Journal of Mathematical Research with Applications
Sept., 2024, Vol. 44, No. 5, pp. 669–680
DOI:10.3770/j.issn:2095-2651.2024.05.008
Http://jmre.dlut.edu.cn

Nonlinear Potential Analysis in Morrey Spaces on Carnot Groups

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Abstract In this paper, the nonlinear potential theory in the Morrey spaces on Euclidean spaces and the Lebesgue spaces on the Carnot group are studied. According to the methods of abstract harmonic analysis in Heisenberg group and abstract potential theory in carnot group, we mainly give some characterizations for Riesz, Bessel and Wolff potentials, and the corresponding capacities in the Morrey spaces on Carnot group. Meanwhile, we also interpret the relation among Riesz and Bessel type capacities and Housdorff content in the Morrey spaces on Carnot group. All these results above generalize the related ones in the Morrey spaces on Euclidean spaces and the Lebesgue spaces on the Carnot group.

Keywords Riesz potential; Bessel potential; Wolff potential; Morrey space; Carnot group

MR(2020) Subject Classification 22E30; 31C15; 35J60; 46E35

1. Introduction

A Carnot group (i.e., stratified homogeneous group) is a simply connected nilpotent Lie group $\mathbb{G} \equiv (\mathbb{R}^N, \circ)$ whose Lie algebra $\mathcal{G}$ admits a stratification. That is to say, there exist linear subspaces $V_1, \ldots, V_k$ of $\mathcal{G}$ so that the direct sum vector space decomposition below

$$\mathcal{G} = V_1 \bigoplus \cdots \bigoplus V_k,\ [V_1, V_i] = V_{i+1} \text{ for } i = 1, 2, \ldots, k-1 \text{ and } [V_1, V_k] = \{0\}$$

holds, where $[V_1, V_i]$ is the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ with $X \in V_1$ and $Y \in V_i$.

In fact, the subLaplacian operator $\mathcal{L} = \sum_{j=1}^m X_j^2$ is the second-order partial differential operator on $\mathbb{G}$ and the intrinsic gradient $\nabla_{\mathcal{L}}$ associated with $\mathcal{L}$ can be written as

$$\nabla_{\mathcal{L}} = (X_1, \ldots, X_m),$$

where $\{X_1, \ldots, X_m\}$ is a family of vector fields to form a linear basis of the first layer of $\mathcal{G}$.

The dilations $\delta_t : \mathbb{R}^N \to \mathbb{R}^N (t > 0)$ is a family of automorphisms of group $\mathbb{G}$ satisfying

$$\delta_t(x_1, \ldots, x_N) = (t^{\alpha_1} x_1, \ldots, t^{\alpha_N} x_N),$$

Received November 4, 2023; Accepted April 8, 2024
Supported by the Natural Science Foundation of Ningxia (Grant No. 2023AAC03001) and the National Natural Science Foundation of China (Grant No. 12261068).
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where $1 = \alpha_1 = \cdots = \alpha_m < \alpha_{m+1} \le \cdots \le \alpha_N$ are integers and $m = \dim(V_1)$.

If $\gamma(a) = x$, $\gamma(b) = y \in \mathbb{G}$ and $\gamma'(t) \in V_1$ for all t, then the curve $\gamma : [a, b] \to \mathbb{G}$ is called horizontal. Define the Carnot-Caratheodory distance between x and y by

$$d_{CC}(x, y) = \inf_{\gamma} \int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}} \mathrm{d}t,$$

where the infimum is taken over all horizontal curves γ connecting to x and y. Accordingly, the Carnot-Caratheodory ball is denoted by $B_{CC}(x, r) = \{y \in \mathbb{G} : d_{CC}(x, y) < r\}$. From the left invariant properties we infer that

$$d_{CC}(zx, zy) = d_{CC}(x, y),\ B_{CC}(x, r) = xB_{CC}(e, r) \text{ for } x, y, z \in \mathbb{G} \text{ and } r > 0$$

and

$$d_{CC}(\delta_t(x), \delta_t(y)) = t d_{CC}(x, y) \text{ for } x, y \in \mathbb{G} \text{ and } t > 0.$$

For $x \in \mathbb{G}$ and $r > 0$, denote by $B(x, r) = \{y \in \mathbb{G} : \rho(y^{-1} \circ x) = |y^{-1} \circ x| < r\}$ the $\mathbb{G}$-ball with x and radius r, and by $B(e, r) = \{y \in \mathbb{G} : \rho(y) < r\}$ the open ball centered at the identity element e of $\mathbb{G}$ with radius r, where the continuous function $\rho : \mathbb{G} \to [0, \infty)$ is a homogenous norm on $\mathbb{G}$ and satisfies $\rho(x^{-1}) = \rho(x)$, $\rho(\delta_t x) = t\rho(x)$ for all $x \in \mathbb{G}$. Moreover, there exists a constant $\kappa \ge 1$ such that $\rho(x, y) \le \kappa(\rho(x) + \rho(y))$ for all $x, y \in \mathbb{G}$. Here we remark that the pseudometric $\rho(x, y) = |x^{-1} \circ y|$ is equivalent to the metric d_{CC} in the following sense

$$C^{-1}\rho(x, y) \le d_{CC}(x, y) \le C\rho(x, y) \text{ for } x, y \in \mathbb{G} \text{ and } C > 1$$

and satisfies

$$\rho(zx, zy) = \rho(x, y),\ D(x, r) = xD(e, r) \text{ for } x, y, z \in \mathbb{G} \text{ and } r > 0,$$

where $D(x, r) = \{y \in \mathbb{G} : \rho(x, y) < r\}$ is the metric ball associated with ρ. For convenience, we will use d and $B(x, r)$ instead of d_{CC} and $B_{CC}(x, r)$, respectively.

