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Properties and Characteristics of Certain Subclass of Close-to-Convex Harmonic Mappings

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Abstract In this paper, we study a subclass of close-to-convex harmonic mappings whose analytic parts are starlike mappings. We derive some properties and characteristics for this class, such as the bounds of Toeplitz determinants, bounds of Hankel determinants, Zalcman functional and Bohr's inequality.

Keywords univalent harmonic mapping; close-to-convex harmonic mapping; Bohr radius; Toeplitz determinant; Hankel determinant

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1. Introduction

Let $\mathcal{A}$ denote the class of functions h of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk $\mathbb{D} := \{z : |z| < 1\}$. Also, let $\mathcal{G}(\beta)$ be the subclass of $\mathcal{A}$ whose members satisfy the inequality

$$\mathrm{Re}(1 + \frac{zh''(z)}{h'(z)}) < \beta, \quad \beta > 1;\ z \in \mathbb{D}. \tag{1.2}$$

For convenience, we denote by $\mathcal{G}(3/2) =: \mathcal{G}$. The class $\mathcal{G}$ plays an important role in the geometry function theory.

In 1995, Ponnusamy and Rajasekaran [1] proved that the class $\mathcal{G}(\beta)$ is starlike in $\mathbb{D}$ for $\beta \in (1, 3/2]$ (see also Singh and Singh [2]). In 2013, Obradović et al. [3] pointed out that the

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class $\mathcal{G}(\beta)$ is not univalent in $\mathbb{D}$ for $\beta \in (3/2, +\infty)$, but they did not give detailed proof about the non-univalency of the class $\mathcal{G}(\beta)$. Later, Kargar et al. [4] proved that the class $\mathcal{G}(\beta)$ is not univalent in $\mathbb{D}$ for $\beta \in [2, +\infty)$ through a counterexample. Recently, Wang et al. [5] gave a counterexample to clarify the non-univalency of the class $\mathcal{G}(\beta)$ for $\beta \in (3/2, 2)$. For more recent results involving the class starlike functions, one can also refer to Kanas et al. [6], Maharana et al. [7] and Wang et al. [8].

Let $\mathcal{H}$ denote the class of normalized and sense-preserving harmonic mappings $f = h + \overline{g}$, which are given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}. \tag{1.3}$$

Lewy [9] once proved that $f = h + \overline{g}$ is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f = |h'|^2 - |g'|^2 \neq 0$ in $\mathbb{D}$. Note that the harmonic mapping f is sense-preserving if $J_f > 0$ or $|h'| > |g'|$ in $\mathbb{D}$, or its dilatation $\omega_f = g'/h'$ has the property $|\omega_f| < 1$ in $\mathbb{D}$. Let $\mathcal{S}_{\mathcal{H}}$ be the subclass of $\mathcal{H}$ consisting of univalent mappings. We observe that $\mathcal{S}_{\mathcal{H}}$ reduces to the familiar class $\mathcal{S}$ of normalized univalent analytic functions, if their co-analytic parts $g \equiv 0$.

Let $\mathcal{P}$ denote the class of analytic functions p in $\mathbb{D}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.4}$$

such that $\mathrm{Re}(p(z)) > 0$ in $\mathbb{D}$.

We recall the following sufficient condition for close-to-convexity of harmonic mappings, which was due to Abu-Muhanna and Ponnusamy [10] (see also [11, 12]).

Theorem 1.1 *Let h and g be normalized analytic functions in $\mathbb{D}$ such that*

$$\mathrm{Re}(1 + \frac{zh''(z)}{h'(z)}) < \frac{3}{2}$$

and

$$g'(z) = \lambda z^n h'(z), \quad 0 < |\lambda| \leq \frac{1}{n+1}; \, n \in \mathbb{N} := \{1, 2, 3, \ldots\}.$$

Then the harmonic mapping $f = h + \overline{g}$ is univalent and close-to-convex in $\mathbb{D}$.

Motivated essentially by Theorem 1.1, we introduce and investigate the following subclass $\mathcal{F}(\beta, \lambda, \gamma, n)$ of harmonic mappings.

Definition 1.2 *A harmonic mapping $f = h + \overline{g} \in \mathcal{H}$ is said to be in the class $\mathcal{F}(\beta, \lambda, \gamma, n)$, if h and g satisfy the conditions*

$$\mathrm{Re}(1 + \frac{zh''(z)}{h'(z)}) < \beta, \quad 1 < \beta \leq \frac{3}{2}, \tag{1.5}$$

and

$$g'(z) = (\lambda z^n + \gamma)h'(z), \;\; \lambda, \gamma \in \mathbb{C} \text{ with } |\gamma| < 1, \; 0 < |\gamma| + (n+1)|\lambda| \leq 1; \, n \in \mathbb{N}. \tag{1.6}$$

In Section 2, we will show that the class $\mathcal{F}(\beta, \lambda, \gamma, n)$ is a subclass of close-to-convex harmonic mappings. We also observe that the class $\mathcal{F}(\beta, \lambda, 0, n)$ was investigated by Wang et al. [5]. In this paper, we aim at deriving several new results for the class $\mathcal{F}(\beta,\lambda,\gamma,n)$.

