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Nodal Solutions with a Prescribed Number of Nodes for Quasilinear Schrödinger Equations with a Cubic Term

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Abstract This paper is concerned with the existence of nodal solutions for the following quasilinear Schrödinger equation with a cubic term

$$\begin{cases} -\Delta u+V(|x|)u-\frac{1}{2}\Delta(|u|^2)u=\lambda|u|^2u, & \text{in } \mathbb{R}^N,\\ u\to 0, & \text{as } |x|\to\infty, \end{cases}$$

where $N\geq 3$, $\lambda>0$, the function $V(|x|)$ is a radially symmetric and positive potential. By using the variational method and energy comparison method, for any given integer $k\geq 1$, the above equation admits a radial nodal solution $U_{k,4}^{\lambda}$ having exactly k nodes via a limit approach. Furthermore, the energy of $U_{k,4}^{\lambda}$ is monotonically increasing in k and for any sequence $\{\lambda_n\}$, up to a subsequence, $\lambda_n^{\frac{1}{2}}U_{k,4}^{\lambda_n}$ converges strongly to some $\bar{U}_{k,4}^{0}$ as $\lambda_n\to+\infty$, which is a radial nodal solution with exactly k nodes of the classical Schrödinger equation

$$\begin{cases} -\Delta u+V(|x|)u=|u|^2u, & \text{in } \mathbb{R}^N,\\ u\to 0, & \text{as } |x|\to\infty. \end{cases}$$

Our results extend the existing ones in the literature from the super-cubic case to the cubic case.

Keywords quasilinear Schrödinger equations; nodal solutions; limit approach; variational method

MR(2020) Subject Classification 35A15; 35J20; 35J50

1. Introduction

We consider the existence of nodal solutions to the following quasilinear Schrödinger equation in the whole space $\mathbb{R}^N$

$$\begin{cases} -\Delta u+V(x)u-\dfrac{1}{2}\Delta(|u|^2)u=f(u), & x\in\mathbb{R}^N,\\ u\to 0, & \text{as } |x|\to\infty. \end{cases} \tag{1.1}$$

The solution of Eq. (1.1) is related to the standing waving solutions of a more general quasilinear problems of this form

$$i\partial_t z=-\Delta z+V(x)z-f(|z|^2)z-\kappa\Delta h(|z|^2)h'(|z|^2)z=0, \tag{1.2}$$

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where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $V : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $\kappa > 0$ is a real number and f, h are real functions. Eq. (1.2) was used to describe several physical phenomena corresponding to various type of the function h. For example, the case $h(s) = s$ corresponds to the time evolution of the condensate wave function in super-fluid film. One can refer to [1–5] and references therein for more details and applications. Based on the work of Poppenberg [6], Eq. (1.1) has attracted much attention in the mathematical literature. We can see [7–24] for the existence and properties of ground state solutions, positive solutions, multiple solutions and sign-changing solutions to (1.1) by using the variational method. In the variational formulation, the main mathematical difficulties lie in the presence of nonlinear functional $\int_{\mathbb{R}^N} u^2|\nabla u|^2$, which is homogeneous of order 4 and non-convex. Besides, another difficulty is caused by the loss of compactness due to the whole space $\mathbb{R}^N$.

In the past decades, great progress has been made in studying sign-changing solutions of Eq. (1.1). Corresponding to the model $f(u) = \lambda|u|^{q-2}u$ with $\lambda > 0$, Eq. (1.1) is reduced to the following equation

$$-\Delta u + V(|x|)u - \frac{1}{2}\Delta(|u|^2)u = \lambda|u|^{q-2}u, \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

For $q \in [4, 22^*)$, Liu, Wang and Wang [16] established the existence of both one-sign and nodal ground state solutions for (1.3) by the Nehari method. In [25], Deng, Peng and Wang proved that for any given integer $k \geq 1$, (1.3) admits radial nodal solutions with exactly k nodes by using a minimization argument and the energy comparison method for $q \in (4, 22^*)$ with $22^* = \frac{4N}{N-2}$. Later, Deng, Peng and Wang [26] extended this result to the critical growth case. One can also refer to [27–29] for more related results on sign-changing solutions. Now a natural question arises whether or not (1.3) has such type nodal solutions with the nodal characterization when $q \leq 4$? To the best of our knowledge, this problem still remains unsolved.