By the left translation and dilation, we know that

$$|B(x, r)| = r^Q|B(x, 1)| = r^Q|B(0, 1)|,$$

where the homogeneous dimension Q of $\mathbb{G}$ is denoted by

$$Q = \sum_{j=1}^{m} j \dim(V_j).$$

The classical Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{G})$ on $\mathbb{G}$ is defined by the following norm

$$\| f \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})} := \sup_{x\in\mathbb{G}, r>0} r^{\frac{-\lambda}{p}} \| f \|_{L^p(B(x,r))} < \infty$$

for $0 \le \lambda \le Q$ and $1 \le p \le \infty$. For $\lambda = 0$ and $\lambda = Q$, $\mathcal{L}^{p,0}(\mathbb{G}) = L^p(\mathbb{G})$ and $\mathcal{L}^{p,Q}(\mathbb{G}) = L^\infty(\mathbb{G})$, respectively. As for $\lambda < 0$ and $\lambda > Q$, $\mathcal{L}^{p,\lambda}(\mathbb{G}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{G}$. Here we point out that the closure of C_0 on $\mathbb{G}$ is $\mathcal{L}_0^{p,\lambda}(\mathbb{G})$ but not $\mathcal{L}^{p,\lambda}(\mathbb{G})$.

Define the Riesz kernel on Carnot group by $I_\alpha(x) = d(x)^{\alpha - Q}$. A generalized Bessel kernel

$J_\alpha(x)$ on Carnot groups for $\Re\alpha > 0$ is denoted by

$$J_\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} \exp(-t) h(x,t) \mathrm{d}t,$$

where the function $h(x,t)$ is the fundamental solution of the operator $\mathcal{L} + \partial/\partial t$ on $\mathbb{G} \times (0,\infty)$. Here, we point out that $J_\alpha(x) \leq cI_\alpha(x)$ for $0 < \alpha < Q$ and $x \in \mathbb{G}$.

The αth fractional Riesz and Bessel potentials of f are determined by

$$\mathcal{I}_\alpha f(x) = \int_{\mathbb{G}} I_\alpha(y^{-1}x) f(y) \mathrm{d}y, \quad 0 < \alpha < Q$$

and

$$\mathcal{J}_\alpha f(x) = \int_{\mathbb{G}} J_\alpha(y^{-1}x) f(y) \mathrm{d}y, \quad 0 < \alpha < Q,$$

respectively. If we denote $\mu = f dx$, then $\mathcal{I}_\alpha f = \mathcal{I}_\alpha \mu$ and $\mathcal{J}_\alpha f = \mathcal{J}_\alpha \mu$.

In this article our object is to characterize the nonlinear potential theory in the Morrey spaces on homogenous Carnot group. For the nonlinear potential theory in the another function spaces for the Euclidean setting, we may refer to Adams ([1] for Riesz potential), Adams and Hedberg ([2] for Lebesgue space, Besov space and Lizorkin-Triebel space, etc), Heinonen et al. ([3] for Sobolev space and weighted Sobolev space, etc), Hedberg and Wolff ([4] for the geometric properties of potential) and Filippis and Stroffolini ([5] for nonlinear potentials). Since Morrey [6,7] introduced the function space later named by himself, there has a great deal of advances along this line. In Morrey or Morrey type spaces the boundednesses of various classical operators have been largely investigated, refer to Adams and Xiao [8–10], Burenkov et al. [12, 13], Guliyev et al. [14] and references therein. Unfortunately, although Kalita [15] and Zorko [7] ever provided the predual spaces of Morrey spaces, their duality is still hard to character. For this Adams and Xiao [8] put forward the another predual spaces of Morrey spaces and proved the equivalences with Kalita and Zorko's ones.

For the characterizations of classical operators in the abstract harmonic analysis, refer to some books by Folland and Stein [16], Thangavelu [17] and Varopoulos et al. [18]. In the generalized Morrey spaces on the Heisenberg group, Guliyev et al. [19, 20] ever studied the boundednesses of Riesz potential and fractional maximal operator. For the properties of Lebesgue space or Sobolev space on Carnot group in abstract potential theory, refer to Arya1 and Kumar [21], Bonfiglioli et al. [22, 23], Capolli et al. [24], Gafofalo and Rotz [25], Lu [26, 27], Vodoṕyanov et al. [28, 29], and Rigot [30] and references therein. Up to now we still know a little about the properties of the Morrey space on Carnot group [19, 31, 32]. Stimulated by the preceding statements, we will obtain some properties from Adams and Xiao [8] in the Morrey spaces on Carnot group and simultaneously develop the results from Lu [27] and Vodoṕyanov et al. [29] on Carnot group. Preciously, our purpose is to character Riesz, Bessel and Wolff potentials, and the corresponding capacities in the Morrey spaces on Carnot group. Meanwhile, we also interpret the relation among Riesz and Bessel type capacities and Housdorff content in the Morrey spaces on Carnot group.

2. Main results and the proofs

To begin with, define Riesz and Bessel type capacities in Morrey space. By the approximation method of the identity in harmonic analysis, we know that the capacities in $\mathcal{L}^{p,\lambda}(\mathbb{G})$ and $\mathcal{L}_0^{p,\lambda}(\mathbb{G})$ are comparable and we only consider the former. For a set $E \subset \mathbb{G}$ and $0 \le \alpha < Q$, let

$$\mathcal{R}_{\alpha,p}^{\lambda}(E) = \inf\{\| f \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})}^{p}: \mathcal{I}_{\alpha} f \ge 1 \text{ on } E \text{ and } f \ge 0\}.$$

Similarly, the Bessel type capacity is defined by

$$\mathcal{B}_{\alpha,p}^{\lambda}(E) = \inf\{\| f \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})}^{p}: \mathcal{J}_{\alpha} f \ge 1 \text{ on } E \text{ and } f \ge 0\}.$$