Recently, the Toeplitz determinants and Hankel determinants of functions in the class $\mathcal{S}$ or its subclasses have attracted many researchers' attention [13]. Among them, the symmetric Toeplitz determinant $|T_q(n)|$ for subclasses of $\mathcal{S}$ with small values of n and q, is investigated by [14–17].

Let h be given by (1.1). Then, the q-th Hankel determinant is defined for $q\geq 1$ and $n\geq 0$ by

$$H_q(n)(h):=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}. \tag{1.7}$$

We easily find that

$$H_2(2)(h)=a_2a_4-a_3^2.$$

The symmetric Toeplitz determinant $T_q(n)$ for analytic functions h is defined as follows:

$$T_q(n)[h]:=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_n & \cdots & a_{n+q-2}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n\end{vmatrix}, \tag{1.8}$$

where $n,q\in\mathbb{N}$ and $a_1=1$. In particular, for functions in starlike and convex classes, $T_2(2)[h],T_3(1)[h]$ and $T_3(2)[h]$ were studied by Ali et al. [14].

The Zalcman conjecture was posed in the early 1970s by Zalcman that if $h\in\mathcal{S}$, then

$$|a_n^2-a_{2n-1}|\leq(n-1)^2$$

for $n\geq 2$. The Zalcman conjecture reduces to the celebrated Bieberbach conjecture $|a_n|\leq n$ for $h\in\mathcal{A}$. Ma [18] generalized the Zalcman functional as follows:

$$J_{m,n}(h):=a_ma_n-a_{m+n-1}$$

for $m,n\in\mathbb{N}\backslash\{1\}$, and conjectured that if $h\in\mathcal{S}$, then for $m,n\in\mathbb{N}\backslash\{1\}$,

$$|J_{m,n}(h)|\leq(n-1)(m-1).$$

Particularly, we know that $J_{2,3}(h)=a_2a_3-a_4$ and $J_{3,3}(h)=a_3^2-a_5$.

Let $\mathcal{B}$ be the class of analytic functions f in $\mathbb{D}$ such that $|f(z)|<1$ for all $z\in\mathbb{D}$, and let $\mathcal{B}_0=\{f\in\mathcal{B}:f(0)=0\}$. In 1914, Bohr [19] proved that if $f\in\mathcal{B}$ is of the form $f(z)=\sum_{n=0}^{\infty}a_nz^n$, then the majorant series $M_f(r)=\sum_{n=0}^{\infty}|a_n||z|^n$ of f satisfies

$$M_{f_0}(r)=\sum_{n=1}^{\infty}|a_n||z|^n\leq 1-|a_0|=d(f(0),\partial f(\mathbb{D})) \tag{1.9}$$

for all $z\in\mathbb{D}$ with $|z|=r\leq 1/3$, where $f_0(z)=f(z)-f(0)$. Bohr actually obtained the inequality (1.9) for $|z|\leq 1/6$. Moreover, Wiener, Riesz and Schur, independently, established the Bohr inequality (1.9) for $|z| \leq 1/3$ (known as Bohr radius for the class $\mathcal{B}$) and proved that $1/3$ is the best possible.

In this paper, we aim at determining the estimates for Toeplitz determinants, Hankel determinants and Zalcman functional of the class $\mathcal{F}(\beta, \lambda, \gamma, n)$. Moreover, we will derive the Bohr's inequality for the class $\mathcal{F}(\beta, \lambda, 0, n)$.

2. Preliminary results

To prove our main results, we need the following lemmas.

Lemma 2.1 ([10]) *Suppose that $h \in \mathcal{G}$ and satisfies the condition $g'(z) = \omega(z)h'(z)$ in $\mathbb{D}$, where $\omega : \mathbb{D} \to \mathbb{D}$ is analytic, $W(z) = z(1+\omega(z))$ is starlike in $\mathbb{D}$. Then the harmonic mapping $f = h + \overline{g}$ is close-to-convex and univalent in $\mathbb{D}$.*

The following lemma shows that the class $\mathcal{F}(\beta, \lambda, \gamma, n)$ is a subclass of close-to-convex harmonic mappings.