Motivated by this, this paper is devoted to the existence of nodal solutions for (1.3) in the cubic case $q = 4$. Namely, we consider the following quasi-linear Schrödinger equation

$$\begin{cases} -\Delta u + V(|x|)u - \frac{1}{2}\Delta(|u|^2)u = \lambda|u|^2u, & x \in \mathbb{R}^N, \\ u \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.4}$$

where $N \geq 3$ and the potential V satisfies the following condition:

(V): the potential $V \in C(\mathbb{R}^N, \mathbb{R})$ is bounded from below by a positive constant V_0.

Let $H_r^1(\mathbb{R}^N)$ be the radial Sobolev subspace of $H^1(\mathbb{R}^N)$ and

$$X := \Big\{u \in H_r^1(\mathbb{R}^N) \Big| \int_{\mathbb{R}^N} V(|x|)u^2 < +\infty, \int_{\mathbb{R}^N} |\nabla u|^2|u|^2 < +\infty\Big\}$$

with the norm $\|u\| = (\int_{\mathbb{R}^N} |\nabla u|^2 + V(|x|)u^2)^{1/2}$. As usual, we define the energy functional $I_{\lambda,4} : X \to \mathbb{R}$ associated with (1.4) by

$$I_{\lambda,4}(u) := \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2 - \frac{\lambda}{4}\int_{\mathbb{R}^N} |u|^4. \tag{1.5}$$

As stated in [25], for any fixed $u \in X$, the functional $I_{\lambda,4}$ has the Gateaux derivative along any direction $v \in C_0^\infty(\mathbb{R}^N)$ at u, that is,

$$\langle I'_{\lambda,4}(u), v\rangle = \int_{\mathbb{R}^N} ((1+u^2)\nabla u \nabla v + |\nabla u|^2 uv + V(|x|)uv - \lambda |u|^2 uv).$$

Then $u \in X$ is called a weak solution of (1.4) if and only if $\langle I'_{\lambda,4}(u), v\rangle = 0$ for all $v \in C_0^\infty(\mathbb{R}^N)$. By considering the ground state energy

$$m := \inf_{u\in\mathcal{N}} I_{\lambda,4}(u) \tag{1.6}$$

constraint on the Nehari manifold

$$\mathcal{N} = \{u \in X\backslash\{0\} : \langle I'_{\lambda,4}(u), u\rangle = 0\}, \tag{1.7}$$

it was proved in [17, Theorem 1] that there is a positive ground state solution $U_0 \in \mathcal{N}$ of (1.4) such that

$$I_{\lambda,4}(U_0) = m > 0. \tag{1.8}$$

For $k \in \mathbb{N}_+$ and $0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := +\infty$, define $\mathbf{r}_k = (r_1, \ldots, r_k)$ and $\Gamma_k = \{\mathbf{r}_k = (r_1, \ldots, r_k) \in (0,\infty)^k : 0 =: r_0 < r_1 < \ldots < r_k < r_{k+1} := \infty\}$. Let

$$B_1^{\mathbf{r}_k} := \{x \in \mathbb{R}^N : 0 \le |x| < r_1\},\ B_i^{\mathbf{r}_k} := \{x \in \mathbb{R}^N : r_{i-1} < |x| < r_i\},\quad i = 2, \ldots, k+1.$$

Obviously, $B_1^{\mathbf{r}_k}$ is a ball, $B_2^{\mathbf{r}_k}, \ldots, B_k^{\mathbf{r}_k}$ are annulus and $B_{k+1}^{\mathbf{r}_k}$ is the complement of a ball. To state our results, we introduce a Nehari type set

$$\begin{aligned}\mathcal{N}_{k,4} = \big\{u \in X : &\text{ there exists } \mathbf{r}_k \in \Gamma_k \text{ such that } u_i \ne 0 \text{ in } B_i^{\mathbf{r}_k},\\ &\langle I'_{\lambda,4}(u), u_i\rangle = 0, \text{ for all } i \in \{1, \ldots, k+1\}\big\},\end{aligned} \tag{1.9}$$

and consider the infimum level

$$c_{k,4} = \inf_{u\in\mathcal{N}_{k,4}} I_{\lambda,4}(u), \tag{1.10}$$

where $u_i = u$ in $B_i^{\mathbf{r}_k}$ and $u_i = 0$ on $\partial B_i^{\mathbf{r}_k}$.