It is known that Zorko [7] and Kalita [32] ever characterized the predual spaces of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$. Later Adams and Xiao [8] proved the predual space of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ obtained by themselves is equivalent to the ones of Zorko and Kalita. In the rest of this article we follow the notations of Adams and Xiao. From the min-max theorem we remark that the dual ones of Riesz and Bessel type capacities in Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{G})$ are

$${}_{\mathcal{R}}Cap_{\alpha,p'}^{\lambda}(E) = \sup\{\mu(E) : \mu \in \mathcal{M}^{+}(E) \text{ and } \| \mathcal{I}_{\alpha}\mu \|_{\mathcal{H}^{p',\lambda}(\mathbb{G})} \le 1\}$$

and

$${}_{\mathcal{B}}Cap_{\alpha,p'}^{\lambda}(E) = \sup\{\mu(E) : \mu \in \mathcal{M}^{+}(E) \text{ and } \| \mathcal{J}_{\alpha}\mu \|_{\mathcal{H}^{p',\lambda}(\mathbb{G})} \le 1\},$$

respectively, where $\mathcal{M}^{+}(E)$ consists of all the nonnegative Radon measures μ supported in E, and the predual space of $\mathcal{L}^{p,\lambda}(\mathbb{G})$ is denoted by $\mathcal{H}^{q,\lambda}(\mathbb{G})$ via the norm

$$\| g \|_{\mathcal{H}^{p',\lambda}(\mathbb{G})} := \inf_{\omega} \| g\omega^{1/p} \|_{L^{p'}(\mathbb{G})} < \infty,$$

here $1/p + 1/p' = 1$ and ω is the nonnegative function on $\mathbb{G}$ satisfying

$$\| \omega \|_{L^1(\Lambda_{\lambda}^{(\infty)})} < 1.$$

Let $h(r)$ be an increasing positive function for all $r > 0$ and satisfy $h(0) = 0$. Now we recall the Hausdorff measure on $\mathbb{G}$ with respect to h. For $\rho > 0$, the set function $\Lambda_h^{(\rho)}$ is denoted by

$$\Lambda_h^{(\rho)}(E) := \inf \sum h(r_j),$$

where the infimum is taken over all countable coverings of $E \subset \mathbb{G}$ by open balls with radius $r_j \le \rho$. Clearly, $\Lambda_h^{(\rho_1)}(E) \ge \Lambda_h^{(\rho_2)}(E)$ if $\rho_2 \ge \rho_1$, so $\lim_{\rho\to 0} \Lambda_h^{(\rho)}(E) = \Lambda_h(E) \ge \infty$ holds. When $\rho = \infty$, we call $\Lambda_h^{(\infty)}$ Hausdorff capacity with respect to h. If $h(r) = r^d$ for $d \in (0, Q]$, we write $\Lambda_h^{(\rho)}$ as the d-dimensional Hausdorff measure $\Lambda_d^{(\rho)}$. In addition, we also recall that the predual spcae of Morrey spaces $\mathcal{L}^{1,\lambda}(\mathbb{G})$ is exactly $L^1(\Lambda_{\lambda}^{(\infty)})$ with the norm

$$\| g \|_{L^1(\Lambda_{\lambda}^{(\infty)})} := \int_{\mathbb{G}} |g(x)| \mathrm{d}\Lambda_{\lambda}^{(\infty)}(x) = \int_0^{\infty} \Lambda_{\lambda}^{(\infty)}(\{x \in \mathbb{G} : |g(x)| \ge t\}) \mathrm{d}t < \infty.$$

According to the min-max theorem, we also obtain that

$${}_{\mathcal{R}}Cap_{\alpha,p'}^{\lambda}(E) = \{\mathcal{R}_{\alpha,p}^{\lambda}(E)\}^{1/p}$$

and

$${}_{\mathcal{B}}Cap_{\alpha,p'}^{\lambda}(E) = \{\mathcal{B}_{\alpha,p}^{\lambda}(E)\}^{1/p}.$$

Immediately, from the definitions of capacity we obtain the following properties.

Proposition 2.1 *If $E_1 \subset E_2$, then*

$$\mathcal{R}^{\lambda}_{\alpha,p}(E_1)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(E_1)) \leq \mathcal{R}^{\lambda}_{\alpha,p}(E_2)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(E_2)).$$

Since the capacities above are outer measures, we also provide the next results.

Proposition 2.2 *(I) For $E_j \subset \mathbb{G}$, $j \in \mathbb{Z}_+$, let $E = \bigcup_{j=1}^{\infty} E_j$. Then*

$$\mathcal{R}^{\lambda}_{\alpha,p}(E)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(E)) \leq \sum_{j=1}^{\infty} \mathcal{R}^{\lambda}_{\alpha,p}(E_j)(\text{res.}\,\sum_{j=1}^{\infty} \mathcal{B}^{\lambda}_{\alpha,p}(E_j));$$

(II) For $E \subset \mathbb{G}$

$$\mathcal{R}^{\lambda}_{\alpha,p}(E)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(E)) = \inf\{\mathcal{R}^{\lambda}_{\alpha,p}(G)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(G)) : G \supset E \text{ and } G \text{ open}\};$$

(III) For $E \subset \mathbb{G}$, $\mathcal{R}^{\lambda}_{\alpha,p}(E)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(E)) = 0$ if and only if there exists a nonnegative function $f \in \mathcal{L}^{p,\lambda}(\mathbb{G})$ such that $E \subset \{x : \mathcal{I}_\alpha f(x)(\text{res.}\,\mathcal{J}_\alpha f(x)) = \infty\}$.

Proposition 2.3 *For $E \subset \mathbb{G}$, $\mathcal{R}^{\lambda}_{\alpha,p}(E)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(E)) = 0$ if and only if there exists a nonnegative function $f \in \mathcal{L}^{p,\lambda}(\mathbb{G})$ such that $E \subset \{x : \mathcal{I}_\alpha f(x)(\text{res.}\,\mathcal{J}_\alpha f(x)) = \infty\}$.*

A property is called $\mathcal{R}^{\lambda}_{\alpha,p}(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p})$-quasieverywhere or $\mathcal{R}^{\lambda}_{\alpha,p}(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p})$-q.e., if it holds except on a set E satisfying $\mathcal{R}^{\lambda}_{\alpha,p}(E)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(E)) = 0$. For $E \subset \mathbb{G}$, if

$$\begin{aligned}\mathcal{R}^{\lambda}_{\alpha,p}(E)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(E)) &= \sup\{\mathcal{R}^{\lambda}_{\alpha,p}(F)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(F)) : F \subset E \text{ and } F \text{ compact}\}\\ &= \inf\{\mathcal{R}^{\lambda}_{\alpha,p}(G)(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p}(G)) : G \supset E \text{ and } G \text{ open}\},\end{aligned}$$

we call E an $\mathcal{R}^{\lambda}_{\alpha,p}(\text{res.}\,\mathcal{B}^{\lambda}_{\alpha,p})$-capacitable set.