Lemma 2.2 *If $f \in \mathcal{F}(\beta, \lambda, \gamma, n)$, then f is a close-to-convex harmonic mapping.*

Proof Assume that $f \in \mathcal{F}(\beta, \lambda, \gamma, n)$. Then

$$W(z) = z(1 + \omega(z)) = z + z\gamma + \lambda z^{n+1}. \tag{2.1}$$

It follows from (1.6) and (2.1) that

$$\begin{aligned} |\frac{zW'(z)}{W(z)} - 1| &= |\frac{z + z\gamma + (n+1)z^{n+1}\lambda - (z + z\gamma + \lambda z^{n+1})}{z + z\gamma + \lambda z^{n+1}}| \\ &= |\frac{\lambda n z^{n+1}}{z + z\beta + \lambda z^{n+1}}| = |\frac{n\lambda z^n}{1 + \gamma + \lambda z^n}| \\ &< \frac{n|\lambda|}{1 - |\gamma| - |\lambda|} \leq 1. \end{aligned} \tag{2.2}$$

Thus, by Lemma 2.1 and Eq. (2.2), we deduce that the assertion of Lemma 2.2 is true. □

Lemma 2.3 ([3]) *If $h = z + \sum_{k=2}^{\infty} a_k z^k$ satisfies the condition (1.2) with $1 < \beta \leq 3/2$, then*

$$|a_k| \leq \frac{2(\beta - 1)}{(k-1)k}, \quad k \geq 2 \tag{2.3}$$

with the extremal function given by

$$h(z) = \int_0^z (1 - t^{k-1})^{\frac{2(\beta-1)}{k-1}} \mathrm{d}t, \quad k \geq 2. \tag{2.4}$$

Lemma 2.4 ([20, p. 41]) *For a function $p \in \mathcal{P}$ of the form (1.2), the sharp inequality $|p_n| \leq 2$ holds for each $n \geq 1$. Equality holds for the function $p(z) = (1+z)/(1-z)$.*

Lemma 2.5 ([21, Theorem 1]) *Let $p(z) \in \mathcal{P}$ be of the form (1.2) and $\mu \in \mathbb{C}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leq 2 \max\{1, |2\mu - 1|\}, \quad 1 \leq k \leq n-1. \tag{2.5}$$

If $|2\mu - 1| \geq 1$, then the inequality is sharp for the function $p(z) = (1+z)/(1-z)$ or its rotations.

If $|2\mu - 1| < 1$, then the inequality is sharp for $p(z) = (1 + z^n)/(1 - z^n)$ or its rotations.

Lemma 2.6 ([22, Lemma 2.3]) *Let $p(z) \in \mathcal{P}$. If $0 \le B \le 1$ and $B(2B-1) \le D \le B$, then*

$$|p_3 - 2Bp_1p_2 + Dp_1^3| \le 2.$$

Lemma 2.7 ([5]) *Let $f \in \mathcal{F}(\beta, \lambda, 0, n)$. Then*

$$L(\beta, \lambda, n, r) \le |f(z)| \le R(\beta, \lambda, n, r), \tag{2.6}$$

where

$$L(\beta, \lambda, n, r) := r[|\lambda|(\frac{r}{n+2} - \frac{1}{n+1})r^n - \frac{r}{2} + 1]$$

and

$$R(\beta, \lambda, n, r) := r[|\lambda|(\frac{r}{n+2} + \frac{1}{n+1})r^n + \frac{r}{2} + 1].$$

The inequalities are sharp.

3. Toeplitz determinants for the class $\mathcal{F}(\beta, \lambda, \gamma, n)$

In this section, we will give several estimates for Toeplitz determinants $|T_q(n)[\cdot]|$ of functions in the class $\mathcal{F}(\beta, \lambda, \gamma, n)$.

Theorem 3.1 *Let $f \in \mathcal{F}(\beta, \lambda, \gamma, n)$ be of the form (1.3). Then the coefficients a_k $(k \ge 2)$ of h satisfy (2.3) and the coefficients b_k of g satisfy*

$$|b_k| \le \begin{cases} |\gamma|, & k = 1, \\ |\gamma|\frac{2(\beta-1)}{(k-1)k}, & k = 2, \dots, n, \\ \frac{|\lambda|}{n+1} + |\gamma|\frac{2(\beta-1)}{n(n+1)}, & k = n+1, \\ \frac{|\lambda|}{k}\frac{2(\beta-1)}{(k-n-1)} + |\gamma|\frac{2(\beta-1)}{(k-1)k}, & k \ge n+2. \end{cases} \tag{3.1}$$