Now we are in position to illustrate our main results. First we give the existence result.

Theorem 1.1 *Suppose that $\lambda > 0$ and (V) holds. Then for each $k \in \mathbb{N}_+$, problem (1.4) admits a radial nodal solution $U_{k,4} \in \mathcal{N}_{k,4}$ having exactly k nodes such that $I_{\lambda,4}(U_{k,4}) = c_{k,4}$.*

The main difficulty in the proof of Theorem 1.1 lies in that the term $\frac{1}{2}\Delta(|u|^2)u$ is homogeneous of order 3 and non-convex, which competes complicatedly with the cubic term $\lambda|u|^2u$. So the nonempty of $\mathcal{N}_{k,4}$ is not obvious. Even if the nonempty is proved, it is still difficult to show Theorem 1.1 by the Nehari manifold method and gluing method, which depends heavily on the super-cubic condition as stated in [25]. Hence some new ideas are necessary. We shall prove it by the limit approach.

The next result shows the monotonicity and an estimate of the energy of nodal solutions obtained in Theorem 1.1.

Theorem 1.2 *Under the assumptions of Theorem 1.1, the energy of $U_{k,4}$ is strictly increasing in k, i.e.,*

$$I_{\lambda,4}(U_{k+1,4}) > I_{\lambda,4}(U_{k,4}) \text{ for any } k \in \mathbb{N}_+.$$

Moreover, $I_{\lambda,4}(U_{k+1,4}) > (k+1)I_{\lambda,4}(U_0)$, where U_0 is the ground state solution appearing in (1.8).

Clearly, since $U_{k,4}$ obtained in Theorem 1.1 depends on the parameter λ, we shall denote $U_{k,4}$ by $U_{k,4}^{\lambda}$ sometimes to emphasize the dependence. In the following, the asymptotic behavior of $U_{k,4}^{\lambda}$ is studied as $\lambda \to +\infty$.

Theorem 1.3 *Under the assumptions of Theorem 1.1, for any sequence $\{\lambda_n\}$ with $\lambda_n \to +\infty$ as $n \to \infty$, there exists a subsequence, still denoted by $\{\lambda_n\}$, such that $\lambda_n^{\frac{1}{2}} U_{k,4}^{\lambda_n}$ converges to $\bar{U}_{k,4}^{0}$ strongly in X as $n \to \infty$, where $\bar{U}_{k,4}^{0}$ is a least energy radial nodal solution among all the radial nodal solutions with exactly k nodes of the following equation*

$$-\Delta u + V(|x|)u = |u|^2 u. \tag{1.11}$$

Remark 1.4 The above results are still left open for the sub-cubic case $q \in (2,4)$.

The remainder of this paper is organized as follows. In Section 2, we present some elementary results which are useful in the proofs of our main results. In Section 3, we prove the nonempty of Nehari type set $\mathcal{N}_{k,4}$ by construction method and then obtain Theorem 1.1 by applying the limit approach. The energy comparison and asymptotic behavior of the nodal solutions of (1.4) will be investigated in Section 4.

2. Preliminary results

In this section, first we give some notations and elementary results.

- For fixed $\mathbf{r}_k \in \Gamma_k$ and a family of $\{B_i^{\mathbf{r}_k}\}_{i=1}^{k+1}$, we denote

$$H_i^{\mathbf{r}_k} = \{u \in H_0^1(B_i^{\mathbf{r}_k}) : u(x) = u(|x|), u(x) = 0 \text{ if } x \notin B_i^{\mathbf{r}_k}\}$$

for $i = 1, \ldots, k+1$. It is easy to see that $H_i^{\mathbf{r}_k}$ is a radial Hilbert space with norm denoted by $\|\cdot\|$. Sometimes $\|\cdot\|$ can be written as $\|\cdot\|_i$ (or $\|\cdot\|_{B_i^{\mathbf{r}_k}}$) if we emphasize the integral domain $B_i^{\mathbf{r}_k}$.

