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Existence of Meromorphic Solutions of Fermat-Type Functional Equations

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Abstract The functional equation $f(z)^n + g(z)^n = 1$ can be interpreted as the Fermat-type equations over function field. In this paper, by using Nevanlinna theory of meromorphic functions, we investigate the existence of meromorphic solutions of hyper-order strictly less than 1 to the Fermat-type functional equation $(a_0 f(z) + a_1 f(z+c))^3 + (b_0 f(z) + b_1 f(z+c))^3 = e^{\alpha z+\beta}$, where a_0, a_1, b_0, b_1, α, β, c are complex constants and $c \neq 0$.

Keywords Fermat-type functional equation; Nevanlinna theory; Meromorphic solution; Weierstrass $\wp$-function

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1. Introduction

It has always been a well-known and interesting problem to investigate the existence and forms of meromorphic solutions for Fermat-type functional equations

$$f(z)^n + g(z)^n = 1, \tag{1.1}$$

which can be regarded as the Fermat diophantine equation

$$x^n + y^n = 1 \tag{1.2}$$

over function fields, where $n \geq 2$ is an integer. Wiles [1] in 1995 pointed out that (1.2) does not admit nontrivial solutions in rational numbers for $n \geq 3$, and does admit nontrivial rational solutions for $n = 2$. In fact, Montel [2], Iyer [3] and Gross [4, 5] had earlier established results discussing the existence of solutions for Fermat type functional equation (1.1). For the convenience, we summarize their results as follows.

Theorem 1.1 *The solutions f and g of (1.1) are characterized as follows:*

(i) For $n = 2$, the entire solutions are $f(z) = \cos(h(z))$, $g(z) = \sin(h(z))$, where $h(z)$ is an entire solution; the meromorphic solutions of (1.1) are of the form

$$f(z) = \frac{2\psi(z)}{1+\psi^2(z)}, \quad g(z) = \frac{1-\psi^2(z)}{1+\psi^2(z)},$$

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where $\psi(z)$ is a nonconstant meromorphic solutions;

(ii) For $n > 2$, there are no non-constant entire solutions of (1.1);

(iii) For $n = 3$, the meromorphic solutions of (1.1) are of the form

$$f(z) = \frac{1}{2\wp(h)}(1 + \frac{\wp'(h)}{\sqrt{3}}), \quad g(z) = \frac{\omega}{2\wp(h)}(1 - \frac{\wp'(h)}{\sqrt{3}}),$$

where $\omega^3 = 1$ and h is a nonconstant entire function, $\wp$ denotes the Weierstrass $\wp$-function with periods ω_1 and ω_2 defined as

$$\wp(z;\omega_1,\omega_2) := \frac{1}{z^2} + \sum_{u,v\in\mathbb{Z};u^2+v^2\neq 0}\{\frac{1}{(z+uw_1+vw_2)^2} - \frac{1}{(uw_1+v\omega_2)^2}\},$$

and it satisfies $(\wp')^2 = 4\wp^3 - 1$;

(iv) For $n > 3$, there are no non-constant meromorphic solutions of (1.1).

From Theorem 1.1, the case $n > 3$ is clear. Hence, some scholars further considered equations such that $g(z)$ has a special relation with $f(z)$ and some generalization of (1.1). For example, Yang and Li [6] investigated the existence of entire solutions to the equation $f^2(z) + f'(z)^2 = 1$, and obtained the transcendental entire solution of $f^2(z) + f'(z)^2 = 1$ has the form $f(z) = \frac{1}{2}(Ae^{\alpha z} + \frac{1}{A}e^{-\alpha z})$, where A, α are non-zero complex constants. Due to the development of the Nevanlinna theory of difference operators [7–9], there has been a recent study on whether the derivative f' of f can be replaced by the shift $f(z+c)$ or difference operator $\Delta_c(f)$. The difference analogs of the Fermat-type functional equations have been investigated by many scholars [10–17]. In 2016, Lü and Han [15] derived the following result.

Theorem 1.2 *The difference equation $f^3(z)+f(z+c)^3 = 1$ does not have meromorphic solutions of finite order.*

Furthermore, Ma et al. [14] considered the above equation with the difference operators $\Delta_c f = f(z + c) - f(z)$ in the place of $f(z + c)$.