Proof By comparing the coefficients of each power of z on both sides of (1.6), we obtain

$$b_1 = \gamma,\ 2b_2 = 2\gamma a_2, \dots, kb_k = k\gamma a_k, \quad k \le n \tag{3.2}$$

and

$$(n+1)b_{n+1} = \lambda + \gamma(n+1)a_{n+1}, \dots, (n+m)b_{n+m} = \lambda k a_m + \gamma(n+m)a_{n+m}, \quad m \in \mathbb{N}. \tag{3.3}$$

Thus, by Lemma 2.3, (3.2) and (3.3), we conclude that the assertion of Theorem 3.1 holds. □

Theorem 3.2 *Let $f \in \mathcal{F}(\beta, \lambda, \gamma, n)$. Then*

$$|T_2(m)[h]| \le \frac{8(\beta-1)(m^2+1)}{m^2(m^2-1)^2} \tag{3.4}$$

and

$$|T_2(m)[g]| \leq \begin{cases} |\gamma|^2+[|\gamma|(\beta-1)]^2, & m=1, \\ |\gamma|^2\{[\frac{2(\beta-1)}{(m-1)m}]^2+[\frac{2(\beta-1)}{m(m+1)}]^2\}, & 2\leq m\leq n-1, \\ [|\gamma|\frac{2(\beta-1)}{(n-1)n}]^2+[\frac{|\lambda|}{n+1}+|\gamma|\frac{2(\beta-1)}{n(n+1)}]^2, & m=n, \\ [\frac{|\lambda|}{n+1}+|\gamma|\frac{2(\beta-1)}{n(n+1)}]^2+[\frac{2(\beta-1)|\lambda|}{n+2}+|\gamma|\frac{2(\beta-1)}{(n+1)(n+2)}]^2, & m=n+1, \\ [\frac{2(\beta-1)|\lambda|}{m(m-n-1)}+|\gamma|\frac{2(\beta-1)}{(m-1)m}]^2+[\frac{2(\beta-1)|\lambda|}{(m+1)(m-n)}+|\gamma|\frac{2(\beta-1)}{m(m+1)}]^2, & m\geq n+2. \end{cases} \tag{3.5}$$

Proof Suppose that $f \in \mathcal{F}(\beta,\lambda,\gamma,n)$. By Lemma 2.3, we get

$$|T_2(m)[h]| = |a_m^2 - a_{m+1}^2| \leq |a_m^2| + |a_{m+1}^2| \leq \frac{8(\beta-1)(m^2+1)}{m^2(m^2-1)^2}. \tag{3.6}$$

In view of (3.1), we obtain the assertion (3.5) of Theorem 3.2. □

Theorem 3.3 *Suppose that $f \in \mathcal{F}(\beta,\lambda,\gamma,1)$ be of the form (1.3). Then*

$$|b_3-\delta b_2^2| \leq \frac{2(\beta-1)|\lambda|}{3}+\frac{|\delta||\lambda|^2}{4}+|\gamma|\{\frac{\beta-1}{3}+|\delta|[\gamma(\beta-1)^2+|\lambda|(\beta-1)]\}. \tag{3.7}$$

Proof Let $f \in \mathcal{F}(\beta,\lambda,\gamma,1)$. In view of (3.2) and (3.3), we know that

$$\begin{cases} b_1=\gamma, \\ b_2=\frac{1}{2}\lambda+\gamma a_2, \\ b_3=\frac{2}{3}\lambda a_2+\gamma a_3. \end{cases} \tag{3.8}$$

From (2.3) and (3.8), we obtain

$$\begin{aligned} |b_3-\delta b_2^2| &= |\frac{2}{3}\lambda a_2+\gamma a_3-\delta(\frac{1}{2}\lambda+\gamma a_2)^2| \\ &\leq |\frac{2}{3}\lambda a_2-\frac{1}{4}\delta\lambda^2|+|\gamma||a_3-\delta(a_2^2\gamma+\lambda a_2)| \\ &\leq \frac{2(\beta-1)|\lambda|}{3}+\frac{|\delta||\lambda|}{4}+|\gamma|\{\frac{\beta-1}{3}+|\delta|[\gamma(\beta-1)^2+|\lambda|(\beta-1)]\}. \end{aligned} \tag{3.9}$$

Therefore, we complete the proof of Theorem 3.3. □

Theorem 3.4 *Let $f \in \mathcal{F}(\beta,\lambda,\gamma,1)$. Then*

$$|T_3(1)[h]| \leq \frac{1}{9}(4\beta^3+7\beta^2-26\beta+24) \tag{3.10}$$

and

$$|T_3(1)[g]| \leq [\frac{1}{3}(\beta-1)+|\gamma|]\{|\gamma|[|\gamma|+\frac{1}{3}(\beta-1)]+\frac{1}{8}[1+|\gamma|(4\beta-5)]^2\}. \tag{3.11}$$