Lemma 2.1 ([17]) *Let $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ satisfy $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then*

$$\liminf_{n} \int_{\mathbb{R}^n} |\nabla u_n|^2 |u_n|^2 \mathrm{d}x \geq \int_{\mathbb{R}^n} |\nabla u|^2 |u|^2 \mathrm{d}x.$$

For $1 < s, t < \infty$, we define the space $L^s(\mathbb{R}^N) \bigcap L^t(\mathbb{R}^N)$ with the norm $|u|_{s \bigwedge t} := |u|_{L^s} + |u|_{L^t}$. It is easy to see that $L^s(\mathbb{R}^N) \bigcap L^t(\mathbb{R}^N)$ is a Banach space. Then we have the following lemma.

Lemma 2.2 *Let $\{q_n\}_{n\geq 1} \subset (4, 22^*)$ be a sequence such that $q_n \searrow 4$ as $n \to \infty$. Moreover, for any $s \in (2, 22^*)$, the sequence $\{u_n\}_{n\geq 1} \subset L^s(\mathbb{R}^N)$ satisfies that $u_n \to u_0$ in $L^s(\mathbb{R}^N)$. Then up to a subsequence,*

$$\int_{\mathbb{R}^n} |u_n|^{q_n} \mathrm{d}x \to \int_{\mathbb{R}^n} |u_0|^4 \mathrm{d}x, \quad \text{as } n \to \infty. \tag{2.1}$$

Proof Going if necessary to a subsequence, $u_n \to u_0$ a.e., in $\mathbb{R}^N$. Then $|u_n(x)|^{q_n} \to |u_0(x)|^4$ a.e., in $\mathbb{R}^N$ as $n \to \infty$. For some $\bar{s} \in (4, 22^*)$, we can assume that $q_n \in (4, \bar{s})$ for each n. So

$u_n \to u_0$ in $L^4(\mathbb{R}^N)\bigcap L^{\bar{s}}(\mathbb{R}^N)$ as $n\to\infty$, and up to a subsequence,

$$|u_{n+1}-u_n|_{4\bigwedge \bar{s}} < \frac{1}{2^n} \ \text{ for any } \ n\geq 1.$$

Let

$$g(x) = |u_1(x)| + \sum_{n=1}^{\infty} |u_{n+1}(x)-u_n(x)|.$$

Then $|u_0(x)|, |u_n(x)| \leq g(x)$ for all $n\geq 1$, and $g\in L^4(\mathbb{R}^N)\bigcap L^{\bar{s}}(\mathbb{R}^N)$. Notice that

$$\int_{\mathbb{R}^n} |u_n(x)|^{q_n}\mathrm{d}x \leq \int_{\mathbb{R}^n} |g(x)|^{q_n}\mathrm{d}x \leq C\int_{\mathbb{R}^n} |g(x)|^4\mathrm{d}x + C\int_{\mathbb{R}^n} |g(x)|^{\bar{s}}\mathrm{d}x < \infty.$$

By using the Lebesgue dominated convergence theorem, we obtain (2.1). The proof is completed. □

In the sequel, we shall prove the nonempty of the Nehari type set $\mathcal{N}_{k,4}$ and then show the properties of $\mathcal{N}_{k,4}$.

Lemma 2.3 $\mathcal{N}_{k,4}\neq\emptyset$, *where* $\mathcal{N}_{k,4}$ *is defined in* (1.9).

