([x,y,z]_\mathfrak{g})=[\alpha_\mathfrak{g}(x),\alpha_\mathfrak{g}(y),\alpha_\mathfrak{g}(z)]_\mathfrak{g}$ such that*

$$[x,y,z]_\mathfrak{g}+[y,x,z]_\mathfrak{g}=0, \tag{2.1}$$

$$\circlearrowleft_{x,y,z}[x,y,z]_\mathfrak{g}=0, \tag{2.2}$$

$$[\alpha_\mathfrak{g}(a),\alpha_\mathfrak{g}(b),[x,y,z]_\mathfrak{g}]_\mathfrak{g}=[[a,b,x]_\mathfrak{g},\alpha_\mathfrak{g}(y),\alpha_\mathfrak{g}(z)]_\mathfrak{g}+[\alpha_\mathfrak{g}(x),[a,b,y]_\mathfrak{g},\alpha_\mathfrak{g}(z)]_\mathfrak{g}+ [\alpha_\mathfrak{g}(x),\alpha_\mathfrak{g}(y),[a,b,z]_\mathfrak{g}]_\mathfrak{g}, \tag{2.3}$$

where $x,y,z,a,b\in\mathfrak{g}$ and $\circlearrowleft_{x,y,z}$ denotes the summation over the cyclic permutations of x,y,z, that is $\circlearrowleft_{x,y,z}[x,y,z]_\mathfrak{g}=[x,y,z]_\mathfrak{g}+[z,x,y]_\mathfrak{g}+[y,z,x]_\mathfrak{g}$. A Hom-Lie triple system as above *may be denoted by the triple* $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$. *In particular, the Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ *is said to be regular, if* $\alpha_{\mathfrak{g}}$ *is nondegenerate.*

(ii) A homomorphism between two Hom-Lie triple systems $(\mathfrak{g}_1, [-,-,-]_1, \alpha_1)$ *and* $(\mathfrak{g}_2, [-,-,-]_2, \alpha_2)$ *is a linear map* $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *satisfying*

$$\varphi(\alpha_1(x)) = \alpha_2(\varphi(x)), \quad \varphi([x,y,z]_1) = [\varphi(x), \varphi(y), \varphi(z)]_2, \quad \forall\, x,y,z \in \mathfrak{g}_1.$$

Remark 2.2 (i) Let $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ be a Hom-Lie algebra. Then $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ is a Hom-Lie triple system, where $[x,y,z]_{\mathfrak{g}} = [[x,y]_{\mathfrak{g}}, \alpha_{\mathfrak{g}}(z)]_{\mathfrak{g}}$, $\forall\, x,y,z \in \mathfrak{g}$.

(ii) A Hom-Lie triple system $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ with $\alpha_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}$ is nothing but a Lie triple system.

Definition 2.3 ([9]) *A representation of a Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ *on a Hom-vector space* (V, β) *is a bilinear map* $\theta : \mathfrak{g} \times \mathfrak{g} \to \mathrm{End}(V)$, *such that for all* $x,y,a,b \in \mathfrak{g}$

$$\theta(\alpha_{\mathfrak{g}}(x), \alpha_{\mathfrak{g}}(y)) \circ \beta = \beta \circ \theta(x,y), \tag{2.4}$$

$$\begin{aligned}&\theta(\alpha_{\mathfrak{g}}(a), \alpha_{\mathfrak{g}}(b))\theta(x,y) - \theta(\alpha_{\mathfrak{g}}(y), \alpha_{\mathfrak{g}}(b))\theta(x,a) - \theta(\alpha_{\mathfrak{g}}(x), [y,a,b]_{\mathfrak{g}}) \circ \beta + \\ &D(\alpha_{\mathfrak{g}}(y), \alpha_{\mathfrak{g}}(a))\theta(x,b) = 0,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\theta(\alpha_{\mathfrak{g}}(a), \alpha_{\mathfrak{g}}(b))D(x,y) - D(\alpha_{\mathfrak{g}}(x), \alpha_{\mathfrak{g}}(y))\theta(ab) + \theta([x,y,a]_{\mathfrak{g}}, \alpha_{\mathfrak{g}}(b)) \circ \beta + \\ &\theta(\alpha_{\mathfrak{g}}(a), [x,y,b]_{\mathfrak{g}}) \circ \beta = 0,\end{aligned} \tag{2.6}$$

where $D(x,y) = \theta(y,x) - \theta(x,y)$. *We also denote a representation of* $\mathfrak{g}$ *on* (V,β) *by* $(V, \beta; \theta)$.

Proposition 2.4 ([9]) *Let* $(V, \beta; \theta)$ *be a representation of a Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$. *Define a trilinear operation* $[-,-,-]_{\ltimes} : (\mathfrak{g} \oplus V) \otimes (\mathfrak{g} \oplus V) \otimes (\mathfrak{g} \oplus V) \to \mathfrak{g} \oplus V$ *and a linear map* $\alpha_{\mathfrak{g}} \oplus \beta : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ *by*

$$\begin{aligned}&[(a,u), (b,v), (c,w)]_{\ltimes} = ([a,b,c]_{\mathfrak{g}}, D(a,b)w + \theta(b,c)u - \theta(a,c)v), \\ &\alpha_{\mathfrak{g}} \oplus \beta(a,u) = (\alpha_{\mathfrak{g}}(a), \beta(u)), \text{ for } (a,u), (b,v), (c,w) \in \mathfrak{g} \oplus V.\end{aligned}$$

Then $(\mathfrak{g} \oplus V, [-,-,-]_{\ltimes}, \alpha_{\mathfrak{g}} \oplus \beta)$ *is a Hom-Lie triple system. This Hom-Lie triple system is called the semi-product Hom-Lie triple system and denoted by* $\mathfrak{g} \ltimes V$.

Next we recall the cohomology theory on Hom-Lie triple systems given in [9]. Let $(V, \beta; \theta)$ be a representation of a Hom-Lie triple system $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$. Denote the n-cochains of $\mathfrak{g}$ with coefficients in V by

$$\mathcal{C}^1_{\mathrm{HLts}}(\mathfrak{g}, V) := \{f \in \mathrm{Hom}(\mathfrak{g}, V) \mid \beta(f(a)) = f(\alpha_{\mathfrak{g}}(a))\},$$

$$\begin{aligned}\mathcal{C}^{n+1}_{\mathrm{HLts}}(\mathfrak{g}, V) := \Big\{&f \in \mathrm{Hom}(\overbrace{\wedge^2\mathfrak{g} \otimes \cdots \otimes \wedge^2\mathfrak{g}}^{n \text{ times}} \otimes \mathfrak{g}, V) \mid \beta \circ f = f \circ (\alpha_{\mathfrak{g}}^{\wedge 2} \otimes \cdots \otimes \alpha_{\mathfrak{g}}^{\wedge 2} \otimes \alpha_{\mathfrak{g}}), \\ &f(a_1 \wedge b_1, \ldots, a_{n-1} \wedge b_{n-1}, a, b, c) + f(a_1 \wedge b_1, \ldots, a_{n-1} \wedge b_{n-1}, b, a, c) = 0, \\ &\circlearrowleft_{a,b,c} f(a_1 \wedge b_1, \ldots, a_{n-1} \wedge b_{n-1}, a, b, c) = 0\Big\}.\end{aligned}$$

The coboundary operator $\delta : \mathcal{C}^n_{\mathrm{HLts}}(\mathfrak{g}, V) \to \mathcal{C}^{n+1}_{\mathrm{HLts}}(\mathfrak{g}, V)$ is denoted by

$$\delta f(\mathfrak{A}_1, \ldots, \mathfrak{A}_n, c)$$

$$= \theta(\alpha_{\mathfrak{g}}^{n-1}(b_n), \alpha_{\mathfrak{g}}^{n-1}(c)) f(\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1} a_n) - \theta(\alpha_{\mathfrak{g}}^{n-1}(a_n), \alpha_{\mathfrak{g}}^{n-1}(c)) f(\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, b_n) +$$
$$\sum_{i=1}^{n} (-1)^{i+n} D(\alpha_{\mathfrak{g}}^{n-1}(a_i), \alpha_{\mathfrak{g}}^{n-1}(b_i)) f(\mathfrak{A}_1, \ldots, \widehat{\mathfrak{A}}_i, \ldots, \mathfrak{A}_n \ldots, c) +$$
$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} (-1)^{i+n+1} f(\alpha_{\mathfrak{g}}(\mathfrak{A}_1), \ldots, \widehat{\mathfrak{A}}_i, \ldots, [a_i, b_i, a_j]_{\mathfrak{g}} \wedge \alpha_{\mathfrak{g}}(b_j) + \alpha_{\mathfrak{g}}(a_j) \wedge [a_i, b_i, b_j]_{\mathfrak{g}}, \ldots,$$
$$\alpha_{\mathfrak{g}}(\mathfrak{A}_n), \alpha_{\mathfrak{g}}(c)) + \sum_{i=1}^{n} (-1)^{i+n+1} f(\alpha_{\mathfrak{g}}(\mathfrak{A}_1), \ldots, \widehat{\mathfrak{A}}_i, \ldots, \alpha_{\mathfrak{g}}(\mathfrak{A}_n), [a_i, b_i, c]_{\mathfrak{g}}),$$

where $\mathfrak{A}_i = a_i \wedge b_i \in \wedge^2 \mathfrak{g}, 1 \le i \le n, c \in \mathfrak{g},\ f \in \mathcal{C}^n_{\mathrm{HLts}}(\mathfrak{g}, V),\ \alpha_{\mathfrak{g}}(\mathfrak{A}_i) = \alpha_{\mathfrak{g}}(a_i) \wedge \alpha_{\mathfrak{g}}(b_i)$.