Proof For $f \in \mathcal{F}(\beta,\lambda,\gamma,1)$, we see that

$$p(z)=\frac{1}{\beta-1}(\beta-1-\frac{zh''(z)}{h'(z)}) \in \mathcal{P}, \quad 1<\beta\leq\frac{3}{2};\ z\in\mathbb{D}.$$

It follows that

$$n(n-1)a_n=(1-\beta)\sum_{k=1}^{n-1}ka_kp_{n-k}, \quad n\geq 2. \tag{3.12}$$

From (3.12), we obtain

$$\begin{cases} a_2 = \frac{1}{2}(1-\beta)p_1, \\ a_3 = \frac{1}{6}(1-\beta)[(1-\beta)p_1^2 + p_2], \\ a_4 = \frac{1}{24}(1-\beta)[(1-\beta)^2p_1^3 + 3(1-\beta)p_1p_2 + 2p_3], \\ a_5 = \frac{1}{120}(1-\beta)[(1-\beta)^3p_1^4 + 6(1-\beta)^2p_1^2p_2 + 8(1-\beta)p_1p_3 + 3(1-\beta)p_2^2 + 6p_4]. \end{cases} \tag{3.13}$$

By virtue of Lemmas 2.4, 2.5 and (3.13), we get

$$\begin{aligned} |T_3(1)[h]| =& |1 - 2a_2^2 + 2a_2^2a_3 - a_3^2| \\ \le& 1 + 2|a_2^2| + |a_3||a_3 - 2a_2^2| \\ \le& 1 + \frac{1}{2}(1-\beta)^2p_1^2 + \frac{1}{36}(1-\beta)^2|p_2 - (\beta-1)p_1^2||p_2 + 2(\beta-1)p_1^2| \\ \le& \frac{1}{9}(4\beta^3 + 7\beta^2 - 26\beta + 24). \end{aligned} \tag{3.14}$$

In view of Lemmas 2.4, 2.5, (3.8) and (3.13), we get

$$\begin{aligned} |T_3(1)[g]| =& |(b_1 - b_3)[(b_1 + b_3)b_1 - 2b_2^2]| \\ \le& |\gamma + \frac{1}{3}\lambda(\beta-1)p_1 + \frac{1}{6}(\beta-1)[(1-\beta)p_1^2 + p_2]\gamma| \times \\ & |\gamma\{\gamma + \frac{1}{3}\lambda(1-\beta)p_1 + \frac{1}{6}\gamma(1-\beta)[(1-\beta)p_1^2 + p_2]\} - 2[\frac{1}{2}\lambda + \frac{1}{2}(1-\beta)\gamma p_1]^2| \\ \le& [\frac{1}{3}(\beta-1) + |\gamma|]\{|\gamma|[|\gamma| + \frac{1}{3}(\beta-1)] + \frac{1}{8}[1 + |\gamma|(4\beta-5)]^2\}. \end{aligned} \tag{3.15}$$

The proof of Theorem 3.4 is thus completed. □

Theorem 3.5 *Let $f \in \mathcal{F}(\beta, \lambda, 0, 2)$. Then*

$$|T_3(2)[h]| \le \frac{35}{108}(\beta-1)^3(2\beta^2 - 4\beta + 7) \tag{3.16}$$

and

$$|T_3(2)[g]| = |2b_3^2b_4| \le \frac{1}{243}(\beta-1). \tag{3.17}$$

Proof Suppose that $f \in \mathcal{F}(\beta, \lambda, 0, 2)$. It follows that

$$T_3(2)[h] = (a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4).$$

In view of (3.13), Lemmas 2.4 and 2.6, we find that

$$\begin{aligned} |a_2 - a_4| \le& \frac{1}{2}|(1-\beta)p_1| + \frac{1}{24}|(1-\beta)[(1-\beta)^2p_1^3 + 3(1-\beta)p_1p_2 + 2p_3]| \\ \le& \frac{7}{6}(\beta-1). \end{aligned} \tag{3.18}$$

Next, we shall maximize $|a_2^2 - 2a_3^2 + a_2a_4|$. With the help of (3.13), Lemmas 2.4 and 2.5, we get

$$\begin{aligned} |a_2^2 - 2a_3^2 + a_2a_4| =& \frac{(\beta-1)^2}{144}| - 5(\beta-1)^2p_1^4 + 36p_1^2 + 7(\beta-1)p_1^2p_2 - 8p_2^2 + 6p_1p_3| \\ \le& \frac{(\beta-1)^2}{144}[5(\beta-1)^2|p_1|^4 + 36|p_1|^2 + 8|p_2||p_2 - \frac{7}{8}(\beta-1)p_1^2| + 6|p_1||p_3|] \\ \le& \frac{5}{18}(\beta-1)^2(2\beta^2 - 4\beta + 7). \end{aligned} \tag{3.19}$$