Proof We take $\mathbf{r}_k=(r_1,\ldots,r_k)\in\Gamma_k$ and $u=\sum_{i=1}^{k+1}u_i\in X$ with $u_i\in H_i^{\mathbf{r}_k}$. For each i, let

$$\delta_0 = \max_{1\leq i\leq n} (\frac{2\int_{B_i^{\mathbf{r}_k}}|u_i|^2|\nabla u_i|^2}{\lambda\int_{B_i^{\mathbf{r}_k}}|u_i|^4})^{1/2}.$$

Then define a radial function by

$$v_i(x) = \delta_i u_i(\frac{x}{\delta_0}), \quad i=1,\ldots,k+1,$$

where $\delta_i\in(0,+\infty)$ is determined later. Then $\mathrm{supp}(v_i)\subset\{x: \delta_0 r_{i-1}\leq|x|\leq\delta_0 r_i\}$. Let $\widetilde{\mathbf{r}}_k=(\delta_0 r_1,\ldots,\delta_0 r_k)$. A direct calculation gives that

$$\begin{aligned}\Big\langle I'_{\lambda,4}\Big(\sum_{i=1}^{k+1}v_i\Big),v_i\Big\rangle =&\|v_i\|^2+2\int_{B_i^{\widetilde{\mathbf{r}}_k}}v_i^2|\nabla v_i|^2-\lambda\int_{B_i^{\widetilde{\mathbf{r}}_k}}|v_i|^4\\ =&\delta_i^2\delta_0^{N-2}\|\nabla u_i\|^2_{L^2(B_i^{\mathbf{r}_k})}+\delta_i^2\delta_0^N\int_{B_i^{\mathbf{r}_k}}V(\delta_0|x|)u_i^2+\\ &2\delta_i^4\delta_0^{N-2}\int_{B_i^{\mathbf{r}_k}}|u_i|^2|\nabla u_i|^2-\lambda\delta_i^4\delta_0^N\int_{B_i^{\mathbf{r}_k}}|u_i|^4\\ =&:g_i(\delta_i).\end{aligned} \tag{2.2}$$

In view of the condition (V), there exists a small number $\delta>0$ and a large number $L>0$ such that $g_i(\delta)>0$ and $g_i(L)<0$. So we can find a $\bar{\delta}_i\in(\delta,L)$ such that $g_i(\bar{\delta}_i)=0$. Set $\delta_i=\bar{\delta}_i$. Then $\langle I'_{\lambda,4}(\sum_{i=1}^{k+1}v_i),v_i\rangle=0$, which shows that $\sum_{i=1}^{k+1}v_i\in\mathcal{N}_{k,4}$. The desired result is proved. □

The following result gives useful properties of $\mathcal{N}_{k,4}$.

Lemma 2.4 *Assume that* $\mathbf{r}_k\in\Gamma_k$ *and* $\sum_{i=1}^{k+1}u_i\in\mathcal{N}_{k,4}$. *Then the following statements are true.*

(i) Each component u_i *is bounded away from zero, that is, there exists* $\rho>0$ *depending on* $\|\sum_{i=1}^{k+1}u_i\|$ *such that*

$$\|u_i\|^2+\int_{B_i^{\mathbf{r}_k}}|u_i|^2|\nabla u_i|^2\geq\rho \ \text{ for all } i=1,\ldots,k+1.$$

(ii)

$$I_{\lambda,4}\Big(\sum_{i=1}^{k+1} t_i u_i\Big) < I_{\lambda,4}\Big(\sum_{i=1}^{k+1} u_i\Big), \quad \forall\, t_i \in (0,1)\cup(1,+\infty). \tag{2.3}$$

Proof (i) Observe that

$$\Big\langle I'_{\lambda,4}\Big(\sum_{i=1}^{k+1} u_i\Big), u_i\Big\rangle = \|u_i\|^2 + 2\int_{B_i^{\mathbf{r}_k}} |u_i|^2|\nabla u_i|^2 - \lambda\int_{B_i^{\mathbf{r}_k}} |u_i|^4 = 0.$$

Let $\eta = \|u_i\|^2 + 2\int_{B_i^{\mathbf{r}_k}} |u_i|^2|\nabla u_i|^2$ and $\theta = \frac{22^*-4}{22^*-2} \in (0,1)$. Then by using Hölder inequality and Sobolev inequality, we obtain

$$\begin{aligned}\eta &\le \lambda\Big(\int_{B_i^{\mathbf{r}_k}} |u_i|^2\Big)^{\theta}\Big(\int_{B_i^{\mathbf{r}_k}} |u_i|^{22^*}\Big)^{1-\theta}\\ &\le \lambda\Big(\int_{B_i^{\mathbf{r}_k}} |u_i|^2\Big)^{\theta}\Big(\int_{B_i^{\mathbf{r}_k}} |u_i|^2|\nabla u_i|^2\Big)^{\frac{N(1-\theta)}{N-2}}\\ &\le C\eta^{\theta}\eta^{\frac{N(1-\theta)}{N-2}}.\end{aligned} \tag{2.4}$$