Theorem 1.3 *The difference equation $f^3(z) + (\Delta_c f)^3 = 1$ does not have meromorphic solutions of finite order.*

Ahamed [10] in 2019 proved a result generalizing the results of Theorems 1.2 and 1.3.

Theorem 1.4 *The difference equation $f^3(z) + (L_c f)^3 = 1$ does not have meromorphic solutions of finite order, where $L_c(f) = c_1 f(z + c) + c_0 f(z)$, c_1 $(\neq 0)$, $c_0 \in \mathbb{C}$.*

If we take into account the meromorphic solutions of the equations $f^n(z) + (f')^n = \gamma^n$ for $n \geq 4$ and $\gamma \not\equiv 0$, then Theorem 1.1 tells us both $\frac{f}{\gamma}$ and $\frac{f'}{\gamma}$ must be constants. As a result, if we assume $f = c_1\gamma$ and $f' = c_2\gamma$, then we have $c_1^n + c_2^n = 1$. Clearly, $c_1 \neq 0$; otherwise $f \equiv 0$, hence $\gamma = 0$. Similar to this, $c_2 \neq 0$; otherwise f and γ will be constants. Therefore, when $c_1c_2 \neq 0$, then γ cannot have any zeros and poles. Hence $\gamma^n(z) = e^{\alpha z+\beta}$ where $\alpha = n\frac{c_2}{c_1}$. Motivated by these observations, by replacing $f'(z)$ with $f(z + c)$ in the above equation, Han and Lü [18] proved

Theorem 1.5 *The equation $f^3(z) + f^3(z + c) = e^{\alpha z+\beta}$ does not admit meromorphic solutions* *of finite order, where α, β are two constants.*

Lü and Guo [16] proceeded to consider and derived the following result.

Theorem 1.6 *Let $\alpha, \beta, c\ (\not\equiv 0)$ be complex constants. Then the functional equation*

$$f^3(z) + f^3(z+c) = e^{\alpha z+\beta} \tag{1.3}$$

does not admit non-constant meromorphic solutions of hyper-order strictly less than 1.

Here, the order and hyper-order of a meromorphic function f are defined as follows

$$\rho(f) := \limsup_{r\to+\infty} \frac{\log^+ T(r,f)}{\log r}, \ \ \rho_2(f) = \limsup_{r\to+\infty} \frac{\log^+ \log^+ T(r,f)}{\log r},$$

where $T(r,f)$ denotes the Nevanlinna characteristic function of f.

Remark 1.7 We point out that non-constant meromorphic solutions should be "non-trivial meromorphic solutions" in Theorem 1.6, since $f^3(z) + f^3(z+c) = e^{\alpha z+\beta}$ has meromorphic solutions $f(z) = de^{\frac{1}{3}(\alpha z+\beta)}$, where d is a nonzero constant and $d^3(1+e^{\alpha c}) = 1$. These solutions are called trivial meromorphic solutions, which was given in [18]. So, the non-trivial solution referred to in this article is not the solution of this form.

Below, we give an example to illustrate that Eq. (1.3) may have a non-trivial solution f with $\rho_2(f) \geq 1$.

Example 1.8 Let $c = \pi$ and α, β be fixed constants satisfying $e^{\alpha c} = 1$. Consider

$$f(z) = \frac{1}{2} \frac{1 + \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))} e^{\frac{\alpha z+\beta}{3}},$$

where $h(z) = \cos z + \sin z$. Then, a routine computation leads to

$$f(z+c) = \frac{\omega}{2} \frac{1 - \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))} e^{\frac{\alpha z+\beta}{3}},$$

where $\omega = e^{\frac{\alpha c}{3}}$. Further, via (iii) of Theorem 1.1, one has

$$f^3(z) + f^3(z+c) = e^{\alpha z+\beta},$$

which implies that the above equation admits non-trivial meromorphic solution. Obviously, $\rho_2(f) = 1$.

Motivated from the above results, we are interested to investigate for the nonconstant meromorphic solutions of general difference equations. Henceforth, we recall that (1.3) can be written as a generalized Fermat-type equation

$$(a_0 f(z) + a_1 f(z+c))^3 + (b_0 f(z) + b_1 f(z+c))^3 = e^{\alpha z+\beta}, \tag{1.4}$$

where a_0, a_1, b_0, $b_1 \in \mathbb{C}$. Clearly, the shift $f(z+c)$ and difference operator $\Delta_c f$ are the particular cases of $a_0 f(z) + a_1 f(z+c)$. With this setting, in this paper, we will establish a combined result regarding Theorems 1.3–1.6. Before stating the main result of this paper, we make some remarks.