Therefore, combining (3.18) with (3.19), we obtain the inequality (3.16). By noting that for $f \in \mathcal{F}(\beta, \lambda, 0, 2)$, we have

$$\begin{cases} b_3 = \frac{1}{3}\lambda a_1, \\ b_4 = \frac{1}{2}\lambda a_2. \end{cases} \tag{3.20}$$

By means of Lemma 2.4, we get the assertion (3.17). □

4. Hankel determinants for the class $\mathcal{F}(\beta, \lambda, \gamma, 2)$

In this section, we will give the upper bound for the second order Hankel determinants $|H_2(2)[\cdot]|$ of functions in the class $\mathcal{F}(\beta, \lambda, \gamma, 2)$.

Theorem 4.1 *Let $f \in \mathcal{F}(\beta, \lambda, \gamma, 2)$. Then*

$$|H_2(2)[h]| \leq \frac{5}{18}(\beta - 1)^2 \tag{4.1}$$

and

$$|H_2(2)[g]| \leq \frac{5}{18}|\gamma|^2(\beta - 1)^2 + \frac{2}{27}(|\gamma| - |\gamma|^2)(\beta - 1) + \frac{1}{81}(1 - |\gamma|)^2. \tag{4.2}$$

Proof By means of Lemmas 2.4, 2.5 and (3.13), we have

$$\begin{aligned} |a_2a_4 - a_3^2| =& | - \frac{1}{144}p_1^4(1 - \beta)^4 + \frac{1}{144}p_1^2p_2(1 - \beta)^3 + \frac{1}{24}p_1p_3(1 - \beta)^2 - \frac{1}{36}p_2^2(1 - \beta)^2| \\ =& | - \frac{(\beta - 1)^2}{144}[(\beta - 1)^2p_1^4 + (\beta - 1)p_1^2p_2 - 6p_1p_3 + 4p_2^2]| \\ =& | - \frac{(\beta - 1)^2}{144}\{4(p_2 - \frac{1}{2}(\beta - 1)p_1^2)^2 - 6p_1[p_3 - \frac{5}{6}(\beta - 1)p_1p_2]\}| \\ \leq& \frac{5}{18}(\beta - 1)^2. \end{aligned} \tag{4.3}$$

In view of (3.2) and (3.3), we know that

$$\begin{cases} b_2 = \gamma a_2, \\ b_3 = \gamma a_3 + \frac{\lambda}{3}, \\ b_4 = \gamma a_4 + \frac{1}{2}a_2\lambda. \end{cases} \tag{4.4}$$

By virtue of Lemmas 2.4, 2.5, (3.8), (4.3) and (4.4), we obtain

$$\begin{aligned} |b_2b_4 - b_3^2| =& |a_2\gamma(\frac{1}{2}\lambda a_2 + a_4\gamma) - (\frac{1}{3}\lambda + a_3\gamma)^2| \\ \leq& |a_2a_4\gamma^2 - a_3\gamma^2| + |\frac{1}{2}\lambda\gamma a_2^2 - \frac{2}{3}\lambda\gamma a_3 - \frac{1}{9}| \\ \leq& \frac{5}{18}|\gamma|^2(\beta - 1)^2 + \frac{|\lambda|}{18}|3\gamma\{\frac{3}{4}(1 - \beta)^2p_1^2 - \frac{2}{3}(1 - \beta)[(1 - \beta)p_1^2 + p_2]\} - 2\lambda| \\ \leq& \frac{5}{18}|\gamma|^2(\beta - 1)^2 + \frac{2}{27}(|\gamma| - |\gamma|^2)(\beta - 1) + \frac{1}{81}(1 - |\gamma|)^2. \end{aligned} \tag{4.5}$$

Therefore, we deduce that the assertion of Theorem 4.1 holds. □

5. Zalcman functional for the class $\mathcal{F}(\beta,\lambda,0,1)$

We note that

$$|J_{2,2}(h)|=|H_2(1)[h]|=|a_3-a_2^2|$$

and

$$|J_{2,2}(g)|=|b_3-b_2^2|$$

from the class $\mathcal{F}(\beta,\lambda,0,1)$ were considered by Wang et al. [5]. In what follows, we will give the estimates of Zalcman functional $|J_{3,3}|(\cdot)$ for the class $\mathcal{F}(\beta,\lambda,0,1)$.