Remark 2.5 The cochain complex of $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ with coefficients in $(V, \beta; \theta)$ is adapted from [9]. There the coboundary operator δ_{hom} has degree 2 sending $C^n_{\alpha_{\mathfrak{g}},\beta}(\mathfrak{g}, V)$ to $C^{n+2}_{\alpha_{\mathfrak{g}},\beta}(\mathfrak{g}, V)$, where

$$C^n_{\alpha_{\mathfrak{g}},\beta}(\mathfrak{g}, V) := \big\{ f \in \mathrm{Hom}(\overbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}^{n \text{ times}}, V) \mid \beta(f(x_1, \ldots, x_n)) = f(\alpha_{\mathfrak{g}}(x_1), \ldots, \alpha_{\mathfrak{g}}(x_n)),$$
$$f(x_1, \ldots, x, x, x_n) = 0, \quad \circlearrowleft_{x,y,z} f(x_1, \ldots, x_{n-3}, x, y, z) = 0 \big\}.$$

The present paper gives the different n-th cochains. In particular, the coboundary operator δ has degree 1.

Let $(\mathfrak{g}, \alpha_{\mathfrak{g}})$ be a Hom-vector space. For each $n \ge 1$, consider the spaces $C^n_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g})$ by

$$C^1_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g}) := \{ f \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) \mid \alpha_{\mathfrak{g}}(f(a)) = f(\alpha_{\mathfrak{g}}(a)) \},$$
$$C^{n+1}_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g}) := \big\{ f \in \mathrm{Hom}(\overbrace{\wedge^2 \mathfrak{g} \otimes \cdots \otimes \wedge^2 \mathfrak{g}}^{n \text{ times}} \otimes \mathfrak{g}, \mathfrak{g}) \mid \alpha_{\mathfrak{g}} \circ f = f \circ (\alpha_{\mathfrak{g}}^{\wedge 2} \otimes \cdots \otimes \alpha_{\mathfrak{g}}^{\wedge 2} \otimes \alpha_{\mathfrak{g}}),$$
$$f(a_1 \wedge b_1, \ldots, a_{n-1} \wedge b_{n-1}, a, b, c) + f(a_1 \wedge b_1, \ldots, a_{n-1} \wedge b_{n-1}, b, a, c) = 0 \big\}.$$

Then the graded vector space $C^{*+1}_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g}) = \oplus_{n=0}^{+\infty} C^{n+1}_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g})$ carries a graded Lie bracket defined as follows. For $P \in C^{p+1}_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g})$ and $Q \in C^{q+1}_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g})$, the bracket $[P, Q]_{\mathrm{Hlts}} \in C^{p+q+1}_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g})$ given by $[P, Q]_{\mathrm{Hlts}} = P \diamond Q - (-1)^{pq} Q \diamond P$, where

$$P \diamond Q(\mathfrak{A}_1, \ldots, \mathfrak{A}_{p+q}, c)$$
$$= \sum_{\sigma \in \mathbb{S}(p,q)} (-1)^{pq+\sigma} P(\alpha_{\mathfrak{g}}^n(\mathfrak{A}_{\sigma(1)}), \ldots, \alpha_{\mathfrak{g}}^n(\mathfrak{A}_{\sigma(p)}), Q(\mathfrak{A}_{\sigma(p+1)}, \ldots, \mathfrak{A}_{\sigma(p+q)}, c)) +$$
$$\sum_{k=1}^{p} \sum_{\sigma \in \mathbb{S}(k-1,q)} (-1)^{(k-1)q+\sigma} P(\alpha_{\mathfrak{g}}^n(\mathfrak{A}_{\sigma(1)}), \ldots, \alpha_{\mathfrak{g}}^n(\mathfrak{A}_{\sigma(k-1)}), \alpha_{\mathfrak{g}}^n(a_{k+q}) \wedge$$
$$Q(\mathfrak{A}_{\sigma(k)}, \ldots, \mathfrak{A}_{\sigma(k+q-1)}, b_{k+q}) + Q(\mathfrak{A}_{\sigma(k)}, \ldots, \mathfrak{A}_{\sigma(k+q-1)}, a_{k+q}) \wedge$$
$$\alpha_{\mathfrak{g}}^n(b_{k+q}), \alpha_{\mathfrak{g}}^n(\mathfrak{A}_{k+q+1}), \ldots, \alpha_{\mathfrak{g}}^n(\mathfrak{A}_{p+q}), \alpha_{\mathfrak{g}}^n(c))$$

for any $\mathfrak{A}_i = a_i \wedge b_i \in \wedge^2 \mathfrak{g},\ i = 1, \ldots, p+q,\ c \in \mathfrak{g}$, where $\alpha_{\mathfrak{g}}^n(\mathfrak{A}_i) = \alpha_{\mathfrak{g}}^n(a_i) \wedge \alpha_{\mathfrak{g}}^n(b_i)$, $\mathbb{S}_{(i,n-i)}$ denotes the set of $(i, n-i)$-unshuffles, i.e., the permutation $\sigma \in \mathbb{S}_n$ satisfies $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$.

It is important to note that the coboundary operator for the cohomology of the Hom-Lie triple system $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ with coefficients in itself is simply given by

$$\delta f = (-1)^{n-1}[\pi, f]_{\mathrm{Hlts}}, \quad \forall f \in C^n_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g}),$$

where π corresponds to the Hom-Lie triple system bracket $[-,-,-]_{\mathfrak{g}}$.

Proposition 2.6 *Assume that $\pi \in C^2_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g})$ satisfies $\circlearrowleft_{a,b,c} \pi(a, b, c) = 0$, $\forall a, b, c \in \mathfrak{g}$. Then $(\mathfrak{g}, \pi, \alpha_{\mathfrak{g}})$ is a Hom-Lie triple system if and only if π is a Maurer-Cartan element of the graded Lie algebra $(C^*_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g}), [-,-]_{\mathrm{Hlts}})$, i.e., $[\pi, \pi]_{\mathrm{Hlts}} = 0$.*

Proof Let $\pi \in C^2_{\mathrm{H}}(\mathfrak{g}, \mathfrak{g})$. We have

$$\begin{aligned}\pi \diamond \pi(\mathfrak{A}_1, \mathfrak{A}_2, c) = &- \pi(\alpha_{\mathfrak{g}}(\mathfrak{A}_1), \pi(\mathfrak{A}_2, c)) + \pi(\alpha_{\mathfrak{g}}(a_2) \wedge \pi(\mathfrak{A}_1, b_2), \alpha_{\mathfrak{g}}(c)) + \\ &\pi(\pi(\mathfrak{A}_1, a_2) \wedge \alpha_{\mathfrak{g}}(b_2), \alpha_{\mathfrak{g}}(c)) + \pi(\alpha_{\mathfrak{g}}(\mathfrak{A}_2), \pi(\mathfrak{A}_1, c)),\end{aligned}$$

which implies that π defines a Hom-Lie triple system structure on $\mathfrak{g}$ if and only if $[\pi, \pi]_{\mathrm{Hlts}} = 0$. □

Now, we recall some known results on bidifferential graded Lie algebras given in [15].

Definition 2.7 ([15]) *A differential graded Lie algebra is a triple $(\mathfrak{g} = \oplus_{i=0}^{+\infty} \mathfrak{g}^i, [-,-], d)$ consisting of a graded Lie algebra $(\mathfrak{g} = \oplus_{i=0}^{+\infty} \mathfrak{g}^i, [-,-])$ and a differential $d : \mathfrak{g} \to \mathfrak{g}$ satisfying $d(\mathfrak{g}_k) \subseteq \mathfrak{g}_{k+1}$ and*

$$d[a, b] = [da, b] + (-1)^k [a, db], \quad \forall a, b \in \mathfrak{g}_k.$$

In particular, any graded Lie algebra is a differential graded Lie algebra with $d = 0$. Furthermore, an element $\pi \in \mathfrak{g}_1$ is called a Maurer-Cartan element of the differential graded Lie algebra $(\mathfrak{g} = \oplus_{i=0}^{+\infty} \mathfrak{g}^i, [-,-], d)$ if it satisfies $d\pi + \frac{1}{2}[\pi, \pi] = 0$.