Let $\rho = \frac{1}{2}C^{\frac{N-2}{2(\theta-1)}} > 0$. Then we deduce from (2.4) that $\|u_i\|^2 + \int_{B_i^{\mathbf{r}_k}} |u_i|^2|\nabla u_i|^2 \ge \rho$. Hence (i) is finished.

(ii) This together with the fact,

$$\frac{dI_{\lambda,4}(\sum_{i=1}^{k+1} t_i u_i)}{dt_i} t_i = \Big\langle I'_{\lambda,4}\Big(\sum_{i=1}^{k+1} t_i u_i\Big), t_i u_i\Big\rangle - t_i^4\Big\langle I'_{\lambda,4}\Big(\sum_{i=1}^{k+1} u_i\Big), u_i\Big\rangle = t_i^2(1-t_i^2)\|u_i\|^2,$$

gives that

$$\frac{dI_{\lambda,4}(\sum_{i=1}^{k+1} t_i u_i)}{dt_i} > 0 \text{ for } t_i \in (0,1)$$

and

$$\frac{dI_{\lambda,4}(\sum_{i=1}^{k+1} t_i u_i)}{dt_i} < 0 \text{ for } t_i \in (1,+\infty).$$

So $t_i = 1$ is the global maximum point of $I_{\lambda,4}(\sum_{i=1}^{k+1} t_i u_i)$. Thus (2.3) follows immediately and the proof is completed. □

3. Existence of nodal solutions

This section is devoted to the proof of Theorem 1.1 by the limit approach and subtle analysis. Before that, we introduce the energy functional of (1.3), which is defined by

$$I_{\lambda,q}(u) := \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 \mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q \mathrm{d}x.$$

For $k \in \mathbb{N}_+$, we define a Nehari type set

$$\begin{aligned}\mathcal{N}_{k,q} :=\{u \in\ X : \text{there exists}\, \mathbf{r}_k \text{ s.t.} u_i \neq 0 \text{ in } B_i^{\mathbf{r}_k} \text{ and}\\ \langle I'_{\lambda,q}(u), u_i\rangle = 0,\ \forall\, i = 1,\ldots,k+1\}\end{aligned}$$

and the corresponding least energy level $c_{k,q} = \inf_{u\in\mathcal{N}_{k,q}} I_{\lambda,q}(u)$. We collect the following result that shows the existence of nodal solutions of (1.3) with exactly k nodes.

Proposition 3.1 ([25, Theorem 1.1]) *For each $k \in \mathbb{N}_+$ and $q \in (4, 22^*)$, Eq. (1.3) admits a nontrivial radial nodal solution $U_{k,q} \in \mathcal{N}_{k,q}$ with exactly k nodes $0 < r_{1,q} < \cdots < r_{k,q} < +\infty$ such that $I_{\lambda,q}(U_{k,q}) = c_{k,q}$.*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 For each $k \in \mathbb{N}_+$ and $q \in (4, 22^*)$, let $U_{k,q}$ be defined as in Proposition 3.1 with $\mathbf{r}_{k,q} = (r_{1,q}, \ldots, r_{k,q})$ and $U_{k,q} = \sum_{i=1}^{k+1}(U_{k,q})_i$. We divide the whole proof into several steps.