Theorem 5.1 *Let $f\in\mathcal{F}(\beta,\lambda,0,1)$. Then*

$$|J_{3,3}(h)|\le\frac{1}{360}(\beta-1)(17\beta+19) \tag{5.1}$$

and

$$|J_{3,3}(g)|\le\frac{1}{45}(\beta-1)(5\beta-2). \tag{5.2}$$

Proof We find from (3.13) that

$$\begin{aligned}J_{3,3}(h)=&\frac{1}{360}(\beta-1)[7(\beta-1)^3p_1^4-2(\beta-1)^2p_1^2p_2+(\beta-1)p_2^2-24(\beta-1)p_1p_3+18p_4]\\=&\frac{1}{360}(\beta-1)\{\frac{7}{4}(\beta-1)[p_2-2(\beta-1)p_1^2]^2-\frac{3}{4}(\beta-1)p_2[p_2-2(\beta-1)p_1^2]-\\&\frac{7}{4}(\beta-1)p_1[p_3-2(\beta-1)p_1p_2]+18[p_4-\frac{89}{72}(\beta-1)p_1p_3]\}.\end{aligned} \tag{5.3}$$

By using Lemmas 2.4 and 2.5, we obtain the bound for the Zalcman functional $J_{3,3}(h)$. Moreover, in view of (3.2) and (3.3), we know that

$$\begin{cases}b_4=\frac{3}{4}\lambda a_3,\\ b_5=\frac{4}{5}\lambda a_4.\end{cases} \tag{5.4}$$

Then, from (3.8), (5.4), Lemmas 2.4 and 2.6, we get

$$|J_{3,3}(g)|=|b_3^2-b_5|\le|(\frac{2}{3}\lambda a_2)^2|+\frac{4}{5}|\lambda a_4|\le\frac{1}{45}(\beta-1)(5\beta-2). \tag{5.5}$$

Thus, we complete the proof of Theorem 5.1. □

6. Bohr inequality for the class $\mathcal{F}(\beta,\lambda,0,n)$

In this section, we will derive the Bohr inequality for the class $\mathcal{F}(\beta,\lambda,0,n)$.

Theorem 6.1 *Let $f\in\mathcal{F}(\beta,\lambda,0,n)$ with $1\le\beta<3/2$ and $0\le\lambda<1/(n+1)$. Then the inequality*

$$|z|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)|z|^n\le d(f(0),\partial f(\mathbb{D})) \tag{6.1}$$

holds for $|z|=r\le r_f$, where r_f is the smallest root in $(0,1)$ of

$$F_n(r):=R(\beta,\lambda,n,r)-L(\beta,\lambda,n,1)=0,$$

where $R(\beta,\lambda,n,r)$ and $L(\beta,\lambda,n,1)$ are given in Lemma 2.7. The radius r_f is sharp.

Proof By Lemma 2.7, the Euclidean distance between $f(0)$ and the boundary of $f(\mathbb{D})$ shows that

$$d(f(0),\partial f(\mathbb{D}))=\liminf_{|z|\to 1}|f(z)-f(0)|\geq L(\beta,\lambda,n,1). \tag{6.2}$$

We note that r_f is the root of the equation $R(\beta,\lambda,n,r)=L(\beta,\lambda,n,1)$ in $(0,1)$. The existence of the root is ensured by the relationship $R(\beta,\lambda,n,1)>L(\beta,\lambda,n,1)$ with (2.6). For $0<r\leq r_f$, it is evident that $R(\beta,\lambda,n,r)\leq L(\beta,\lambda,n,1)$. In view of Theorem 3.1 and Eq. (6.2) for $|z|=r\leq r_f$, we have

$$\begin{aligned}|z|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)|z|^n&\leq r_f+(|a_2|+|b_2|)r_f^2+\sum_{n=3}^{\infty}(|a_n|+|b_n|)r_f^n\\&=R(\beta,\lambda,n,r_f)\leq L(\beta,\lambda,n,1)\leq d(f(0),\partial f(\mathbb{D})).\end{aligned}$$

To show that the sharpness of the radius r_f, we consider the function $f=f_{\beta,\lambda,0,n}$, which is defined in Lemma 2.7. By noting that $f_{\beta,\lambda,0,n}$ belongs to $\mathcal{F}(\beta,\lambda,0,n)$, since the left side of the growth inequality in Lemma 2.7 holds for $f=f_{\beta,\lambda,0,n}$ or its rotations, we have

$$d(f(0),\partial f(\mathbb{D}))=L(\beta,\lambda,n,1).$$

Therefore, the function $f=f_{\beta,\lambda,0,n}$ for $|z|=r_f$ gives

$$\begin{aligned}|z|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)|z|^n&=r_f+(|a_2|+|b_2|)r_f^2+\sum_{n=3}^{\infty}(|a_n|+|b_n|)r_f^n\\&=R(\beta,\lambda,n,r_f)=L(\beta,\lambda,n,1)=d(f(0),\partial f(\mathbb{D})),\end{aligned}$$

which reveals that the radius r_f is the best possible. □

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References

[1] S. PONNUSAMY, S. RAJASEKARAN. *New sufficient conditions for starlike and univalent functions*. Soochow J. Math., 1995, **21**(2): 193–201.