Example 1.9 Let $c=\pi$ and α,β be fixed constants satisfying $e^{\alpha c}=1$. Consider

$$f(z)=\frac{1}{2\wp a_0(h(z))}(1+\frac{\wp'(h(z))}{\sqrt{3}})e^{\frac{1}{3}(\alpha z+\beta)},\ f(z+c)=\frac{(a_0\omega-b_0)-(a_0\omega+b_0)\frac{\wp'(h(z))}{\sqrt{3}}}{2a_0b_1\wp(h(z))},$$

where $h(z)=\cos z$. Then, a routine computation leads to $b_0f(z)+b_1f(z+c)=\frac{\omega}{2\wp(h(z))}(1-\frac{\wp'(h(z))}{\sqrt{3}})$. Further, via (iii) of Theorem 1.1, one has

$$(a_0f(z))^3+(b_0f(z)+b_1f(z+c))^3=e^{\alpha z+\beta},$$

which implies that the above equation admits non-trivial meromorphic solution. Obviously, $\rho_2(f)=1$.

From Example 1.9, we know that Eq. (1.4) does have meromorphic solutions of infinite order. But as we can see, in the example above, the hyper-order of solutions of the equation is 1. Therefore, the following question is inevitable.

Question 1. Does there exist meromorphic solutions of hyper-order less than 1 of Eq. (1.4)?

In this paper, we will discuss this question with the help of some ideas from [11] and [15].

Theorem 1.10 *Let $a_0, a_1, b_0, b_1\in\mathbb{C}$ with $(1\pm\frac{i}{\sqrt{3}})b_j\neq(1\mp\frac{i}{\sqrt{3}})\omega a_j$ $(w^3=1,\ j=0,1)$. Then when $a_0b_1-a_1b_0\neq 0$, the Fermat-type equation (1.4) does not admit non-trivial meromorphic solutions of hyper-order strictly less than 1.*

Remark 1.11 It is pointed out that the restriction on hyper-order of f is necessary in Theorem 1.10. This can be shown by the following example.

Example 1.12 Let $c=\pi i$ and α,β be fixed constants satisfying $e^{\alpha c}=1$. Consider

$$f(z)=\frac{1}{2\eta\wp(h(z))}(\eta_1+\eta_2\frac{\wp'(h(z))}{\sqrt{3}})e^{\frac{1}{3}(\alpha z+\beta)},$$

$$f(z+c)=-\frac{1}{2\eta\wp(h(z))}(\eta_3+\eta_4\frac{\wp'(h(z))}{\sqrt{3}})e^{\frac{1}{3}(\alpha z+\beta)},$$

where $\eta=a_0b_1-a_1b_0$, $\eta_1=b_1-\omega a_1$, $\eta_2=b_1+\omega a_1$, $\eta_3=b_0-\omega a_0$, $\eta_4=b_0+\omega a_0$, $h(z)=e^z$. Then, a routine computation leads to $a_0f(z)+a_1f(z+c)=\frac{1}{2\wp(h(z))}(1+\frac{\wp'(h(z))}{\sqrt{3}})e^{\frac{1}{3}(\alpha z+\beta)}$, $b_0f(z)+b_1f(z+c)=\frac{\omega}{2\wp(h(z))}(1-\frac{\wp'(h(z))}{\sqrt{3}})e^{\frac{1}{3}(\alpha z+\beta)}$. Further, via (iii) of Theorem 1.1, one has

$$(a_0f(z)+a_1f(z+c))^3+(b_0f(z)+b_1f(z+c))^3=e^{\alpha z+\beta},$$

which implies that the above equation admits non-trivial meromorphic solution. Obviously, $\rho_2(f)=1$.

Remark 1.13 The condition $a_0b_1-a_1b_0\neq 0$ is also necessary in Theorem 1.10. Next, we give a brief explanation.

If $a_0b_1-a_1b_0=0$, i.e.,

$$\begin{pmatrix} a_0 & a_1\\ b_0 & b_1\end{pmatrix}=0,$$

then it yields that $a_0 = kb_0$, $a_1 = kb_1$, where $k \in \mathbb{C}$. In this case, we can rewrite (1.4) as

$$k_1(a_0 f(z) + a_1 f(z+c))^3 = e^{\alpha z+\beta},$$

where $k_1 \in \mathbb{C}$. Consider

$$a_0 f(z) + a_1 f(z+c) = e^{\alpha_1 z+\beta_1}. \tag{1.5}$$

Now, we give an example to illustrate the solution f of (1.5) may satisfy $\rho_2(f) \geq 1$.