Step 1. Prove

$$\limsup_{q\searrow 4_+} c_{k,q} \le c_{k,4} < +\infty. \tag{3.1}$$

In view of Lemma 2.3 and [25, Lemma 2.6], for any $U^4 := \sum_{i=1}^{k+1} u_i^4 \in \mathcal{N}_{k,4}$, there exists a unique $(t_{1,q}, \ldots, t_{k+1,q}) \in (\mathbb{R}_{>0})^{k+1}$ such that

$$U^q := \sum_{i=1}^{k+1} t_{i,q} u_i^4 \in \mathcal{N}_{k,q},$$

where $t_{i,q}$ satisfies

$$t_{i,q}^2\|u_i^4\|^2 + 2t_{i,q}^4 \int_{B_i^{\mathbf{r}_{k,q}}} |u_i^4|^2|\nabla u_i^4|^2 = \lambda t_{i,q}^q \int_{B_i^{\mathbf{r}_{k,q}}} |u_i^4|^q, \quad i = 1, \ldots, k+1. \tag{3.2}$$

This implies that $t_{i,q} \ge \big(\frac{\|u_i^4\|^2}{\lambda \int_{B_i^{\mathbf{r}_{k,q}}} |u_i^4|^q}\big)^{\frac{1}{q-2}}$ and for all $i \in \{1, \ldots, k+1\}$,

$$\liminf_{q\searrow 4_+} t_{i,q} \ge \Big(\frac{\|u_i^4\|^2}{\lambda \int_{\mathbb{R}^N} |u_i^4|^4}\Big)^{\frac{1}{2}} := \tilde{\delta} > 0. \tag{3.3}$$

We claim that $\{t_{i,q}\}_q$ is bounded for q nearby 4_+. In fact, suppose on the contrary that there is $i_q \in \{1, \ldots, k+1\}$ such that $t_{i_q,q} \to +\infty$ as $q \searrow 4_+$. Since $U^4 \in \mathcal{N}_{k,4}$, it follows from (3.2) that

$$\begin{aligned}
0 &= t_{i_q,q}^{2-q}\|u_{i_q}^4\|^2 + 2t_{i_q,q}^{4-q} \int_{\mathbb{R}^N} |u_{i_q}^4|^2|\nabla u_{i_q}^4|^2 - \lambda \int_{\mathbb{R}^N} |u_{i_q}^4|^q \\
&\le t_{i_q,q}^{2-q}\|u_{i_q}^4\|^2 + 2\int_{\mathbb{R}^N} |u_{i_q}^4|^2|\nabla u_{i_q}^4|^2 - \lambda \int_{\mathbb{R}^N} |u_{i_q}^4|^q \\
&\to 2\int_{\mathbb{R}^N} |u_{i_q}^4|^2|\nabla u_{i_q}^4|^2 - \lambda \int_{\mathbb{R}^N} |u_{i_q}^4|^4 \quad (\text{as } q \searrow 4_+) \\
&= -\|u_{i_q}^4\|^2 < 0,
\end{aligned}$$

which leads to a contradiction. The claim holds. So by (3.3), there exists $(t_{1,4}, \ldots, t_{k+1,4}) \in (\mathbb{R}_{>0})^{k+1}$ and a sequence $\{q_n\}_n$ with $q_n \searrow 4_+$ as $n \to \infty$ such that

$$(t_{1,q_n}, \ldots, t_{k+1,q_n}) \to (t_{1,4}, \ldots, t_{k+1,4}), \text{ as } n \to +\infty.$$

This together with (3.2), yields that

$$t_{i,4}^2\|u_i^4\|^2 + 2t_{i,4}^4\int_{\mathbb{R}^N}|u_i^4|^2|\nabla u_i^4|^2 = \lambda t_{i,4}^4\int_{\mathbb{R}^N}|u_i^4|^4, \tag{3.4}$$

which implies $\sum_{i=1}^{k+1} t_{i,4}u_i^4 \in \mathcal{N}_{k,4}$. On the other hand, $\sum_{i=1}^{k+1} u_i^4 \in \mathcal{N}_{k,4}$. Hence by Lemma 2.3, it follows easily that

$$(t_{1,4},\dots,t_{k+1,4}) = (1,\dots,1).$$

Thus, we obtain

$$\limsup_{n\to\infty} I_{\lambda,q_n}(U^{q_n}) = \limsup_{n\to\infty} I_{\lambda,q_n}\Big(\sum_{i=1}^{k+1} t_{i,q_n}u_i^4\Big) = I_{\lambda,4}\Big(\sum_{i=1}^{k+1} u_i^4\Big) = I_{\lambda,4}(U^4).$$

Therefore,

$$\limsup_{q_n\searrow 4_+} c_{k,q_n} \le \limsup_{q_n\searrow 4_+} I_{\lambda,q_n}(U^{q_n}) = I_{\lambda,4}(U^4).$$

The arbitrariness of U^4 yields (3.1). Step 1 is finished.