Proposition 2.4 *Suppose that $\{f_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathcal{L}^{p,\lambda}(\mathbb{G})$ with limit f. Then there exists a subsequence $\{f_{j_n}\}_{n=1}^{\infty}$ such that $\lim \mathcal{I}_\alpha f_{j_n}(x) = f(x)$ $\mathcal{R}^{\lambda}_{\alpha,p}$-q.e.(res. $\lim \mathcal{J}_\alpha f_{j_n}(x) = f(x)$ $\mathcal{B}^{\lambda}_{\alpha,p}$-q.e.) and uniformly outside an open set of arbitrarily small $\mathcal{R}^{\lambda}_{\alpha,p}$ (res. $\mathcal{B}^{\lambda}_{\alpha,p}$)-capacity.*

For an arbitrary nonnegative measure μ, define the modified maximal function by

$$\mathcal{M}_{\alpha,\rho}\mu(x) = \sup\{r^{\alpha-Q}\mu(B(x,r)) : 0 < r \leq \rho\},$$

where $0 \leq \alpha < Q$ and $\rho > 0$. If $\rho = \infty$, $\mathcal{M}_{\alpha,\rho}\mu$ is just the maximal function $\mathcal{M}_\alpha\mu$. Here we remark that Morrey spaces $\mathcal{L}^{1,\lambda}(\mathbb{G})$ consists of all Radon measures μ on $\mathbb{G}$ satisfying

$$\| \mu \|_{\mathcal{L}^{1,\lambda}(\mathbb{G})} = \sup\{\mathcal{M}_{Q-\lambda,\rho}|\mu|(x) : x \in \mathbb{G} \text{ and } \rho > 0\} < \infty,$$

where $|\mu|$ stands for the total variation measure of μ.

For the characteristic function $\chi_B(x,r)$ on the set $B(x,r)$, let $d\mu_1 = \chi_{B(x,r)}d\mu$. Since

$$\| \mathcal{I}_\alpha\mu \|_{L^p(\mathbb{G})} \lesssim \| \mathcal{M}_\alpha\mu \|_{L^p(\mathbb{G})}$$

(refer to [26] and [28]), we know that

$$\| \mathcal{I}_\alpha\mu \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})} = \sup_{x\in\mathbb{G},r>0} r^{\frac{-\lambda}{p}} \| \mathcal{I}_\alpha\mu_1 \|_{L^p(\mathbb{G})} \lesssim \sup_{x\in\mathbb{G},r>0} r^{\frac{-\lambda}{p}} \| \mathcal{M}_\alpha\mu_1 \|_{L^p(\mathbb{G})} = \| \mathcal{M}_\alpha\mu \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})},$$

and so we obtain the next theorem.

Theorem 2.5 *For $1 < p < \infty$ and $0 < \alpha, \lambda < Q$,* $\| \mathcal{I}_\alpha \mu \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})} \lesssim \| \mathcal{M}_\alpha \mu \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})}$.

Now we infer the estimate for the αth fractional Bessel potential and αth fractional maximal function in Morrey space on Carnot group, refer to [9] for the Euclidean setting.

Theorem 2.6 *For $1 < p < \infty$ and $0 < \alpha, \lambda < Q$,* $\| \mathcal{J}_\alpha \mu \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})} \lesssim \| \mathcal{M}_\alpha \mu \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})}$.

Proof Set $\rho = 1$. Following the similar method of the truncation function for [29, Theorem 2] and the sharp estimate of the fundamental solution to heat equations in [29, Theorem 1], we know that

$$\| \mathcal{J}_\alpha \mu_1 \|_{\mathcal{L}^p(\mathbb{G})} \lesssim \| \mathcal{M}_{\alpha,1} \mu_1 \|_{L^p(\mathbb{G})} .$$

Moreover,

$$\| \mathcal{J}_\alpha \mu \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})} \lesssim \| \mathcal{M}_{\alpha,1} \mu \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})} . \quad \square$$

Define the Wolff potential as follows

$$\mathcal{W}^\lambda_{\alpha,p}\mu(x) = \int_0^1 \Big[\frac{\mu(B(x,s))}{s^{Q-\lambda+p(\lambda-\alpha)}}\Big]^{p'-1}\frac{\mathrm{d}s}{s}, \quad \alpha p \le Q + (p-1)\lambda.$$

To apply the dyadic decomposition, we will use subdivisions of $\mathbb{G}$ into dyadic cubes, and subdivide $\mathbb{G}$ so that each cube in one generation is split into 2^Q cubes of the next generation. For such dyadic cube $\mathcal{Q}$, let $|\mathcal{Q}|$ be the volume of $\mathcal{Q}$, $\ell(\mathcal{Q})$ be its sidelength and $\chi(x)$ be the characteristic function of $\mathcal{Q}$. Besides, The cube concentric to $\mathcal{Q}$ with sidelength $3\ell(\mathcal{Q})$ is denoted by $\widetilde{\mathcal{Q}}$. Define

$$\mathbb{W}^\lambda_{\alpha,p}\mu(x) = \sum_{\ell(\mathcal{Q})\le 1} \Big(\frac{\mu(\varphi_\mathcal{Q})}{\ell(\mathcal{Q})^{Q-\lambda+p(\lambda-\alpha)}}\Big)^{p'-1}\varphi_\mathcal{Q},$$

here, for a C^∞ function $\varphi_\mathcal{Q}$, $\chi_\mathcal{Q} \le \varphi_\mathcal{Q} \le \chi_{\widetilde{\mathcal{Q}}}$ and $\mu(\varphi_\mathcal{Q}) = \int_\mathbb{G} \varphi_\mathcal{Q}\mathrm{d}\mu$.