Example 1.14 Let $f(z) = e^{e^{2z}+z} + \frac{1}{2}$. When $c = \pi i$, a simple calculation shows that $f(z+c) = -e^{e^{2z}+z} + \frac{1}{2}$. Then $f(z)$ satisfies equation (1.5) with $a_0 = a_1 = 1, \alpha_1 = \beta_1 = 0$. Clearly, $\rho_2(f) = 1$.

We assume that the reader is familiar with the basic results and notations of Nevanlinna's Value Distribution Theory, such as the first and second main theorems, the characteristic function $T(r,f)$, the proximity function $m(r,f)$ and the counting function $N(r,f)$.

2. Preliminary lemmas

In order to prove our result, we need the following lemmas.

Lemma 2.1 ([19]) *Let g be entire and suppose that $0 < \mu < \rho(g) \leq \infty$. Then, there exist a real number R_0 and a sequence r_k which leads to infinity such that*

$$N(r_k, \frac{1}{g-a}) \geq (r_k)^{\mu}$$

for all $a \in \mathbb{C}$ which satisfy

$$R_0 < |a| \leq \exp(r_k)^{\mu}.$$

Lemma 2.2 ([20]) *Let g be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{g'(z+c)}{g(z+c)} \longrightarrow 0 \quad \text{and} \quad \frac{g(z+c)}{g(z)} \longrightarrow 1 \quad \text{as} \quad z \longrightarrow \infty \quad \text{in} \quad \mathbb{C}\backslash E,$$

uniformly in c for $|c| \leqslant h$. Further, E may be chosen so that for large z not in E the function g has no zeros or poles in $|\zeta - z| \leqslant h$.

Lemma 2.3 ([21]) *Let $f(z)$ be a nonconstant meromorphic function and $c \neq 0 \in \mathbb{C}$. If $\rho_2(f) < 1$, then there exists a set $E \subset (0,\infty)$ with finite logarithmic measure, when $r \to \infty$ and $r \notin E$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + S(r,f).$$

Lemma 2.4 ([11]) *Take complex numbers α, β, c, a_i, b_i, $i = 0,1,2$ with $c \neq 0$, and assume*

$$\operatorname{rank}\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} = 2.$$

If the equation

$$(a_0 f(z) + a_1 f(z+c) + a_2 f'(z))^3 + (b_0 f(z) + b_1 f(z+c) + b_2 f'(z))^3 = e^{\alpha z+\beta}$$

has meromorphic solutions of finite order, then it has only entire solutions of the following form

$$f(z) = Ae^{\frac{\alpha z+\beta}{3}} + Ce^{Dz},$$

where A, C, D *are constants. Moreover, if we define constants* c_0, c_1 *by* $c_0^3 + c_1^3 = 1$, *then* A, D *are completely determined by* a_i, b_i, α, c_0, c_1 *as follows:*

(1) $A = \frac{b_2c_0 - a_2c_1}{a_0b_2 - a_2b_0}$, $C = 0$ *if* $a_0b_1 - a_1b_0 = 0$, $b_1a_2 - a_1b_2 = 0$,

(2) $A = \frac{b_1c_0 - a_1c_1}{a_0b_1 - a_1b_0}$, $C = 0$ *if* $a_0b_1 - a_1b_0 \neq 0$, $b_1a_2 - a_1b_2 = 0$,

(3) $A = \frac{3(b_1c_0 - a_1c_1)}{(b_1a_2 - a_1b_2)(\alpha - 3D)}$, $C \in \mathbb{C}$, $D = \frac{a_1b_0 - a_0b_1}{b_1a_2 - a_1b_2}$ *if* $b_1a_2 - a_1b_2 \neq 0$.

3. Proof of Theorem 1.10

This section is devoted to proving our theorem.