Definition 2.8 ([15]) *Let $(\mathfrak{g} = \oplus_{i=0}^{+\infty} \mathfrak{g}^i, [-,-], d_1)$ and $(\mathfrak{g} = \oplus_{i=0}^{+\infty} \mathfrak{g}^i, [-,-], d_2)$ be two differential graded Lie algebras. The quadruple $(\mathfrak{g} = \oplus_{i=0}^{+\infty} \mathfrak{g}^i, [-,-], d_1, d_2)$ is called a bidifferential graded Lie algebra if d_1 and d_2 satisfy*

$$d_1 \circ d_2 + d_2 \circ d_1 = 0.$$

Definition 2.9 ([15]) *Let $(\mathfrak{g} = \oplus_{i=0}^{+\infty} \mathfrak{g}^i, [-,-], d_1, d_2)$ be a bidifferential graded Lie algebra. A pair $(\pi_1, \pi_2) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1$ is said to be a Maurer-Cartan element of the bidifferential graded Lie algebra $(\mathfrak{g}, [-,-], d_1, d_2)$, if π_1 and π_2 are Maurer-Cartan elements of the differential graded Lie algebras $(\mathfrak{g}, [-,-], d_1)$ and $(\mathfrak{g}, [-,-], d_2)$, respectively, and $d_1\pi_2 + d_2\pi_1 + [\pi_1, \pi_2] = 0$.*

Proposition 2.10 ([15]) *Let $(\mathfrak{g}, [-,-], d_1, d_2)$ be a bidifferential graded Lie algebra and (π_1, π_2) be its Maurer-Cartan element. Then $(\mathfrak{g}, [-,-], \widetilde{d}_1, \widetilde{d}_2)$ is a bidifferential graded Lie algebra, where $\widetilde{d}_1$ and $\widetilde{d}_2$ are given by*

$$\widetilde{d}_1 \mu = d_1 \mu + [\pi_1, \mu], \widetilde{d}_2 \mu = d_1 \mu + [\pi_2, \mu], \quad \forall \mu \in \mathfrak{g}.$$

Moreover, for all $\widetilde{\pi}_1, \widetilde{\pi}_2 \in \mathfrak{g}_1$, $(\pi_1 + \widetilde{\pi}_1, \pi_2 + \widetilde{\pi}_2)$ is a Maurer-Cartan element of the bidifferential graded Lie algebra $(\mathfrak{g}, [-,-], d_1, d_2)$ if and only if the pair $(\widetilde{\pi}_1, \widetilde{\pi}_2)$ is a Maurer-Cartan element of the bidifferential graded Lie algebra $(\mathfrak{g}, [-,-], \widetilde{d}_1, \widetilde{d}_2)$.

Next we introduce the concept of a compatible Hom-Lie triple system and provide its Maurer-Cartan characterization. We also define the representation of a compatible Hom-Lie triple system.

Definition 2.11 *A compatible Hom-Lie triple system is a quadruple* $(\mathfrak{g},[-,-,-]_{\mathfrak{g}}^1,[-,-,-]_{\mathfrak{g}}^2,\alpha_{\mathfrak{g}})$ *in which* $(\mathfrak{g},[-,-,-]_{\mathfrak{g}}^1,\alpha_{\mathfrak{g}})$ *and* $(\mathfrak{g},[-,-,-]_{\mathfrak{g}}^2,\alpha_{\mathfrak{g}})$ *are Hom-Lie triple system satisfying*

$$\begin{aligned}&[\alpha_{\mathfrak{g}}(a),\alpha_{\mathfrak{g}}(b),[x,y,z]_{\mathfrak{g}}^2]_{\mathfrak{g}}^1+[\alpha_{\mathfrak{g}}(a),\alpha_{\mathfrak{g}}(b),[x,y,z]_{\mathfrak{g}}^1]_{\mathfrak{g}}^2\\&=[[a,b,x]_{\mathfrak{g}}^1,\alpha_{\mathfrak{g}}(y),\alpha_{\mathfrak{g}}(z)]_{\mathfrak{g}}^2+[[a,b,x]_{\mathfrak{g}}^2,\alpha_{\mathfrak{g}}(y),\alpha_{\mathfrak{g}}(z)]_{\mathfrak{g}}^1+[\alpha_{\mathfrak{g}}(x),[a,b,y]_{\mathfrak{g}}^1,\alpha_{\mathfrak{g}}(z)]_{\mathfrak{g}}^2+\\&\quad[\alpha_{\mathfrak{g}}(x),[a,b,y]_{\mathfrak{g}}^2,\alpha_{\mathfrak{g}}(z)]_{\mathfrak{g}}^1+[\alpha_{\mathfrak{g}}(x),\alpha_{\mathfrak{g}}(y),[a,b,z]_{\mathfrak{g}}^1]_{\mathfrak{g}}^2+[\alpha_{\mathfrak{g}}(x),\alpha_{\mathfrak{g}}(y),[a,b,z]_{\mathfrak{g}}^2]_{\mathfrak{g}}^1.\end{aligned}\tag{2.7}$$

A homomorphism between two compatible Hom-Lie triple systems $(\mathfrak{g}_1,[-,-,-]_{\mathfrak{g}_1}^1,[-,-,-]_{\mathfrak{g}_1}^2,\alpha_1)$ and $(\mathfrak{g}_2,[-,-,-]_{\mathfrak{g}_2}^1,[-,-,-]_{\mathfrak{g}_2}^2,\alpha_2)$ is a linear map $\varphi:\mathfrak{g}_1\to\mathfrak{g}_2$ satisfying

$$\varphi(\alpha_1(x))=\alpha_2(\varphi(x)),\ \varphi([x,y,z]_{\mathfrak{g}_1}^i)=[\varphi(x),\varphi(y),\varphi(z)]_{\mathfrak{g}_2}^i\ \text{ for } i=1,2 \text{ and }\ x,y,z\in\mathfrak{g}_1.$$

Furthermore, if φ is nondegenerate, then φ is called an isomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$.

Example 2.12 Let $\mathfrak{g}$ be a 2-dimensional vector space with a basis $\{\varepsilon_1,\varepsilon_2\}$. If we define two non-zero trilinear operations $[-,-,-]_{\mathfrak{g}}^1,[-,-,-]_{\mathfrak{g}}^2$ and a linear map $\alpha_{\mathfrak{g}}$ on $\mathfrak{g}$ as follows

$$[\varepsilon_1,\varepsilon_2,\varepsilon_1]_{\mathfrak{g}}^1=\varepsilon_1,[\varepsilon_1,\varepsilon_2,\varepsilon_2]_{\mathfrak{g}}^1=-\varepsilon_2,[\varepsilon_1,\varepsilon_2,\varepsilon_1]_{\mathfrak{g}}^2=-\varepsilon_2,[\varepsilon_1,\varepsilon_2,\varepsilon_2]_{\mathfrak{g}}^2=\varepsilon_1,\ \alpha_{\mathfrak{g}}(\varepsilon_1)=\varepsilon_1,\alpha_{\mathfrak{g}}(\varepsilon_2)=-\varepsilon_2,$$

then $(\mathfrak{g},[-,-,-]_{\mathfrak{g}}^1,[-,-,-]_{\mathfrak{g}}^2,\alpha_{\mathfrak{g}})$ is a 2-dimensional compatible Hom-Lie triple system.

Example 2.13 Let $\mathfrak{g}$ be a 3-dimensional vector space with a basis $\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$. Define two non-zero trilinear operations $[-,-,-]_{\mathfrak{g}}^1,[-,-,-]_{\mathfrak{g}}^2$ and a linear map $\alpha_{\mathfrak{g}}$ on $\mathfrak{g}$ by

$$[\varepsilon_1,\varepsilon_2,\varepsilon_2]_{\mathfrak{g}}^1=\varepsilon_3,[\varepsilon_2,\varepsilon_1,\varepsilon_2]_{\mathfrak{g}}^2=-\varepsilon_3,\ \alpha_{\mathfrak{g}}(\varepsilon_1)=-\varepsilon_1,\alpha_{\mathfrak{g}}(\varepsilon_2)=\varepsilon_2,\alpha_{\mathfrak{g}}(\varepsilon_3)=-\varepsilon_3.$$

Then $(\mathfrak{g},[-,-,-]_{\mathfrak{g}}^1,[-,-,-]_{\mathfrak{g}}^2,\alpha_{\mathfrak{g}})$ is a 3-dimensional compatible Hom-Lie triple system.

Example 2.14 Let $(V,\beta;\theta)$ be a representation of a Hom-Lie triple system $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},\alpha_{\mathfrak{g}})$. Given a 2-cocycle $f\in\mathcal{C}_{\mathrm{HLts}}^2(\mathfrak{g},V)$, there exists a Hom-Lie triple system structure on the direct sum $\mathfrak{g}\oplus V$ that is defined by

$$[(a,u),(b,v),(c,w)]_{\ltimes+f}=([a,b,c]_{\mathfrak{g}},D(a,b)w+\theta(b,c)u-\theta(a,c)v+f(a,b,c)),$$
$$\alpha_{\mathfrak{g}}\oplus\beta(a,u)=(\alpha_{\mathfrak{g}}(a),\beta(u))\ \text{ for all } a,b,c\in\mathfrak{g},u,v,w\in V.$$

This Hom-Lie triple system is called the f-twisted semi-direct product Hom-Lie triple system. Moreover, the quadruple $(\mathfrak{g}\oplus V,[-,-,-]_{\ltimes},[-,-,-]_{\ltimes+f},\alpha_{\mathfrak{g}}\oplus\beta)$ is a compatible Hom-Lie triple system, where $[-,-,-]_{\ltimes}$ is given by Proposition 2.4.

The proof of the following proposition is straightforward.

Proposition 2.15 *A quadruple* $(\mathfrak{g},[-,-,-]_{\mathfrak{g}}^1,[-,-,-]_{\mathfrak{g}}^2,\alpha_{\mathfrak{g}})$ *is a compatible Hom-Lie triple system if and only if* $([-,-,-]_{\mathfrak{g}}^1,\alpha_{\mathfrak{g}})$ *and* $([-,-,-]_{\mathfrak{g}}^2,\alpha_{\mathfrak{g}})$ *are Hom-Lie triple system structures on* $\mathfrak{g}$ *such that for any* $k_1,k_2\in\mathbb{K}$*, the new trilinear operation*

$$\{-,-,-\}_{\mathfrak{g}}=k_1[-,-,-]_{\mathfrak{g}}^1+k_2[-,-,-]_{\mathfrak{g}}^2$$

define the Hom-Lie triple system structure on Hom-vector space $(\mathfrak{g},\alpha_{\mathfrak{g}})$.

In other words, Proposition 2.15 gives an equivalent definition of the compatible Hom-Lie triple system.