For $1 < p < \infty$, define by $\mathcal{A}_p$ the set of all non-negative functions ω on Q via

$$\sup_\mathcal{Q} \Big(\int_\mathcal{Q} \omega(y)\frac{\mathrm{d}y}{|\mathcal{Q}|}\Big)\Big(\int_\mathcal{Q} (\omega(y))^{\frac{1}{1-p}}\frac{\mathrm{d}y}{|\mathcal{Q}|}\Big)^{p-1} < \infty.$$

Letting $p \to 1$, we denote by $\mathcal{A}_1$ the set of all non-negative functions ω on Q satisfying

$$\sup_\mathcal{Q} \Big(\int_\mathcal{Q} \omega(y)\frac{\mathrm{d}y}{|\mathcal{Q}|}\Big)\Big(\inf_{y\in\mathcal{Q}} \omega(y)\Big)^{-1} < \infty.$$

Note that $\mathcal{A}_1 \subset \cap_{1<p<\infty}\mathcal{A}_p$ and $\mathcal{A}_\infty = \cup_{1<p<\infty}\mathcal{A}_p$. Moreover, put $\mathcal{A}^\lambda_1 = \mathcal{A}_1 \cap \mathcal{W}_\lambda$ for $0 < \lambda < Q$, where $\mathcal{W}_\lambda$ is the set of all non-negative functions ω on Q satisfying

$$\int_\mathcal{Q} \omega \mathrm{d}\Lambda^{(\infty)}_\lambda := \int_0^\infty \Lambda^{(\infty)}_\lambda(\{x \in Q : \omega(x) > t\}) \le 1.$$

By the same procedure of proof of [10, Theorem 21] and [11, Theorem 2.1] for the Euclidean setting and Theorem 2.5 above, and [27, Theorem 2.16], we may establish the next result.

Theorem 2.7 *For $1 < p < \infty$ and $0 < \alpha, \lambda < Q$, let $\alpha p < Q + (p-1)\lambda$, $p' = p/(p-1)$ and μ be any nonnegative measure on $\mathbb{G}$. Then*

$$\| \mathcal{I}_\alpha \mu \|^{p'}_{\mathcal{L}^{p',\lambda}(\mathbb{G})} \approx \int_\mathbb{G} \mathcal{W}^\lambda_{\alpha,p}\mu(y)\mathrm{d}\mu(y) \approx \sup_{\omega\in\mathcal{A}^\lambda_1} \int_\mathbb{G} W^{\omega,1}_{\alpha,p}\mu(y)\mathrm{d}\mu(y)$$

and for some constant $C>0$,

$$C^{-1}\int_{\mathbb{G}}\mathcal{W}^{\lambda}_{\alpha,p}\mu(y)\mathrm{d}\mu(y)\le\|\mathcal{J}_\alpha\mu\|^{p'}_{\mathcal{L}^{p',\lambda}(\mathbb{G})}\le C\int_{\mathbb{G}}\mathcal{W}^{\lambda}_{\alpha,p}\mu(y)\mathrm{d}\mu(y).$$

In addition

$$\|\mathcal{I}_\alpha\mu\|^{p'}_{\mathcal{H}^{p',\lambda}(\mathbb{G})}\approx\inf_{\omega\in\mathcal{W}_\lambda}\int_{\mathbb{G}}(\mathcal{I}_\alpha\mu(y))^{p'}(\omega(y))^{1-p'}\mathrm{d}y\approx\inf_{\omega\in\mathcal{A}_1^\lambda}\int_{\mathbb{G}}W^{\omega,2}_{\alpha,p}\mu(y)\mathrm{d}\mu(y),$$

where

$$W^{\omega,k}_{\alpha,p}\mu(y)=\begin{cases}\int_0^\infty(t^{\alpha p-Q}\mu(B(y,t)))^{\frac{1}{p-1}}(\int_{B(y,t)}\omega(z)\mathrm{d}z)\frac{\mathrm{d}t}{t^{Q+1}}, & k=1;\\ \int_0^\infty(t^{\alpha p-Q}\mu(B(y,t)))^{\frac{1}{p-1}}(\int_{B(y,t)}(\omega(z))^{\frac{1}{1-p}}\mathrm{d}z)\frac{\mathrm{d}t}{t^{Q+1}}, & k=2.\end{cases}$$

Theorem 2.8 *For $1<p<\infty$, $0<\alpha,\lambda<Q$ and given any nonnegative Radon measure μ,*

$$\mathcal{B}^{\lambda}_{\alpha,p}(\{\mathcal{W}^{\lambda}_{\alpha,p}\mu>t\})\lesssim t^{-p+1}\|\mu\|_1,\alpha p\le Q+(p-1)\lambda,$$

where $\|\mu\|_1$ is the total variation of μ.

Proof Let ν be the capacitary measure for the compact subset K of $\{\mathcal{W}^{\lambda}_{\alpha,p}\mu>t\}$ so that $\nu(K)=\mathcal{B}^{\lambda}_{\alpha,p}(K)$. For $x\in\operatorname{supp}\nu$, we put

$$M_\nu\mu(x)=\sup_{r>0}\frac{\mu(B(x,r/5))}{\nu(B(x,r))}.$$

Therefore,

$$\mathcal{W}^{\lambda}_{\alpha,p}\mu(x)=\int_0^1\Big[\frac{\mu(B(x,r))}{r^{Q-\lambda+p(\lambda-\alpha)}}\Big]^{p'-1}\frac{\mathrm{d}r}{r}\le CM_\nu\mu(x)^{p'-1}\mathcal{W}^{\lambda}_{\alpha,p}\nu(x).$$

Since $\mathcal{W}^{\lambda}_{\alpha,p}\nu(x)\le C\mathbb{W}^{\lambda}_{\alpha,p}\nu(x)$, and $\mathbb{W}^{\lambda}_{\alpha,p}\mu(x)\le 1$ on $\operatorname{supp}\nu$, we know that

$$\mathcal{W}^{\lambda}_{\alpha,p}\mu(x)\le CM_\nu\mu(x)^{p'-1}.$$

Then

$$\operatorname{supp}\nu\subset\{t<CM_\nu\mu(x)^{p'-1}\}=\{(\frac{t}{C})^{p-1}<M_\nu\mu(x)\}.$$