[2] R. SINGH, S. SINGH. *Some sufficient conditions for univalence and starlikeness*. Colloq. Math., 1982, **47**(2): 309–314

[3] M. OBRADOVIĆ, S. PONNUSAMY, K. -J. WIRTHS. *Coefficient characterizations and sections for some univalent functions*. Sib. Math. J., 2013, **54**(4): 679–696.

[4] R. KARGAR, N. R. PASCU, A. EBADIAN. *Locally univalent approximations of analytic functions*. J. Math. Anal. Appl., 2017, **453**(2): 1005–1021.

[5] Zhigang WANG, Xinzhong HUANG, Zhihong LIU, et al. *On quasiconformal close-to-convex harmonic mappings involving starlike functions*. Acta Math. Sinica (Chinese Ser.), 2020, **63**(6): 565–576.

[6] S. KANAS, S. MAHARANA, J. K. PRAJAPAT. *Norm of the pre-Schwarzian derivative, Bloch's constant and coefficient bounds in some classes of harmonic mappings*. J. Math. Anal. Appl., 2019, **474**(2): 931–943.

[7] S. MAHARANA, J. K. PRAJAPAT, H. M. SRIVASTAVA. *The radius of convexity of partial sums of convex functions in one direction*. Proc. Nat. Acad. Sci. India Sect. A, 2017, **87**(2): 215–219.

[8] Zhigang WANG, M. HUSSAIN, Xiaoyuan WANG. *On sharp solutions to majorization and Fekete-Szegö problems for starlike functions*. Miskolc Math. Notes, 2023, **24**(2): 1003–1019.

[9] H. LEWY. *On the non-vanishing of the Jacobian in certain one-to-one mappings*. Bull. Amer. Math. Soc., 1936, **42**(10): 689–692.

[10] Y. ABU-MUHANNA, S. PONNUSAMY. *Extreme Points Method and Univalent Harmonic Mappings*. Amer. Math. Soc., Providence, RI, 2016.

[11] Á. FERRADA-SALAS, R. HERNÁNDEZ, M. J. MARTÍN. *On convex combinations of convex harmonic mappings*. Bull. Aust. Math. Soc., 2017, **96**(2): 256–262.

[12] Yong SUN, Yueping JIANG, A. RASILA. *On a subclass of close-to-convex harmonic mappings*. Complex Var. Elliptic Equ., 2016, **61**(12): 1627–1643.

[13] K. O. BABALOLA. *On $H_3(1)$ Hankel determinant for some classes of univalent functions*. a book chapter in "Inequality Theory and Applications", Eds: Y. J. Cho, J. K. Kim and S. S. Dragomir, 7 pages. Nova Science Publishers Inc. 2011.

[14] M. F. ALI, D. K. THOMAS, A. VASUDEVARAO. *Toeplitz determinants whose elements are the coefficients of analytic and univalent functions*. Bull. Aust. Math. Soc., 2018, **97**(2): 253–264.

[15] O. P. AHUJA, K. KHATTER, V. RAVICHANDRAN. *Toeplitz determinants associated with Ma-Minda classes of starlike and convex functions*. Iran. J. Sci. Technol. Trans. A Sci., 2021, **45**(6): 2021–2027.

[16] Dongrong WANG, Huaying HUANG, Boyong LONG. *Coefficient problems for subclasses of close-to-star functions*. Iran. J. Sci. Technol. Trans. A Sci., 2021, **45**(3): 1071–1077.

[17] Xiaoyuan WANG, Zhigang WANG, Jinhua FAN, et al. *Some properties of certain close-to-convex harmonic mappings*. Anal. Math. Phys., 2022, **12**(1): Paper No. 28, 21 pp.

[18] W. MA. *Generalized Zalcman conjecture for starlike and typically real functions*. J. Math. Anal. Appl., 1999, **234**(1): 328–339.

[19] H. BOHR. *A theorem concerning power series*. Proc. Lond. Math. Soc., 1914, **13**: 1–5.

[20] P. DUREN. *Univalent Functions*. Springer-Verlag, 1983.

[21] I. EFRAIMIDIS. *A generalization of Livingston's coefficient inequalities for functions with positive real part*. J. Math. Anal. Appl., 2016, **435**(1): 369–379.

[22] R. M. ALI. *Coefficients of the inverse of strongly starlike functions*. Bull. Malays. Math. Sci. Soc., 2003, **26**(1): 63–71.