Proposition 2.16 *Assume that* $\pi_i \in C^2_{\mathrm{H}}(\mathfrak{g},\mathfrak{g})$ *satisfies* $\circlearrowleft_{a,b,c} \pi_i(a,b,c) = 0$ *for* $i = 1, 2, a, b, c \in \mathfrak{g}$. *Then* $(\mathfrak{g}, \pi_1, \pi_2, \alpha_{\mathfrak{g}})$ *is a compatible Hom-Lie triple system if and only if* (π_1, π_2) *is a Maurer-Cartan element of the bidifferential graded Lie algebra* $(C^*_{\mathrm{H}}(\mathfrak{g},\mathfrak{g}), [-,-]_{\mathrm{Hlts}}, d_1 = 0, d_2 = 0)$.

Proof Let $\pi_1, \pi_2 \in C^2_{\mathrm{H}}(\mathfrak{g},\mathfrak{g})$. Then we have

$$\begin{aligned} \pi_1 \text{ defines a Hom-Lie triple system structure} &\Leftrightarrow [\pi_1,\pi_1]_{\mathrm{Hlts}} = 0,\\ \pi_2 \text{ defines a Hom-Lie triple system structure} &\Leftrightarrow [\pi_2,\pi_2]_{\mathrm{Hlts}} = 0,\\ \text{compatibility equation } (2.7) &\Leftrightarrow [\pi_1,\pi_2]_{\mathrm{Hlts}} = 0. \end{aligned}$$

Therefore, $(\mathfrak{g}, \pi_1, \pi_2, \alpha_{\mathfrak{g}})$ is a compatible Hom-Lie triple system if and only if (π_1, π_2) is a Maurer-Cartan element of the bidifferential graded Lie algebra $(C^*_{\mathrm{H}}(\mathfrak{g},\mathfrak{g}), [-,-]_{\mathrm{Hlts}}, d_1 = 0, d_2 = 0)$. □

So, from Propositions 2.10 and 2.16, we get the following conclusion.

Proposition 2.17 *Let* $(\mathfrak{g}, \pi_1, \pi_2, \alpha_{\mathfrak{g}})$ *be a compatible Hom-Lie triple system. Then*

(i) $(C^*_{\mathrm{H}}(\mathfrak{g},\mathfrak{g}), [-,-]_{\mathrm{Hlts}}, d_{\pi_1}, d_{\pi_2})$ *is a bidifferential graded Lie algebra, where* $d_{\pi_1}Q = [\pi_1, Q]_{\mathrm{Hlts}}$, $d_{\pi_2}Q = [\pi_2, Q]_{\mathrm{Hlts}}$, $\forall\, Q \in C^p_{\mathrm{H}}(\mathfrak{g},\mathfrak{g})$.

(ii) For all $\widetilde{\pi}_1, \widetilde{\pi}_2 \in C^2_{\mathrm{H}}(\mathfrak{g},\mathfrak{g})$ *satisfying* $\circlearrowleft_{a,b,c} \widetilde{\pi}_i(a,b,c) = 0, \forall a,b,c \in \mathfrak{g}, i = 1,2$, *the quadruple* $(\mathfrak{g}, \pi_1 + \widetilde{\pi}_1, \pi_2 + \widetilde{\pi}_2, \alpha_{\mathfrak{g}})$ *is a compatible Hom-Lie triple system if and only if* $(\widetilde{\pi}_1, \widetilde{\pi}_2)$ *is a Maurer-Cartan element of the bidifferential graded Lie algebra* $(C^*_{\mathrm{H}}(\mathfrak{g},\mathfrak{g}), [-,-]_{\mathrm{Hlts}}, d_{\pi_1}, d_{\pi_2})$.

Next, we define representations of a compatible Hom-Lie triple system.

Definition 2.18 *A representation of the compatible Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ *consists of a quadruple* $(V, \beta; \theta_1, \theta_2)$ *such that*

(i) $(V, \beta; \theta_1)$ *is a representation of the Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$;

(ii) $(V, \beta; \theta_2)$ *is a representation of the Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$;

(iii) for any $a_1, a_2, a_3, a_4 \in \mathfrak{g}$, *the following compatibility equations hold:*

$$\begin{aligned} &\theta_1(\alpha_{\mathfrak{g}}(a_1), [a_2,a_3,a_4]^2_{\mathfrak{g}}) \circ \beta + \theta_2(\alpha_{\mathfrak{g}}(a_1), [a_2,a_3,a_4]^1_{\mathfrak{g}}) \circ \beta + \theta_1(\alpha_{\mathfrak{g}}(a_2), \alpha_{\mathfrak{g}}(a_4))\theta_2(a_1,a_3) +\\ &\theta_2(\alpha_{\mathfrak{g}}(a_2), \alpha_{\mathfrak{g}}(a_4))\theta_1(a_1,a_3) - \theta_1(\alpha_{\mathfrak{g}}(a_3), \alpha_{\mathfrak{g}}(a_4))\theta_2(a_1,a_2) - \theta_2(\alpha_{\mathfrak{g}}(a_3), \alpha_{\mathfrak{g}}(a_4))\theta_1(a_1,a_2) -\\ &D_1(\alpha_{\mathfrak{g}}(a_2), \alpha_{\mathfrak{g}}(a_3))\theta_2(a_1,a_4) - D_2(\alpha_{\mathfrak{g}}(a_2), \alpha_{\mathfrak{g}}(a_3))\theta_1(a_1,a_4) = 0, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &D_1(\alpha_{\mathfrak{g}}(a_3), [a_1,a_2,a_4]^2_{\mathfrak{g}}) \circ \beta + D_2(\alpha_{\mathfrak{g}}(a_3), [a_1,a_2,a_4]^1_{\mathfrak{g}}) \circ \beta + D_1([a_1,a_2,a_3]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}}(a_4)) \circ \beta +\\ &D_2([a_1,a_2,a_3]^1_{\mathfrak{g}}, \alpha_{\mathfrak{g}}(a_4)) \circ \beta - D_1(\alpha_{\mathfrak{g}}(a_1), \alpha_{\mathfrak{g}}(a_2))D_2(a_3,a_4) + D_2(\alpha_{\mathfrak{g}}(a_3), \alpha_{\mathfrak{g}}(a_4))D_1(a_1,a_2) +\\ &D_1(\alpha_{\mathfrak{g}}(a_3), \alpha_{\mathfrak{g}}(a_4))D_2(a_1,a_2) - D_2(\alpha_{\mathfrak{g}}(a_1), \alpha_{\mathfrak{g}}(a_2))D_1(a_3,a_4) = 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\theta_1(\alpha_{\mathfrak{g}}(a_3), \alpha_{\mathfrak{g}}(a_4)) \circ D_2(a_1,a_2) + \theta_2(\alpha_{\mathfrak{g}}(a_3), \alpha_{\mathfrak{g}}(a_4)) \circ D_1(a_1,a_2) + \theta_1([a_1,a_2,a_3]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}}(a_4)) \circ \beta +\\ &\theta_2([a_1,a_2,a_3]^1_{\mathfrak{g}}, \alpha_{\mathfrak{g}}(a_4)) \circ \beta - D_1(\alpha_{\mathfrak{g}}(a_1), \alpha_{\mathfrak{g}}(a_2))\theta_2(a_3,a_4) - D_2(\alpha_{\mathfrak{g}}(a_1), \alpha_{\mathfrak{g}}(a_2))\theta_1(a_3,a_4) +\\ &\theta_1(\alpha_{\mathfrak{g}}(a_3), [a_1,a_2,a_4]^2_{\mathfrak{g}}) \circ \beta + \theta_2(\alpha_{\mathfrak{g}}(a_3), [a_1,a_2,a_4]^1_{\mathfrak{g}}) \circ \beta = 0, \end{aligned} \tag{2.10}$$

where $D_k(a_i, a_j) = \theta_k(a_j, a_i) - \theta_k(a_i, a_j)$, $k = 1, 2$.

It follows that any compatible Hom-Lie triple system $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ is a representation of itself, where $V = \mathfrak{g}, \theta_1(a,b)c = [c,a,b]^1_{\mathfrak{g}}, \theta_2(a,b)c = [c,a,b]^2_{\mathfrak{g}}$ and $\beta = \alpha_{\mathfrak{g}}$. This is called the adjoint representation.

Remark 2.19 Let $(V, \beta; \theta_1, \theta_2)$ be a representation of a compatible Hom-Lie triple system $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$. Then $(V, \beta; k_1\theta_1 + k_2\theta_2)$ is a representation of the Hom-Lie triple system $(\mathfrak{g}, k_1[-,-,-]^1_{\mathfrak{g}} + k_2[-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$.

It is similar to the standard case that the following proposition is proved.

Proposition 2.20 *Let $(V, \beta; \theta_1, \theta_2)$ be a representation of a compatible Hom-Lie triple system $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$. Define two trilinear operations $[-,-,-]^1_{\ltimes}, [-,-,-]^2_{\ltimes} : \wedge^2(\mathfrak{g}\oplus V)\otimes(\mathfrak{g}\oplus V) \to \mathfrak{g}\oplus V$ and a linear map $\alpha_{\mathfrak{g}} \oplus \beta : \mathfrak{g}\oplus V \to \mathfrak{g}\oplus V$ by*

$$[(a,u),(b,v),(c,w)]^1_{\ltimes} = ([a,b,c]^1_{\mathfrak{g}}, D_1(a,b)w + \theta_1(b,c)u - \theta_1(a,c)v),$$
$$[(a,u),(b,v),(c,w)]^2_{\ltimes} = ([a,b,c]^2_{\mathfrak{g}}, D_2(a,b)w + \theta_2(b,c)u - \theta_2(a,c)v),$$
$$\alpha_{\mathfrak{g}} \oplus \beta(a,u) = (\alpha_{\mathfrak{g}}(a), \beta(u))$$

for any $(a,u),(b,v),(c,w) \in \mathfrak{g}\oplus V$. Then $(\mathfrak{g}\oplus V, [-,-,-]^1_{\ltimes}, [-,-,-]^2_{\ltimes}, \alpha_{\mathfrak{g}}\oplus\beta)$ is a compatible Hom-Lie triple system. This compatible Hom-Lie triple system is called the semi-product compatible Hom-Lie triple system and denoted by $\mathfrak{g} \ltimes^1_2 V$.