From the covering lemma [33] the $\operatorname{supp}\nu$ can be covered with a union of balls $B_j=B(x_j,r_j)$ so that the balls $\frac{1}{5}B_j=B(x_j,r_j/5)$ are disjoint, and

$$\frac{\mu(B(x,r/5))}{\nu(B(x,r))}>(\frac{t}{C})^{p-1}.$$

Hence, we know that

$$\mathcal{B}^{\lambda}_{\alpha,p}(K)=\nu(K)\le\sum_j\nu(B_j)<(\frac{C}{t})^{p-1}\sum_j\mu(\frac{1}{5}B_j)<(\frac{C}{t})^{p-1}\|\mu\|_1,$$

and so Theorem 2.8 holds. □

Further, by Theorem 2.8 above we may yield the below theorem.

Theorem 2.9 *For a compact set e in $\mathbb{G}$, let μ be a measure supported on e, and $\alpha p\le Q+(p-1)\lambda$.*

(I) If $\mathcal{W}^{\lambda}_{\alpha,p}\mu(x)\le 1$ for all $x\in e$, then $\mathcal{B}^{\lambda}_{\alpha,p}(e)\ge c_1\|\mu\|_1$;

(II) If $\mathcal{W}^{\lambda}_{\alpha,p}\mu(x)\ge 1$ for all $x\in e$, then $\mathcal{B}^{\lambda}_{\alpha,p}(e)\le c_2\|\mu\|_1$, where c_1,c_2 depend only on α,p,λ and Q.

Next we provide the estimates of Riesz and Bessel capacity of a ball. In the Euclidean setting in [8], Adams and Xiao already proved the estimate of Riesz capacity for Morrey space of a ball. Besides, in Carnot group setting, Lu [27] showed the estimate of Bessel capacity for Lebesgue space of a ball. Here we leave out the detailed information for their proofs.

Corollary 2.10 *Let $0<\alpha,\lambda<Q$.*

(I) If $1<p<(Q-\lambda)/\alpha$, then

$$\mathcal{R}^{\lambda}_{\alpha,p}(B(x,r))\sim r^{Q-\lambda-\alpha p}, B(x,r)\subset\mathbb{G}.$$

(II) If $1<p=(Q-\lambda)/\alpha$, then

$$\mathcal{R}^{\lambda}_{\alpha,p}(B(x,r))\sim(\log\frac{1}{r})^{-p}, B(x,r)\subset\mathbb{G},\quad r\to 0.$$

Corollary 2.11 *Let $0<\alpha,\lambda<Q$.*

(I) If $1<p<(Q-\lambda)/\alpha$ and $r\le 1$, then

$$\mathcal{B}^{\lambda}_{\alpha,p}(B(x,r))\sim r^{Q-\lambda-\alpha p}, B(x,r)\subset\mathbb{G}.$$

(II) If $1<p=(Q-\lambda)/\alpha$ and $r\le 1$, then

$$\mathcal{B}^{\lambda}_{\alpha,p}(B(x,r))\sim(\log\frac{2}{r})^{1-p}, B(x,r)\subset\mathbb{G},\quad r\to 0.$$

Now we consider the relation between Housdorff measure and Bessel capacity.

Theorem 2.12 *Let $e\in\mathbb{G}$ be a compact set and $h(\rho)$ a nondecreasing continuous function with $h(0)=0$. If*

$$\int_0^1[\frac{h(s)}{s^{Q-\lambda+p(\lambda-\alpha)}}]^{p'-1}\frac{\mathrm{d}s}{s}<\infty,\tag{2.1}$$

then there exists a constant c so that $\Lambda^{(\infty)}_h(e)\le c\mathcal{B}^{\lambda}_{\alpha,p}(e)$. Moreover, $\Lambda^{(\infty)}_h(e)=0$ when $\mathcal{B}^{\lambda}_{\alpha,p}(e)=0$.

To demonstrate the above theorem, we recall the next lemma.

Lemma 2.13 ([29]) *Take a compact metric space X and a compact set $K\subset X$. There exists a Radon measure ω on K such that $\omega(K)=\lambda^{\rho}_h(K)$ and $\omega(E)\le h(\operatorname{diam}E/2)$ for all $K\subset X$ with $\operatorname{diam}E\le 2\rho$.*

Here the set function $\lambda^{\rho}_h(E)$ is denoted by

$$\lambda^{(\rho)}_h(E):=\inf\sum c_j h(r_j),$$

where the infimum is taken over all countable coverings of $E\subset X$ by open balls with radius $r_j\le\rho$ and $0<c_j\le 1$. Obviously, for all $E\subset X$ and $\rho>0$, $\lambda^{(\rho)}_h(E)\le\Lambda^{(\rho)}_h(E)$ holds.

Proof of Theorem 2.12 From Lemma 2.13, we know that there exists a nonzero measure μ supported on $e\in\mathbb{G}$ such that for $\rho,\ell>0$, $\mu(B(x,\rho))\le h(\rho)$ and $\Lambda^{\infty}_h(e)\le\ell\mu(e)$ hold. Moreover, by (1), it infers that $\mathcal{W}^{\lambda}_{\alpha,p}\mu(x)\le\mathcal{M}$ for every $x\in e$ and the certain constant $\mathcal{M}>0$. Therefore,

$$\mathcal{B}^{\lambda}_{\alpha,p}(e)\ge C\parallel\frac{\mu}{\mathcal{M}^{p-1}}\parallel_1=C\mathcal{M}^{1-p}\parallel\mu\parallel_1.$$

Hence, $\Lambda_h^{(\infty)}(e) \le C^{-1}\ell\mathcal{M}^{p-1}\mathcal{B}^{\lambda}_{\alpha,p}(e)$, and so Theorem 2.12 is true. □

Now we make slightly some changes for the procedure of Vodoṕyanov in [29] and establish the following theorem.