Proof of Theorem 1.10 Suppose that $f(z)$ is a non-trivial solution of (1.4) and $\rho_2 < 1$. Let

$$F(z) = \frac{a_0f(z) + a_1f(z+c)}{e^{\frac{1}{3}(\alpha z+\beta)}}, \tag{3.1}$$

$$G(z) = \frac{b_0f(z) + b_1f(z+c)}{e^{\frac{1}{3}(\alpha z+\beta)}}. \tag{3.2}$$

Then (1.4) can be expressed as $F^3 + G^3 = 1$. By Theorem 1.1, one has

$$F(z) = \frac{1}{2\wp(h(z))}(1 + \frac{\wp'(h(z))}{\sqrt{3}}), \quad G(z) = \frac{\omega}{2\wp(h(z))}(1 - \frac{\wp'(h(z))}{\sqrt{3}}), \tag{3.3}$$

where h is an entire function over $\mathbb{C}$, $\omega^3 = 1$ and $\wp$ denotes the Weierstrass-$\wp$ function satisfying $(\wp')^2 = 4\wp^3 - 1$. From (3.1)–(3.3), we obtain

$$f(z) = \frac{1}{2\eta\wp(h(z))}(\eta_1 + \eta_2\frac{\wp'(h(z))}{\sqrt{3}})e^{\frac{1}{3}(\alpha z+\beta)}, \tag{3.4}$$

$$f(z+c) = -\frac{1}{2\eta\wp(h(z))}(\eta_3 + \eta_4\frac{\wp'(h(z))}{\sqrt{3}})e^{\frac{1}{3}(\alpha z+\beta)}, \tag{3.5}$$

where $\eta = a_0b_1 - a_1b_0$, $\eta_1 = b_1 - \omega a_1$, $\eta_2 = b_1 + \omega a_1$, $\eta_3 = b_0 - \omega a_0$, $\eta_4 = b_0 + \omega a_0$. If $f(z)$ is finite order, then the conclusion follows from Lemma 2.4, which means $f(z)$ is a trivial solution. So, in the following, we assume that $f(z)$ is of infinite order. Combining (3.4) and $(\wp')^2 = 4\wp^3 - 1$, we have

$$\frac{3\eta^2\wp^2(h(z))f^2(z)}{\eta_2^2e^{\frac{2}{3}(\alpha z+\beta)}} - \frac{3\eta\eta_1\wp(h(z))f(z)}{\eta_2^2e^{\frac{1}{3}(\alpha z+\beta)}} + \frac{\eta_1^2}{4\eta_2^2} + \frac{1}{4} = \wp^3(h(z)). \tag{3.6}$$

A routine computation leads to $3T(r,\wp(h)) \leq 2T(r,f) + 2T(r,\wp(h)) + \frac{2}{3}T(r,e^{\alpha z+\beta}) + O(1)$. Since $\rho_2(f) < 1$, we have $\rho_2(\wp(h)) < 1$.

We claim that $\rho(h) < 1$. On the contrary, if $\rho(h) \geq 1$, then for any $\varepsilon > 0$, we have $\rho(h) > 1 - \varepsilon$. Set $u = 1 - \varepsilon$. Thus, $\rho(h) > u$. Let $S = \{z_j\}_{j=1}^{\infty}$ be the set of all the zeros of $\wp$ satisfying $z_j \to \infty$ as $j \to \infty$. By Lemma 2.1 and $\rho(\wp) = 2$, we have $n(r, \frac{1}{\wp}) \geq c_1r^2$, where $c_1 \in \mathbb{C}$. Applying Lemma 2.1 again to $\wp(h)$, we have

$$N(r_k, \frac{1}{\wp(h(z))}) = \sum_{z_j\in S} N(r_k, \frac{1}{h(z) - z_j}) \geq \sum_{R_0<|z_j|\leq \exp(r_k)^u} N(r_k, \frac{1}{h(z) - z_j})$$

$$\geq \sum_{R_0<|z_j|\leq \exp{(r_k)^u}} r_k^u = [n(\exp{(r_k)^u}, \frac{1}{\wp}) - n(R_0, \frac{1}{\wp})]r_k^u$$
$$\geq \frac{1}{2}n(\exp{(r_k)^n}, \frac{1}{\wp})r_k^u \geq c_2(\exp{(r_k)^n})^2, \tag{3.7}$$

where c_2 is a fixed positive constant. It follows from (3.7) that

$$\rho_2(\wp(h)) \geq \lim_{r_k\to\infty}\sup \frac{\log\log T(r_k, \wp(h))}{\log r_k} \geq \lim_{r_k\to\infty}\sup \frac{\log\log N(r_k, \frac{1}{\wp(h)})}{\log r_k}$$
$$\geq \lim_{r_k\to\infty}\sup \frac{\log\log c_2(\exp{(r_k)^n})^2}{\log r_k} = u = 1-\varepsilon,$$

which means $\rho_2(\wp(h)) \geq 1$. A contradiction. Therefore, $\rho(h) < 1$.