3. Cohomology of compatible Hom-Lie triple systems

In this section, we introduce the cohomology of a compatible Hom-Lie triple system with coefficients in a representation by using the construction method in [15, 21].

Next, we always assume that $(V, \beta; \theta_1, \theta_2)$ is a representation of a compatible Hom-Lie triple system $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$.

Let $\delta_1 : \mathcal{C}^n_{\mathrm{HLts}}(\mathfrak{g}, V) \to \mathcal{C}^{n+1}_{\mathrm{HLts}}(\mathfrak{g}, V)$ (resp., $\delta_2 : \mathcal{C}^n_{\mathrm{HLts}}(\mathfrak{g}, V) \to \mathcal{C}^{n+1}_{\mathrm{HLts}}(\mathfrak{g}, V)$) for $n \geq 1$, be the coboundary operator for the cohomology of the Hom-Lie triple system $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ with coefficients in the representation $(V, \beta; \theta_1)$ (resp., of the Hom-Lie triple system $(\mathfrak{g}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ with coefficients in the representation $(V, \beta; \theta_2)$). Then, we have $(\delta_1)^2 = 0$ and $(\delta_2)^2 = 0$.

Next we consider the semidirect product compatible Hom-Lie triple system structure on $\mathfrak{g}\oplus V$ given in Proposition 2.20. Notice that any map $f \in \mathcal{C}^{n-1}_{\mathrm{HLts}}(\mathfrak{g}, V)$ can be lifted to a map $\widetilde{f} \in C^{n-1}_{\mathrm{H}}(\mathfrak{g}\oplus V, \mathfrak{g}\oplus V)$ by

$$\widetilde{f}((\mathfrak{A}_1, \mathfrak{V}_1), \ldots, (\mathfrak{A}_{n-1}, \mathfrak{V}_{n-1}), (c,u)) = (0, f(\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, c))$$

for any $\mathfrak{A}_i = a_i \wedge b_i \in \wedge^2\mathfrak{g}, \mathfrak{V}_i = u_i \wedge v_i \in \wedge^2 V, c \in \mathfrak{g}, u \in V$. Then $f = 0$ if and only if $\widetilde{f} = 0$.

We denote by $\pi_1, \pi_2 \in C^2_{\mathrm{H}}(\mathfrak{g}\oplus V, \mathfrak{g}\oplus V)$ the elements corresponding to the Hom-Lie triple system structures $[-,-,-]^1_{\ltimes}$ and $[-,-,-]^2_{\ltimes}$ on $\mathfrak{g}\oplus V$, respectively. Let

$$\widetilde{\delta}_1 : C^n_{\mathrm{H}}(\mathfrak{g}\oplus V, \mathfrak{g}\oplus V) \to C^{n+1}_{\mathrm{H}}(\mathfrak{g}\oplus V, \mathfrak{g}\oplus V),$$
$$\widetilde{\delta}_2 : C^n_{\mathrm{H}}(\mathfrak{g}\oplus V, \mathfrak{g}\oplus V) \to C^{n+1}_{\mathrm{H}}(\mathfrak{g}\oplus V, \mathfrak{g}\oplus V)$$

denote respectively the coboundary operator for the cohomology of the Hom-Lie triple system $(\mathfrak{g} \oplus V, \pi_1, \alpha_{\mathfrak{g}} \oplus \beta)$ (resp., of the Hom-Lie triple system $(\mathfrak{g} \oplus V, \pi_2, \alpha_{\mathfrak{g}} \oplus \beta)$) with coefficients in itself. Further, for any $f \in \mathcal{C}^{n-1}_{\mathrm{HLts}}(\mathfrak{g}, V)$, we have

$$\widetilde{\delta_1 f} = \widetilde{\delta}_1(\widetilde{f}) = (-1)^{n-1}[\pi_1, \widetilde{f}]_{\mathrm{Hlts}}, \widetilde{\delta_2 f} = \widetilde{\delta}_2(\widetilde{f}) = (-1)^{n-1}[\pi_2, \widetilde{f}]_{\mathrm{Hlts}}.$$

Then, we have the following conclusion.

Lemma 3.1 *The coboundary operators δ_1 and δ_2 satisfy the following compatibility*

$$\delta_1 \circ \delta_2 + \delta_2 \circ \delta_1 = 0.$$

Proof For any $f \in \mathcal{C}^{n-1}_{\mathrm{HLts}}(\mathfrak{g}, V)$, we have

$$\begin{aligned}
&(\widetilde{\delta_1 \circ \delta_2 + \delta_2 \circ \delta_1})(f) = \widetilde{\delta_1(\delta_2 f)} + \widetilde{\delta_2(\delta_1 f)} = (-1)^n[\pi_1, \widetilde{\delta_2 f}]_{\mathrm{H}} + (-1)^n[\pi_2, \widetilde{\delta_1 f}]_{\mathrm{H}}\\
&= -[\pi_1, [\pi_2, \widetilde{f}]_{\mathrm{H}}]_{\mathrm{H}} - [\pi_2, [\pi_1, \widetilde{f}]_{\mathrm{H}}]_{\mathrm{H}} = -[[\pi_1, \pi_2]_{\mathrm{H}}, \widetilde{f}]_{\mathrm{H}} + [\pi_2, [\pi_1, \widetilde{f}]_{\mathrm{H}}]_{\mathrm{H}} - [\pi_2, [\pi_1, \widetilde{f}]_{\mathrm{H}}]_{\mathrm{H}}\\
&= 0.
\end{aligned}$$

Hence the result follows. □

Next, we define the cohomology of a compatible Hom-Lie triple system $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ with coefficients in a represenation $(V, \beta; \theta_1, \theta_2)$. For each $n \geq 1$, we define the set of n-cochains by

$$\mathcal{C}^n_{\mathrm{cHLts}}(\mathfrak{g}, V) := \overbrace{\mathcal{C}^n_{\mathrm{HLts}}(\mathfrak{g}, V) \oplus \cdots \oplus \mathcal{C}^n_{\mathrm{HLts}}(\mathfrak{g}, V)}^{n \text{ times}}.$$

Define a linear map $\delta_c : \mathcal{C}^n_{\mathrm{cHLts}}(\mathfrak{g}, V) \to \mathcal{C}^{n+1}_{\mathrm{cHLts}}(\mathfrak{g}, V)$ for $n \geq 1$ by

$$\begin{aligned}
&\text{if } \ f \in \mathcal{C}^1_{\mathrm{cHLts}}(\mathfrak{g}, V), \quad \delta_c f = (\delta_1 f, \delta_2 f),\\
&\text{if } \ (f_1, \ldots, f_n) \in \mathcal{C}^n_{\mathrm{cHLts}}(\mathfrak{g}, V), \quad n \geq 2, \quad 2 \leq i \leq n-1,\\
&\qquad \delta_c(f_1, \ldots, f_n) = (\delta_1 f_1, \delta_1 f_2 + \delta_2 f_1, \ldots, \underbrace{\delta_1 f_i + \delta_2 f_{i-1}}_{i\text{-th position}}, \ldots, \delta_2 f_n).
\end{aligned}$$

Then, we have the following proposition.

Proposition 3.2 *The pair $(\mathcal{C}^*_{\mathrm{cHLts}}(\mathfrak{g}, V), \delta_c)$ is a cochain complex. So $(\delta_c)^2 = 0$.*

Proof For any $f \in \mathcal{C}^1_{\mathrm{cHLts}}(\mathfrak{g}, V)$, by Lemma 3.1, we have

$$(\delta_c)^2 f = \delta_c(\delta_1 f, \delta_2 f) = (\delta_1\delta_1 f_1, \delta_1\delta_2 f + \delta_2\delta_1 f, \delta_2\delta_2 f) = 0.$$

Given any $(f_1, \ldots, f_n) \in \mathcal{C}^n_{\mathrm{cHLts}}(\mathfrak{g}, V)$ with $n \geq 2$, we have

$$\begin{aligned}
(\delta_c)^2(f_1, \ldots, f_n) =& \delta_c(\delta_1 f_1, \ldots, \delta_1 f_i + \delta_2 f_{i-1}, \ldots, \delta_2 f_n)\\
=& (\delta_1\delta_1 f_1, \delta_1\delta_1 f_2 + \delta_1\delta_2 f_1 + \delta_2\delta_1 f_1, \ldots,\\
& \underbrace{\delta_2\delta_2 f_{i-2} + \delta_2\delta_1 f_{i-1} + \delta_1\delta_2 f_{i-1} + \delta_1\delta_1 f_i}_{3 \leq i \leq n-1}, \ldots,\\
& \delta_2\delta_2 f_{n-1} + \delta_2\delta_1 f_n + \delta_1\delta_2 f_n, \delta_2\delta_2 f_n)\\
=& 0.
\end{aligned}$$

Thus, $(\mathcal{C}^*_{\rm cHLts}(\mathfrak{g},V),\delta_c)$ is a cochain complex. □