Theorem 2.14 *Let $e \in \mathbb{G}$ be a compact set and*

$$h(r) = \begin{cases} r^{Q+(p-1)\lambda-\alpha p}, & \text{for } \alpha p < Q+(p-1)\lambda, \\ (\log_+ 2/r)^{1-p}, & \text{for } \alpha p = Q+(p-1)\lambda. \end{cases}$$

If the Housdorff measure $\Lambda_h(e)$ of e is finite, then $\mathcal{B}^{\lambda}_{\alpha,p}(e) = 0$.

Proof Here we will use the proofs by Contradiction. Assume that $\mathcal{B}^{\lambda}_{\alpha,p}(e) > 0$. For the capacity measure μ_e of e, we see that $\mathcal{U}_{\alpha,p}\mu_e(x) \le 1$ for $x \in \mathrm{supp}\mu_e$, where

$$\mathcal{U}_{\alpha,p}\mu_e(x) = \int_{\mathbb{G}} \mathcal{J}_\alpha(x^{-1}y)[\mathcal{J}_\alpha\mu_e(y)]^{p'-1}\mathrm{d}y,$$

and $\mathcal{W}^{\lambda}_{\alpha,p}\mu_e(x) \le \mathcal{M}$ on $\mathbb{G}$. Set

$$\mathcal{F}_\delta(x) = \int_\delta^1 [\frac{\widetilde{\mu}_e(B(x,s))}{s^{Q-\lambda+p(\lambda-\alpha)}}]^{p'-1}\frac{\mathrm{d}s}{s}$$

for $0 \le \delta \le 1$ and the restriction $\widetilde{\mu}_e$ of μ_e on the any compact set K with $\mu_e(K) > 0$. Thus, $\mathcal{F}_\delta$ is continuous and converges uniformly to $\mathcal{W}^{\lambda}_{\alpha,p}\widetilde{\mu}_e$ on any ball $B \supset K$ when $\delta \to 0$. Now we choose the suitable B so that $\mathcal{F}_\delta(x) = \mathcal{W}^{\lambda}_{\alpha,p}\widetilde{\mu}_e(x) = 0$ for $x \in \mathbb{G}/B$. Then

$$\sum_{j=1}^{\infty}\int_{(j+1)^{-1}}^{j^{-1}} [\frac{\widetilde{\mu}_e(B(x,s))}{s^{Q-\lambda+p(\lambda-\alpha)}}]^{p'-1}\frac{\mathrm{d}s}{s}$$

converges uniformly in $\mathbb{G}$, and is no more than the value $\mathcal{M}$ everywhere. Letting

$$H_j(x) = \int_{(j+1)^{-1}}^{j^{-1}} [\frac{\widetilde{\mu}_e(B(x,s))}{s^{Q-\lambda+p(\lambda-\alpha)}}]^{p'-1}\frac{\mathrm{d}s}{s},$$

we may take a monotone vanishing sequence of positive n_j $(j \in \mathbb{N})$ so that $\sum_{j=1}^{\infty} H_j(x)/n_j$ also converges uniformly in $\mathbb{G}$ to the sum being not greater than $2\mathcal{M}$. Therefore,

$$\int_0^1 [\frac{\widetilde{\mu}_e(B(x,s))}{s^{Q-\lambda+p(\lambda-\alpha)}}]^{p'-1}\frac{\mathrm{d}s}{n(s)s} \le 2\mathcal{M} \tag{2.2}$$

for each $x \in \mathbb{G}$, here $n(s) = n_j$ when $s \in [(j+1)^{-1}, j^{-1}]$ for $j \in \mathbb{N}$. For the dilatation constant κ, let $0 < r \le \frac{1}{4\kappa}$ so that the ball $B_r \subset \mathbb{G}$ with radius r. When $\alpha p < Q+(p-1)\lambda$ for $x \in B_r$, we know that

$$\int_0^1 [\frac{\widetilde{\mu}_e(B(x,s))}{s^{Q-\lambda+p(\lambda-\alpha)}}]^{p'-1}\frac{\mathrm{d}s}{n(s)s} \ge \int_{2\kappa r}^{4\kappa r} [\frac{\widetilde{\mu}_e(B(x,s))}{s^{Q-\lambda+p(\lambda-\alpha)}}]^{p'-1}\frac{\mathrm{d}s}{n(s)s} \ge \ell[\frac{\widetilde{\mu}_e(B_r)}{r^{Q-\lambda+p(\lambda-\alpha)}}]^{p'-1}\frac{1}{n(2\kappa r)}.$$

From (2.2), we imply that

$$\ell\widetilde{\mu}_e(B_r)/n(2\kappa r)^{p-1} \le (2\mathcal{M})^{p-1}r^{Q-\lambda+p(\lambda-\alpha)}.$$

When $\alpha p = Q+(p-1)\lambda$, we may infer that

$$\int_0^1 [\widetilde{\mu}_e(B(x,s))]^{p'-1}\frac{\mathrm{d}s}{n(s)s} \ge \int_{2\kappa r}^1 [\widetilde{\mu}_e(B(x,s))]^{p'-1}\frac{\mathrm{d}s}{n(s)s} \ge \ell[\widetilde{\mu}_e(B_r)]^{p'-1}\frac{\log\frac{1}{2\kappa r}}{n(2\kappa r)}.$$

Similarly, from (2.2), we also obtain that $\ell\widetilde{\mu}_e(B_r)/n(2\kappa r)^{p-1} \le (2\mathcal{M})^{p-1}(\log \frac{1}{2\kappa r})^{1-p}$.