For a set G, we define the function $N(r, G)$ as

$$N(r, G) = \int_0^r \frac{n(t, G) - n(0, G)}{t}\mathrm{d}t + n(0, G)\log r,$$

where $n(r, G)$ is the number of points in $G \cap \{z : |z| < r\}$. Assume that S_1 is a set satisfying $S_1 = \{z|\wp(h(z)) = 0, \wp(h(z + c)) = 0\}$.

Now, we claim that $N(r, S_1) = o(r)$. On the contrary, suppose that $N(r_n, S_1) \geq cr_n$ for some $c \in \mathbb{C}$, and set a sequence $\{r_n\}$ satisfying $r_n \to \infty$ as $n \to \infty$. Let $S_1 = \{a_s\}_{s=1}^{\infty}$, $a_s \to \infty$ as $s \to \infty$. Then $\wp(h(a_s)) = 0$, $\wp(h(a_s + c)) = 0$. Further, we have $(\wp')^2(h(a_s)) = -1$ and $(\wp')^2(h(a_s + c)) = -1$ since $(\wp')^2 = 4\wp^3 - 1$. Combining (3.4) and (3.5) yields

$$(\eta_1 + \eta_2\frac{\wp'(h(z+c))}{\sqrt{3}})e^{\frac{1}{3}(\alpha c)}\ \wp(h(z)) = -(\eta_3 + \eta_4\frac{\wp'(h(z))}{\sqrt{3}})\ \wp(h(z+c)). \tag{3.8}$$

Differentiate (3.8) and apply substitution to observe that

$$e^{\frac{1}{3}(\alpha c)}(\eta_1 + \eta_2\frac{\wp'(h(a_s+c))}{\sqrt{3}})\wp'(h(a_s))h'(a_s) = -(\eta_3 + \eta_4\frac{\wp'(h(a_s))}{\sqrt{3}})\wp'(h(a_s+c))h'(a_s+c).$$

Since $\wp'(h(a_s)) = \pm i$ and $\wp'(h(a_s + c)) = \pm i$, from the equation above, we have one and only one of the following situations occurs

$$\begin{cases} g_1 = \mathrm{e}^{\frac{1}{3}(\alpha c)}(\eta_1 + \eta_2\frac{i}{\sqrt{3}})h'(a_s) + (\eta_3 + \eta_4\frac{i}{\sqrt{3}})h'(a_s+c) = 0, \\ g_2 = -\mathrm{e}^{\frac{1}{3}(\alpha c)}(\eta_1 + \eta_2\frac{i}{\sqrt{3}})h'(a_s) + (\eta_3 - \eta_4\frac{i}{\sqrt{3}})h'(a_s+c) = 0, \\ g_3 = \mathrm{e}^{\frac{1}{3}(\alpha c)}(\eta_1 - \eta_2\frac{i}{\sqrt{3}})h'(a_s) - (\eta_3 + \eta_4\frac{i}{\sqrt{3}})h'(a_s+c) = 0, \\ g_4 = -\mathrm{e}^{\frac{1}{3}(\alpha c)}(\eta_1 - \eta_2\frac{i}{\sqrt{3}})h'(a_s) - (\eta_3 - \eta_4\frac{i}{\sqrt{3}})h'(a_s+c) = 0. \end{cases}$$

Noting that our assumption $(1 \pm \frac{i}{\sqrt{3}})b_j \neq (1 \mp \frac{i}{\sqrt{3}})\omega a_j$ $(w^3 = 1,\ j = 0, 1)$. Then by the simple calculation, we claim that $\eta_1 + \eta_2\frac{i}{\sqrt{3}}$, $\eta_3 + \eta_4\frac{i}{\sqrt{3}}$, $\eta_1 - \eta_2\frac{i}{\sqrt{3}}$ and $\eta_3 - \eta_4\frac{i}{\sqrt{3}}$ are not equal to 0. Assume that $E_j = \{a_s \in S_1|g_j(a_s) = 0\}$ $(j = 1, 2, 3, 4)$. Clearly, $S_1 \subseteq \bigcup_{j=1}^4 E_j$. It follows from the assumption $N(r_n, S_1) \geq cr_n$ that $N(r_n, E_j) \geq c_1r_n$, where c_1 is a positive constant. If $g_j \not\equiv 0$, then $g_j(z) = 0$ for $z \in E_j$. Therefore,