Denote the set of n-cocycles by $\mathcal{Z}^n_{\rm cHLts}(\mathfrak{g},V) := \{f \in \mathcal{C}^n_{\rm cHLts}(\mathfrak{g},V) | \delta_c f = 0\}$, the set of n-coboundaries by $\mathcal{B}^n_{\rm cHLts}(\mathfrak{g},V) := \{\delta_c f \mid f \in \mathcal{C}^{n-1}_{\rm cHLts}(\mathfrak{g},V)\}$, and n-th cohomology group by

$$\mathcal{H}^n_{\rm cHLts}(\mathfrak{g},V) = \frac{\mathcal{Z}^n_{\rm cHLts}(\mathfrak{g},V)}{\mathcal{B}^n_{\rm cHLts}(\mathfrak{g},V)}, \quad n \geq 1.$$

Let $(\mathfrak{g},[-,-,-]^1_{\mathfrak{g}},[-,-,-]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ be a compatible Hom-Lie triple system and $(V,\beta;\theta_1,\theta_2)$ be a representation of it. Then in view of Remark 2.19, $V_+ = (V,\beta;\theta_1+\theta_2)$ is a representation of the Hom-Lie triple system $\mathfrak{g}_+ = (\mathfrak{g},[-,-,-]^1_{\mathfrak{g}}+[-,-,-]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$. Consider the cochain complex $(\mathcal{C}^*_{\rm cHLts}(\mathfrak{g},V),\delta_c)$ of the compatible Hom-Lie triple system $\mathfrak{g}$ with coefficients in the representation V, and the cochain complex $(\mathcal{C}^*_{\rm HLts}(\mathfrak{g}_+,V_+),\delta)$ of the Hom-Lie triple system $\mathfrak{g}_+$ with coefficients in V_+.

For each $n \geq 1$, define a map $\Phi_n : \mathcal{C}^n_{\rm cHLts}(\mathfrak{g},V) \to \mathcal{C}^n_{\rm HLts}(\mathfrak{g}_+,V_+)$ by

$$\Phi_n(f_1,\dots,f_n) = f_1+\cdots+f_n, \forall\ (f_1,\dots,f_n) \in \mathcal{C}^n_{\rm cHLts}(\mathfrak{g},V).$$

Then, we have

$$\begin{aligned}(\delta\circ\Phi_n)(f_1,\dots,f_n) &= \delta(f_1+\cdots+f_n) = \delta_1(f_1,\dots,f_n)+\delta_2(f_1,\dots,f_n)\\ &= \Phi_{n+1}(\delta_1 f_1,\dots,\delta_1 f_i+\delta_2 f_{i-1},\dots,\delta_2 f_n)\\ &= (\Phi_{n+1}\circ\delta_c)(f_1,\dots,f_n).\end{aligned}$$

So we have the following theorem.

Theorem 3.3 *The collection $\{\Phi_n\}_{n\geq 1}$ defines a morphism of cochain complexes from $(\mathcal{C}^*_{\rm cHLts}(\mathfrak{g},V),\delta_c)$ to $(\mathcal{C}^*_{\rm HLts}(\mathfrak{g}_+,V_+),\delta)$. So, it induces a morphism $\mathcal{H}^*_{\rm cHLts}(\mathfrak{g},V) \to \mathcal{H}^*_{\rm HLts}(\mathfrak{g}_+,V_+)$ between corresponding cohomologies.*

4. Linear deformations of compatible Hom-Lie triple systems

In [23], Chen and his collaborators studied the linear deformation of Hom-Lie triple systems. In this section, we study linear deformations of a compatible Hom-Lie triple system. We introduce Nijenhuis operators that generate trivial linear deformations.

Definition 4.1 *Let $(\mathfrak{g},[-,-,-]^1_{\mathfrak{g}},[-,-,-]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ be a compatible Hom-Lie triple system with $\pi_1 = [-,-,-]^1_{\mathfrak{g}}, \pi_2 = [-,-,-]^2_{\mathfrak{g}}$ and $(\mu_1,\mu_2),(\omega_1,\omega_2) \in \mathcal{C}^2_{\rm cHLts}(\mathfrak{g},\mathfrak{g})$ be two 2-cochains. Define two trilinear operations on $\mathfrak{g}$ depending on the parameter t as follows:*

$$\begin{aligned}&[a,b,c]^1_t = \pi_1(a,b,c)+t\mu_1(a,b,c)+t^2\omega_1(a,b,c),\\ &[a,b,c]^2_t = \pi_2(a,b,c)+t\mu_2(a,b,c)+t^2\omega_2(a,b,c),\ \forall\ a,b,c\in\mathfrak{g}.\end{aligned}$$

If for all t, the quadruple $(\mathfrak{g},[-,-,-]^1_t,[-,-,-]^2_t,\alpha_{\mathfrak{g}})$ is a compatible Hom-Lie triple system, then we say that $((\mu_1,\mu_2),(\omega_1,\omega_2))$ generates a linear deformation of the compatible Hom-Lie triple system $(\mathfrak{g},[-,-,-]^1_{\mathfrak{g}},[-,-,-]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$.

The quadruple $(\mathfrak{g}, [-,-,-]_t^1, [-,-,-]_t^2, \alpha_\mathfrak{g})$ is a compatible Hom-Lie triple system is equivalent to saying that

$$[\pi_1 + t\mu_1 + t^2\omega_1, \pi_1 + t\mu_1 + t^2\omega_1]_{\mathrm{Hlts}} = 0,$$
$$[\pi_1 + t\mu_1 + t^2\omega_1, \pi_2 + t\mu_2 + t^2\omega_2]_{\mathrm{Hlts}} = 0,$$
$$[\pi_2 + t\mu_2 + t^2\omega_2, \pi_2 + t\mu_2 + t^2\omega_2]_{\mathrm{Hlts}} = 0,$$

which force that

$$[\pi_1, \mu_1]_{\mathrm{Hlts}} = 0, 2[\pi_1, \omega_1]_{\mathrm{Hlts}} + [\mu_1, \mu_1]_{\mathrm{Hlts}} = 0, [\mu_1, \omega_1]_{\mathrm{Hlts}} = 0, [\omega_1, \omega_1]_{\mathrm{Hlts}} = 0, [\omega_1, \omega_2]_{\mathrm{Hlts}} = 0,$$
$$[\pi_1, \mu_2]_{\mathrm{Hlts}} + [\mu_1, \pi_2]_{\mathrm{Hlts}} = 0, [\pi_1, \omega_2]_{\mathrm{Hlts}} + [\omega_1, \pi_2]_{\mathrm{Hlts}} + [\mu_1, \mu_2]_{\mathrm{Hlts}} = 0, [\mu_1, \omega_2]_{\mathrm{Hlts}} + [\omega_1, \mu_2]_{\mathrm{Hlts}} = 0,$$
$$[\pi_2, \mu_2]_{\mathrm{Hlts}} = 0, 2[\pi_2, \omega_2]_{\mathrm{Hlts}} + [\mu_2, \mu_2]_{\mathrm{Hlts}} = 0, [\mu_2, \omega_2]_{\mathrm{Hlts}} = 0, [\omega_2, \omega_2]_{\mathrm{Hlts}} = 0.$$

Then

$$\begin{aligned}\delta_c(\mu_1, \mu_2) =& (\delta_1\mu_1, \delta_1\mu_2 + \delta_2\mu_1, \delta_2\mu_2)\\ =& ([\pi_1, \mu_1]_{\mathrm{Hlts}}, [\pi_1, \mu_2]_{\mathrm{Hlts}} + [\pi_2, \mu_1]_{\mathrm{Hlts}}, [\pi_2, \mu_2]_{\mathrm{Hlts}}) = 0.\end{aligned}$$

Therefore, $(\mu_1, \mu_2) \in \mathcal{C}^2_{\mathrm{cHLts}}(\mathfrak{g}, \mathfrak{g})$ is a 2-cocycle in the cohomology of the compatible Hom-Lie triple system $\mathfrak{g}$ with coefficients in the adjoint representation. Moreover, (μ_1, μ_2) is called the infinitesimal of $([-,-,-]_t^1, [-,-,-]_t^2)$.

Definition 4.2 *Let $((\mu_1, \mu_2), (\omega_1, \omega_2))$ and $((\mu_1', \mu_2'), (\omega_1', \omega_2'))$ generate linear deformations $(\mathfrak{g}, [-,-,-]_t^1, [-,-,-]_t^2, \alpha_\mathfrak{g})$ and $(\mathfrak{g}, [-,-,-]_t'^1, [-,-,-]_t'^2, \alpha_\mathfrak{g})$ of a compatible Hom-Lie triple system $\mathfrak{g}$. They are said to be equivalent if there exists a linear map $N : \mathfrak{g} \to \mathfrak{g}$ such that $N_t = \mathrm{id}_\mathfrak{g} + tN$ satisfying*

$$\alpha_\mathfrak{g} \circ N_t = N_t \circ \alpha_\mathfrak{g}, \tag{4.1}$$
$$N_t[a, b, c]_t^1 = [N_ta, N_tb, N_tc]_t'^1, \ \ N_t[a, b, c]_t^2 = [N_ta, N_tb, N_tc]_t'^2, \quad \forall a, b, c \in \mathfrak{g}. \tag{4.2}$$

We know that Eqs. (4.1) and (4.2) are equivalent to

$$\alpha_\mathfrak{g} \circ N = N \circ \alpha_\mathfrak{g}, \tag{4.3}$$
$$N\pi_i(a, b, c) + \mu_i(a, b, c) = \pi_i(Na, b, c) + \pi_i(a, Nb, c) + \pi_i(a, b, Nc) + \mu_i'(a, b, c), \tag{4.4}$$
$$\begin{aligned}\omega_i(a, b, c) + N\mu_i(a, b, c) =& \pi_i(Na, Nb, c) + \pi_i(Na, b, Nc) + \pi_i(a, Nb, Nc) + \mu_i'(Na, b, c)+\\ & \mu_i'(a, Nb, c) + \mu_i'(a, b, Nc) + \omega_i'(a, b, c),\end{aligned} \tag{4.5}$$
$$\begin{aligned}N\omega_i(a, b, c) =& \pi_i(Na, Nb, Nc) + \mu_i'(Na, Nb, c) + \mu_i'(Na, b, Nc) + \mu_i'(a, Nb, Nc)+\\ & \omega_i'(Na, b, c) + \omega_i'(a, Nb, c) + \omega_i'(a, b, Nc),\end{aligned} \tag{4.6}$$
$$\mu_i'(Na, Nb, Nc) + \omega_i'(Na, Nb, c) + \omega_i'(Na, b, Nc) + \omega_i'(a, Nb, Nc) = 0, \tag{4.7}$$
$$\omega_i'(Na, Nb, Nc) = 0 \tag{4.8}$$

for any $a, b, c \in \mathfrak{g}$ and $i = 1, 2$.