Now we choose an at most countable covering $\Gamma = \{B_j\}$ of K such that all the radiuses r_j are less than ϵ. Therefore, for the case $\alpha p < Q + (p-1)\lambda$,

$$\ell\frac{\widetilde{\mu}_e(K)}{n(\epsilon)^{p-1}} \le \ell\sum_{B_j\in\Gamma}\frac{\widetilde{\mu}_e(B_j)}{n(\epsilon)^{p-1}} \le (2\mathcal{M})^{p-1}\sum_{B_j\in\Gamma} r_j^{Q-\lambda+p(\lambda-\alpha)}$$

and for the case $\alpha p = Q + (p-1)\lambda$,

$$\ell\frac{\widetilde{\mu}_e(K)}{n(\epsilon)^{p-1}} \le \ell\sum_{B_j\in\Gamma}\frac{\widetilde{\mu}_e(B_j)}{n(\epsilon)^{p-1}} \le (2\mathcal{M})^{p-1}\sum_{B_j\in\Gamma}(\log\frac{1}{2\kappa r_j})^{1-p}.$$

If the Housdorff measure $\Lambda_h(e)$ of e is finite, then $\frac{\widetilde{\mu}_e(K)}{n(\epsilon)^{p-1}}$ is bounded. But, since $\widetilde{\mu}_e(K) > 0$ and $\lim_{\varepsilon\to 0} n(\epsilon) = 0$, $\frac{\widetilde{\mu}_e(K)}{n(\epsilon)^{p-1}}$ is unbounded. Hence, there exists a contradiction, and so Theorem 2.14 holds. □

Theorem 2.15 *For any set $E \subset \mathbb{G}$, let $\widetilde{E} = E \cap B(x,r) \subset \mathbb{G}$ with $x \in \mathbb{G}$ and $r > 0$. Then*

$$|\widetilde{E}|^{(Q-\lambda-\alpha p)/(Q-\lambda)} \lesssim r^{\lambda(Q-\lambda-\alpha p)/(Q-\lambda)}\mathcal{R}^{\lambda}_{\alpha,p}(E)$$

and

$$|\widetilde{E}|^{(Q-\lambda-\alpha p)/(Q-\lambda)} \lesssim r^{\lambda(Q-\lambda-\alpha p)/(Q-\lambda)}\mathcal{B}^{\lambda}_{\alpha,p}(E)$$

for $0 < \alpha, \lambda < Q$ and $1 < p < (Q-\lambda)/\alpha$.

Proof From the definitions of capacity and Hölder inequality, we know that if $\mathcal{I}_\alpha f(x) \ge 1$ on E, then

$$|\widetilde{E}| \le \int_{\widetilde{E}} |\mathcal{I}_\alpha f(x)|\mathrm{d}x \le \Big(\int_{\widetilde{E}} |\mathcal{I}_\alpha f(x)|^{\widetilde{p}}\mathrm{d}x\Big)^{1/\widetilde{p}}|\widetilde{E}|^{1-(Q-\lambda-\alpha p)/\{(Q-\lambda)p\}},$$

where $\widetilde{p} = (Q-\lambda)p/(Q-\lambda-\alpha p)$. Therefore,

$$\begin{aligned}|\widetilde{E}|^{(Q-\lambda-\alpha p)/\{(Q-\lambda)p\}} &\le \Big(\int_{\widetilde{E}} |\mathcal{I}_\alpha f(x)|^{\widetilde{p}}\mathrm{d}x\Big)^{1/\widetilde{p}}\\ &\le \Big(\int_{B(x,r)} |\mathcal{I}_\alpha f(x)|^{\widetilde{p}}\mathrm{d}x\Big)^{1/\widetilde{p}}\\ &\lesssim r^{\lambda(Q-\lambda-\alpha p)/\{(Q-\lambda)p\}} \parallel \mathcal{I}_\alpha f \parallel_{\mathcal{L}^{\widetilde{p},\lambda}(\mathbb{G})}\\ &\lesssim r^{\lambda(Q-\lambda-\alpha p)/\{(Q-\lambda)p\}} \parallel f \parallel_{\mathcal{L}^{p,\lambda}(\mathbb{G})},\end{aligned}$$

here, we apply the boundedness of $\mathcal{I}_\alpha$ from $\mathcal{L}^{p,\lambda}(\mathbb{G})$ to $\mathcal{L}^{\widetilde{p},\lambda}(\mathbb{G})$. Clearly,

$$|\widetilde{E}|^{(Q-\lambda-\alpha p)/(Q-\lambda)} \lesssim r^{\lambda(Q-\lambda-\alpha p)/(Q-\lambda)}\mathcal{R}^{\lambda}_{\alpha,p}(E).$$

Since $J_\alpha(x) \le cI_\alpha(x)$, we also infer that

$$|\widetilde{E}|^{(Q-\lambda-\alpha p)/(Q-\lambda)} \lesssim r^{\lambda(Q-\lambda-\alpha p)/(Q-\lambda)}\mathcal{B}^{\lambda}_{\alpha,p}(E).$$

At last, based on Remark in [1], we provide the following conjecture in Morrey setting.

Conjecture 2.16 Let $0 < \alpha, \lambda < Q$ and $1 < p < (Q-\lambda)/\alpha$. Then there exists a constant

$C = C(\alpha, p, \lambda, Q)$ so that for all $E \subset \mathbb{G}$,

$$C^{-1}\mathcal{R}^{\lambda}_{\alpha,p}(E) \leq \mathcal{B}^{\lambda}_{\alpha,p}(E) \leq C(\mathcal{R}^{\lambda}_{\alpha,p}(E) + \mathcal{R}^{\lambda}_{\alpha,p}(E)^{\frac{Q-\lambda}{Q-\lambda-\alpha p}}).$$

Remark 2.17 In fact, we can easily prove the former inequality. For $J_\alpha(x) \leq cI_\alpha(x)$, if $\mathcal{J}_\alpha f(x) \geq 1$ on E, then $\mathcal{I}_\alpha(cf)(x) \geq 1$ on E. Hence,

$$\mathcal{R}^{\lambda}_{\alpha,p}(E) \leq \| cf \|^p_{\mathcal{L}^{p,\lambda}(\mathbb{G})} = c^p \| f \|^p_{\mathcal{L}^{p,\lambda}(\mathbb{G})},$$

which infers

$$\mathcal{R}^{\lambda}_{\alpha,p}(E) \leq c^p \mathcal{B}^{\lambda}_{\alpha,p}(E).$$

Acknowledgements We thank the referees for their time and comments, particularly, the suggestions of Theorem 2.7 of original manuscript, so that this article is greatly improved.

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