$$N(r_n, \frac{1}{g_j}) \geq N(r_n, E_j) \geq c_1r_n,$$

which implies that

$$\begin{aligned}\rho(g_j) &= \lim_{r\to\infty}\sup\frac{\log T(r,g_j)}{\log r} \geq \lim_{r\to\infty}\sup\frac{\log N(r,\frac{1}{g_j})}{\log r}\\ &\geq \lim_{r\to\infty}\sup\frac{\log N(r_n,\frac{1}{g_j})}{\log r_n} \geq \lim_{r\to\infty}\sup\frac{\log c_1 r_n}{\log r_n} = 1.\end{aligned}$$

i.e.,

$$\rho(g_j) \geq 1.$$

By the form of g_j, one has $T(r,g_j) = O(T(r,h))$, which means $\rho(g_j) \leq \rho(h(z)) < 1$. A contradiction. So, we obtain $g_j \equiv 0$. This implies that one and only one of the following situations can occur

$$\begin{cases} \mathrm{e}^{\frac{1}{3}(\alpha c)}(\eta_1+\eta_2\frac{i}{\sqrt{3}})h'(z) = -(\eta_3+\eta_4\frac{i}{\sqrt{3}})h'(z+c),\\ \mathrm{e}^{\frac{1}{3}(\alpha c)}(\eta_1+\eta_2\frac{i}{\sqrt{3}})h'(z) = (\eta_3-\eta_4\frac{i}{\sqrt{3}})h'(z+c),\\ \mathrm{e}^{\frac{1}{3}(\alpha c)}(\eta_1-\eta_2\frac{i}{\sqrt{3}})h'(z) = (\eta_3+\eta_4\frac{i}{\sqrt{3}})h'(z+c),\\ \mathrm{e}^{\frac{1}{3}(\alpha c)}(\eta_1-\eta_2\frac{i}{\sqrt{3}})h'(z) = -(\eta_3-\eta_4\frac{i}{\sqrt{3}})h'(z+c).\end{cases}$$

Now, we claim that $h'(z)$ is a constant. If not, then the above equations yield that $Ah'(z+c) = h'(z)$, where $A \neq 0 \in \mathbb{C}$. By Lemma 2.2, we have $\rho(h') \geq 1$, which contradicts the fact $\rho(h') = \rho(h) < 1$. So, $h'(z)$ must be a constant. Thus $h(z)$ is a linear function, which implies the order of $\wp(h(z))$ is finite. So does f. A contradiction.

All the discussion above yields that $N(r,S_1) = o(r)$. Now, we assume that $E = \{z|\wp(h(z)) = 0\}$. Obviously, $S_1 \subseteq E$. Let $S_2 = E - S_1$. Then for any $b \in S_2$, we have $\wp(h(b)) = 0$, $\wp(h(b+c)) \neq 0$. In view of $(\wp')^2(h(b)) = -1$ and (3.8), we see that $\wp(h(b+c)) = \infty$. This implies that the zeros of $\wp(h(z))$ are the poles of $\wp(h(z+c))$. Note that the multiple zeros of $\wp(h(z))$ can only occur at zeros of its derivative $\{\wp(h(z))\}'$, that is, the zeros of $h'(z)$ because of $\wp'(h(z)) = \pm i \neq 0$. Hence we obtain

$$\begin{aligned} N(r,\frac{1}{\wp(h)}) &\leq \bar{N}(r,\frac{1}{\wp(h)}) + 2N(r,\frac{1}{h'}) \leq \bar{N}(r,E) + 2N(r,\frac{1}{h'})\\ &\leq \bar{N}(r,S_1) + \bar{N}(r,S_2) + 2T(r,h) \leq \bar{N}(r,\wp(h(z+c))) + o(r) + O(r)\\ &\leq \bar{N}(r,\wp(h(z+c))) + S(r,\wp(h)). \end{aligned} \tag{3.9}$$

In fact, the expression of $F(z) = \frac{1}{2\wp(h(z))}(1+\frac{\wp'(h(z))}{\sqrt{3}})$ in (3.3) yields

$$T(r,F) \leq T(r,\wp'(h) + T(r,\wp(h) + O(1) \leq O(T(r,\wp(h))) + O(1). \tag{3.10}$$