By Eq. (4.4), we have $\mu_i - \mu_i' = \delta_i N$, where N is considered as an element in $\mathcal{C}^1_{\mathrm{cHLts}}(\mathfrak{g}, \mathfrak{g})$. So $(\mu_1 - \mu_1', \mu_2 - \mu_2') = (\delta_1 N, \delta_2 N) = \delta_c N \in \mathcal{B}^2_{\mathrm{cHLts}}(\mathfrak{g}, \mathfrak{g})$. Hence, their cohomology classes are the same in $\mathcal{H}^2_{\mathrm{cHLts}}(\mathfrak{g}, \mathfrak{g})$. To sum up, we have the following result.

Proposition 4.3 *Let* $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ *be a compatible Hom-Lie triple system. Then there is a bijection between the set of all equivalence classes of infinitesimal of linear deformation of* $\mathfrak{g}$ *and the second cohomology group* $\mathcal{H}^2_{\mathrm{cHLts}}(\mathfrak{g}, \mathfrak{g})$.

Now we consider trivial linear deformations in order to introduce the notion of a Nijenhuis operator on a compatible Hom-Lie triple system.

Definition 4.4 *A linear deformation* $(\pi_1 + t\mu_1 + t^2\omega_1, \pi_2 + t\mu_2 + t^2\omega_2)$ *of compatible Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ *is said to be trivial if the deformation is equivalent to the undeformed one* (π_1, π_2).

Therefore, a linear deformation $(\pi_1 + t\mu_1 + t^2\omega_1, \pi_2 + t\mu_2 + t^2\omega_2)$ is trivial if and only if there exists a linear map $N : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$\alpha_{\mathfrak{g}} \circ N = N \circ \alpha_{\mathfrak{g}}, \tag{4.9}$$

$$N\pi_i(a,b,c) + \mu_i(a,b,c) = \pi_i(Na,b,c) + \pi_i(a,Nb,c) + \pi_i(a,b,Nc), \tag{4.10}$$

$$\omega_i(a,b,c) + N\mu_i(a,b,c) = \pi_i(Na,Nb,c) + \pi_i(Na,b,Nc) + \pi_i(a,Nb,Nc), \tag{4.11}$$

$$N\omega_i(a,b,c) = \pi_i(Na,Nb,Nc) \tag{4.12}$$

for any $a,b,c \in \mathfrak{g}$ and $i = 1,2$.

Definition 4.5 *A Nijenhuis operator on a compatible Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ *is a linear map* $N : \mathfrak{g} \to \mathfrak{g}$ *which is a Nijenhuis operator for both the Hom-Lie triple systems* $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ *and* $(\mathfrak{g}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ *, i.e., N satisfies*

$$\begin{aligned}&\alpha_{\mathfrak{g}} \circ N = N \circ \alpha_{\mathfrak{g}},\\ &[Na,Nb,Nc]^i_{\mathfrak{g}} = N([Na,Nb,c]^i_{\mathfrak{g}} + [Na,b,Nc]^i_{\mathfrak{g}} + [Na,Nb,Nc]^i_{\mathfrak{g}}) -\\ &\quad N^2([Na,b,c]^i_{\mathfrak{g}} + [a,Nb,c]^i_{\mathfrak{g}} + [a,b,Nc]^i_{\mathfrak{g}}) + N^3[a,b,c]^i_{\mathfrak{g}} \ \text{ for } \ i = 1,2.\end{aligned}$$

Explicitly, any trivial linear deformation of a compatible Hom-Lie algebra can characterize the Nijenhuis operator. Next, the converse is given by the next result whose proof is straightforward.

Proposition 4.6 *Let* $N : \mathfrak{g} \to \mathfrak{g}$ *be a Nijenhuis operator on a compatible Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$. *Then* $((\mu_1, \mu_2), (\omega_1, \omega_2))$ *generates a trivial linear deformation of the compatible Hom-Lie triple system* $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$*, where*

$$\begin{aligned}\mu_i(a,b,c) &= [Na,b,c]^i_{\mathfrak{g}} + [a,Nb,c]^i_{\mathfrak{g}} + [a,b,Nc]^i_{\mathfrak{g}} - N[a,b,c]^i_{\mathfrak{g}},\\ \omega_i(a,b,c) &= [Na,Nb,c]^i_{\mathfrak{g}} + [Na,b,Nc]^i_{\mathfrak{g}} + [a,Nb,Nc]^i_{\mathfrak{g}} - N\mu_i(a,b,c)\end{aligned}$$

for any $a,b,c \in \mathfrak{g}$ *and* $i = 1,2$.

5. Abelian extensions of compatible Hom-Lie triple systems

In this section, we study abelian extensions of compatible Hom-Lie triple systems and give a classification of equivalence classes of abelian extensions.

Let $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ be a compatible Hom-Lie triple system and (V, α_V) be a

Hom-vector space. Note that (V,α_V) can be considered as a compatible Hom-Lie triple system with both the Hom-Lie triple system brackets on V are trivial.

Definition 5.1 *Let $(\mathfrak{g},[-,-,-]^1_{\mathfrak{g}},[-,-,-]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ and $(V,[-,-,-]^1_V,[-,-,-]^2_V,\alpha_V)$ be two compatible Hom-Lie triple systems. An abelian extension of $(\mathfrak{g},[-,-,-]^1_{\mathfrak{g}},[-,-,-]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ by $(V,[-,-,-]^1_V,[-,-,-]^2_V,\alpha_V)$ is a short exact sequence of homomorphisms of compatible Hom-Lie triple systems:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_V} & & \downarrow{\scriptstyle \alpha_{\hat{\mathfrak{g}}}} & & \downarrow{\scriptstyle \alpha_{\mathfrak{g}}} & & \\ 0 & \longrightarrow & V & \xrightarrow{i} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

Diagram 1 The commutative diagram with short exact rows of compatible Hom-Lie triple systems

such that $\alpha_V(u)=\alpha_{\hat{\mathfrak{g}}}(u)$ and $[-,u,v]^i_{\hat{\mathfrak{g}}}=0$, for $u,v\in V$ and $i=1,2$, i.e., V is an abelian ideal of $\hat{\mathfrak{g}}$.

A section of an abelian extension $(\hat{\mathfrak{g}},[-,-,-]^1_{\hat{\mathfrak{g}}},[-,-,-]^2_{\hat{\mathfrak{g}}},\alpha_{\hat{\mathfrak{g}}})$ of $(\mathfrak{g},[-,-,-]^1_{\mathfrak{g}},[-,-,-]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ by $(V,[-,-,-]^1_V,[-,-,-]^2_V,\alpha_V)$ is a linear map $\sigma:\mathfrak{g}\to\hat{\mathfrak{g}}$ such that $p\circ\sigma=\mathrm{id}_{\mathfrak{g}}$ and $\alpha_{\hat{\mathfrak{g}}}\circ\sigma=\sigma\circ\alpha_{\mathfrak{g}}$. Note that an abelian extension induces a representation of the compatible Hom-Lie triple system $(\mathfrak{g},[-,-,-]^1_{\mathfrak{g}},[-,-,-]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ on (V,α_V) with the maps

$$\theta^i_V(a,b)(u)=[u,\sigma(a),\sigma(b)]^i_{\hat{\mathfrak{g}}},\quad \forall\, a,b\in\mathfrak{g},u\in V \text{ and } i=1,2. \tag{5.1}$$

Clearly, $D^i_V(a,b)(u)=[\sigma(a),\sigma(b),u]^i_{\hat{\mathfrak{g}}}=\theta^i_V(b,a)(u)-\theta^i_V(a,b)(u)$. It is easy to see that the above representation is independent of the choice of the section σ.

Let $\sigma:\mathfrak{g}\to\hat{\mathfrak{g}}$ be a section. Define linear maps $\mu_1,\mu_2\in\mathrm{Hom}(\wedge^2\mathfrak{g}\otimes\mathfrak{g},V)$ by

$$\mu_i(a,b,c)=[\sigma(a),\sigma(b),\sigma(c)]^i_{\hat{\mathfrak{g}}}-\sigma([a,b,c]^i_{\mathfrak{g}}),\quad \forall\, a,b,c\in\mathfrak{g} \text{ and } i=1,2. \tag{5.2}$$

We transfer the compatible Hom-Lie triple system structure on $\hat{\mathfrak{g}}$ to $\mathfrak{g}\oplus V$ by endowing $\mathfrak{g}\oplus V$ with two multiplications $[-,-,-]_{\mu_1}$ and $[-,-,-]_{\mu_2}$ defined by

$$[(a,u),(b,v),(c,w)]_{\mu_i}=([a,b,c]^i_{\mathfrak{g}},\theta^i_V(b,c)u-\theta^i_V(a,c)v+D^i_V(a,b)w+\mu_i(a,b,c)) \tag{5.3}$$

for any $a,b,c\in\mathfrak{g},u,v,w\in V$ and $i=1,2$. Then, it is routine to check that $(\mathfrak{g}\oplus V,[-,-,-]_{\mu_1},[-,-,-]_{\mu_2},\alpha_{\mathfrak{g}}\oplus\alpha_V)$ is a compatible Hom-Lie triple system and $(\mu_1,\mu_2)\in\mathcal{C}^2_{\mathrm{cHLts}}(\mathfrak{g},V)$ is a 2-cocycle in the cohomology of the compatible Hom-Lie triple system $\mathfrak{g}$ with coefficients in the representation $(V,\alpha_V;\theta^1_V,\theta^2_V)$.

Definition 5.2 *Let $(\hat{\mathfrak{g}}_1,[-,-,-]^1_{\hat{\mathfrak{g}}_1},[-,-,-]^2_{\hat{\mathfrak{g}}_1},\alpha_{\hat{\mathfrak{g}}_1})$ and $(\hat{\mathfrak{g}}_2,[-,-,-]^1_{\hat{\mathfrak{g}}_2},[-,-,-]^2_{\hat{\mathfrak{g}}_2},\alpha_{\hat{\mathfrak{g}}_2})$ be two abelian extensions of $(\mathfrak{g},[-,-,-]^1_{\mathfrak{g}},[-,-,-]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ by $(V,[-,-,-]^1_V,[-,-,-]^2_V,\alpha_V)$. They are said to be equivalent if there is an isomorphism of compatible Hom-Lie triple system ζ : $(\hat{\mathfrak{g}}_1,[-,-,-]^1_{\hat{\mathfrak{g}}_1},[-,-,-]^2_{\hat{\mathfrak{g}}_1},\alpha_{\hat{\mathfrak{g}}_1})\to(\hat{\mathfrak{g}}_2,[-,-,-]^1_{\hat{\mathfrak{g}}_2},[-,-,-]^2_{\hat{\mathfrak{g}}_2},\alpha_{\hat{\mathfrak{g}}_2})$ such that the following dia-*

gram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i_1} & \hat{\mathfrak{g}}_1 & \xrightarrow{p_1} & \mathfrak{g} & \longrightarrow & 0 \\ & & \Big\| & & \zeta\Big\downarrow & & \Big\| & & \\ 0 & \longrightarrow & V & \xrightarrow{i_2} & \hat{\mathfrak{g}}_2 & \xrightarrow{p_2} & \mathfrak{g} & \longrightarrow & 0. \end{array}$$

Diagram 2 The commutative diagram of ζ

Next we are ready to classify abelian extensions of a compatible Hom-Lie triple systems.

Theorem 5.3 *Abelian extensions of a compatible Hom-Lie triple systems $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ by $(V, [-,-,-]^1_V, [-,-,-]^2_V, \alpha_V)$ are classified by the second cohomology group $\mathcal{H}^2_{\mathrm{cHLts}}(\mathfrak{g}, V)$ of $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ with coefficients in the representation $(V, \alpha_V; \theta^1_V, \theta^2_V)$.*

Proof Let $(\hat{\mathfrak{g}}, [-,-,-]^1_{\hat{\mathfrak{g}}}, [-,-,-]^2_{\hat{\mathfrak{g}}}, \alpha_{\hat{\mathfrak{g}}})$ be an abelian extension of $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ by $(V, [-,-,-]^1_V, [-,-,-]^2_V, \alpha_V)$. We choose a section $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ to obtain a 2-cocycle (μ_1, μ_2). First, we show that the cohomology class of (μ_1, μ_2) is independent of the choice of σ. Let $\sigma_1, \sigma_2 : \mathfrak{g} \to \hat{\mathfrak{g}}$ be two distinct sections providing 2-cocycles (μ_1, μ_2) and (ν_1, ν_2), respectively. Define linear map $\xi : \mathfrak{g} \to V$ by $\xi(a) = \sigma_1(a) - \sigma_2(a)$. Then, we get

$$(\mu_1, \mu_2) - (\nu_1, \nu_2) = (\delta_1 \xi, \delta_2 \xi) = \delta_c(\xi) \in \mathcal{B}^2_{\mathrm{cHLts}}(\mathfrak{g}, V).$$

So (μ_1, μ_2) and (ν_1, ν_2) are in the same cohomology class in $\mathcal{H}^2_{\mathrm{cHLts}}(\mathfrak{g}, V)$.

Next, assume that $(\hat{\mathfrak{g}}_1, [-,-,-]^1_{\hat{\mathfrak{g}}_1}, [-,-,-]^2_{\hat{\mathfrak{g}}_1}, \alpha_{\hat{\mathfrak{g}}_1})$ and $(\hat{\mathfrak{g}}_2, [-,-,-]^1_{\hat{\mathfrak{g}}_2}, [-,-,-]^2_{\hat{\mathfrak{g}}_2}, \alpha_{\hat{\mathfrak{g}}_2})$ are two equivalent abelian extensions of $(\mathfrak{g}, [-,-,-]^1_{\mathfrak{g}}, [-,-,-]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ by $(V, [-,-,-]^1_V, [-,-,-]^2_V, \alpha_V)$ with the associated isomorphism $\zeta : (\hat{\mathfrak{g}}_1, [-,-,-]^1_{\hat{\mathfrak{g}}_1}, [-,-,-]^2_{\hat{\mathfrak{g}}_1}, \alpha_{\hat{\mathfrak{g}}_1}) \to (\hat{\mathfrak{g}}_2, [-,-,-]^1_{\hat{\mathfrak{g}}_2}, [-,-,-]^2_{\hat{\mathfrak{g}}_2}, \alpha_{\hat{\mathfrak{g}}_2})$ such that the Diagram 2 is commutative. Let σ_1 be a section of $(\hat{\mathfrak{g}}_1, [-,-,-]^1_{\hat{\mathfrak{g}}_1}, [-,-,-]^2_{\hat{\mathfrak{g}}_1}, \alpha_{\hat{\mathfrak{g}}_1})$. As $p_2 \circ \zeta = p_1$, we have

$$p_2 \circ (\zeta \circ \sigma_1) = p_1 \circ \sigma_1 = \mathrm{id}_{\mathfrak{g}},$$

that is, $\zeta \circ \sigma_1$ is a section of $(\hat{\mathfrak{g}}_2, [-,-,-]^1_{\hat{\mathfrak{g}}_2}, [-,-,-]^2_{\hat{\mathfrak{g}}_2}, \alpha_{\hat{\mathfrak{g}}_2})$, denote $\sigma_2 := \zeta \circ \sigma_1$. Since ζ is a isomorphism of compatible Hom-Lie triple system such that $\zeta|_V = \mathrm{id}_V$, then for any $a, b, c \in \mathfrak{g}$, we have

$$\begin{aligned} \nu_1(a,b,c) &= [\sigma_2(a), \sigma_2(b), \sigma_2(c)]^1_{\hat{\mathfrak{g}}_2} - \sigma_2([a,b,c]^1_{\mathfrak{g}_2}) \\ &= [\zeta(\sigma_1(a)), \zeta(\sigma_1(b)), \zeta(\sigma_1(c))]^1_{\hat{\mathfrak{g}}_2} - \zeta(\sigma_1([a,b,c]^1_{\mathfrak{g}_1})) \\ &= \zeta([\sigma_1(a), \sigma_1(b), \sigma_1(c)]^1_{\hat{\mathfrak{g}}_1} - \sigma_1([a,b,c]^1_{\mathfrak{g}_1})) \\ &= \zeta(\mu_1(a,b,c)) = \mu_1(a,b,c). \end{aligned}$$

Similarly, we have $\nu_2 = \mu_2$. Hence, all equivalent abelian extensions give rise to the same element in $\mathcal{H}^2_{\mathrm{cHLts}}(\mathfrak{g}, V)$.

Conversely, given two cohomologous 2-cocycles (μ_1, μ_2) and (ν_1, ν_2) in $\mathcal{H}^2_{\mathrm{cHLts}}(\mathfrak{g}, V)$, we can construct two abelian extensions $(\mathfrak{g} \oplus V, [-,-,-]_{\mu_1}, [-,-,-]_{\mu_2}, \alpha_{\mathfrak{g}} \oplus \alpha_V)$ and $(\mathfrak{g} \oplus V, [-,-,-]_{\nu_1}, [-,-,-]_{\nu_2}, \alpha_{\mathfrak{g}} \oplus \alpha_V)$ via Eq. (5.3). Then there is a linear map $\xi : \mathfrak{g} \to V$ such that

$$(\mu_1, \mu_2) - (\nu_1, \nu_2) = (\delta_1 \xi, \delta_2 \xi) = \delta_c(\xi).$$

Define a linear map $\zeta_\xi : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ by $\zeta_\xi(a, u) := (a, \xi(a) + u)$ for $a \in \mathfrak{g}, u \in V$. It is obvious that ζ_ξ is an isomorphism of these two abelian extensions such that the Diagram 2 is commutative. □

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