Based on the expression of F and the condition $(\wp')^2 = 4\wp^3 - 1$, we have

$$\wp^3(h(z)) = 3F^2(z)\wp^2(h(z)) - 3F(z)\wp(h(z)) + 1.$$

By the same discussion as (3.6), we get $3T(r,\wp(h)) \leq 2T(r,F) + 2T(r,\wp(h)) + O(1)$. i.e.,

$$T(r,\wp(h)) \leq O(T(r,F)). \tag{3.11}$$

Combining (3.10) and (3.11), there exists a set $E_3 \subset [0,\infty)$ with finite logarithmic measure such that

$$O(\log rT(r,\wp(h))) = O(\log rT(r,F)),$$

as $r \to \infty$ and $r \notin E_3$. By (1.4) and (3.3), it yields

$$-G^3 = F^3 - e^{\alpha z+\beta} = (F - e^{\frac{1}{3}(\alpha z+\beta)})(F - \omega e^{\frac{1}{3}(\alpha z+\beta)})(F - \omega^2 e^{\frac{1}{3}(\alpha z+\beta)}),$$

where $\omega^3 = 1$, $\omega \neq 1$. We see that all zeros of $F - e^{\frac{1}{3}(\alpha z+\beta)}$, $F - \omega e^{\frac{1}{3}(\alpha z+\beta)}$, $F - \omega^2 e^{\frac{1}{3}(\alpha z+\beta)}$ are of multiplicities ≥ 3. Hence applying Nevanlinnas first and second theorems to F yields that

$$\begin{aligned} 2T(r,F) &\leq \sum_{m=0}^{2} \bar{N}(r, \frac{1}{F - \omega^m e^{\frac{1}{3}(\alpha z+\beta)}}) + \bar{N}(r,F) + O(\log rT(r,F)) \\ &\leq \frac{1}{3}\sum_{m=0}^{2} N(r, \frac{1}{F - \omega^m e^{\frac{1}{3}(\alpha z+\beta)}}) + N(r,F) + O(\log rT(r,F)) \\ &\leq T(r,F) + N(r,F) + O(\log rT(r,\wp(h))), \end{aligned} \tag{3.12}$$

which immediately implies

$$m(r,F) = O(\log rT(r,\wp(h))).$$

Now we rewrite F into the following form

$$\frac{1}{\wp(h(z))} = 2F - \frac{\wp'(h(z))}{\sqrt{3}\wp(h(z))},$$

which means

$$m(r, \frac{1}{\wp(h)}) \leq m(r,F) + m(r, \frac{\wp'(h)}{\wp(h)}) + O(1).$$

Applying the lemma of logarithmic derivative, we have

$$\begin{aligned} m(r, \frac{\wp'(h)}{\wp(h)}) &\leq m(r, \frac{\wp'(h)h'}{\wp(h)}) + m(r, \frac{1}{h'}) \leq m(r, \frac{\wp'(h)h'}{\wp(h)}) + 2T(r,h) \\ &\leq O(\log rT(r,\wp(h))), \end{aligned}$$

and hence

$$m(r, \frac{1}{\wp(h)}) = O(\log rT(r,\wp(h))). \tag{3.13}$$

Combining (3.9) with (3.13), and noticing that each pole of $\wp(h)$ has multiplicity 2, then Nevanlinnas first main theorem implies

$$\begin{aligned} T(r,\wp(h)) &= T(r, \frac{1}{\wp(h)}) + O(1) = m(r, \frac{1}{\wp(h)}) + N(r, \frac{1}{\wp(h)}) + O(1) \\ &\leq \bar{N}(r, \wp(h(z+c))) + S(r,\wp(h)) \\ &\leq \frac{1}{2}N(r, \wp(h(z+c))) + S(r,\wp(h)) \\ &\leq \frac{1}{2}T(r, \wp(h(z+c))) + S(r,\wp(h)). \end{aligned}$$

By Lemma 2.3, we have

$$T(r, \wp(h(z+c))) = T(r, \wp(h(z))) + S(r,\wp(h)).$$

i.e.,

$$T(r,\wp(h)) = S(r,\wp(h)).$$

We obtain a contradiction. Therefore, we complete the proof of Theorem 1.10. □

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