Step 2. We prove that there exists $U_{k,4}\in X$ such that

$$U_{k,q_n} \to U_{k,4} \neq 0 \text{ strongly in } X, \ \text{ as } n\to\infty. \tag{3.5}$$

We first show that $\{U_{k,q_n}\}$ is uniformly bounded for q_n nearby 4_+. In fact, We conclude from Proposition 3.1 that

$$c_{k,q_n} = I_{\lambda,q_n}(U_{k,q_n}) - \frac{1}{q_n}\langle I'_{\lambda,q_n}(U_{k,q_n}), U_{k,q_n}\rangle \tag{3.6}$$

$$= (\frac{1}{2}-\frac{1}{q_n})\|U_{k,q_n}\|^2 + (\frac{1}{2}-\frac{2}{q_n})|U_{k,q_n}\nabla U_{k,q_n}|_{L^2}^2 \tag{3.7}$$

$$\ge (\frac{1}{2}-\frac{1}{q_n})\|U_{k,q_n}\|^2 \tag{3.8}$$

$$= (\frac{1}{2}-\frac{1}{q_n})\sum_{i=1}^{k+1}\|(U_{k,q_n})_i\|^2. \tag{3.9}$$

Then by using (3.1), we obtain that $\|U_{k,q_n}\|$ and $\|(U_{k,q_n})_i\|$ are uniformly bounded for q_n nearby 4_+.

By Hölder inequality, Sobolev inequality and Young inequality, when n is large enough, we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}|U_{k,q_n}|^2|\nabla U_{k,q_n}|^2 = c_{k,q_n} - \frac{1}{2}\|U_{k,q_n}\|^2 + \frac{\lambda}{q_n}\int_{\mathbb{R}^N}|U_{k,q_n}|^{q_n}\\
&\le c_{k,q_n} + \frac{1}{2}\|U_{k,q_n}\|^2 + \frac{\lambda}{q_n}\int_{\mathbb{R}^N}|U_{k,q_n}|^{\frac{22^*(q_n-2)}{22^*-2}}|U_{k,q_n}|^{\frac{2(22^*-q_n)}{22^*-2}}\\
&\le c_{k,q_n} + \frac{1}{2}\|U_{k,q_n}\|^2 + \frac{\lambda}{q_n}\Big(\int_{\mathbb{R}^N}|U_{k,q_n}|^{22^*}\Big)^{\frac{q_n-2}{22^*-2}}\Big(\int_{\mathbb{R}^N}|U_{k,q_n}|^2\Big)^{\frac{22^*-q_n}{22^*-2}}\\
&\le c_{k,q_n} + \frac{1}{2}\|U_{k,q_n}\|^2 + C\Big(\int_{\mathbb{R}^N}|\nabla U_{k,q_n}^2|^2\Big)^{\frac{N(q_n-2)}{2(N+2)}}\Big(\int_{\mathbb{R}^N}|U_{k,q_n}|^2\Big)^{\frac{4N-(N-2)q_n}{2(N+2)}}\\
&\le c_{k,q_n} + \frac{1}{2}\|U_{k,q_n}\|^2 + \epsilon C_1\int_{\mathbb{R}^N}|\nabla U_{k,q_n}^2|^2 + C(\epsilon)\Big(\int_{\mathbb{R}^N}|U_{k,q_n}|^2\Big)^{\frac{(4-q_n)N+2q_n}{(4-q_n)N+4}},
\end{aligned} \tag{3.10}$$

which is equivalent to

$$(\frac{1}{2}-\epsilon C_1)\int_{\mathbb{R}^N}|U_{k,q_n}|^2|\nabla U_{k,q_n}|^2 \le c_{k,q_n}+\frac{1}{2}\|U_{k,q_n}\|^2+C(\epsilon)\|U_{k,q_n}\|^{\frac{2(4-q_n)N+4q_n}{(4-q_n)N+4}}.$$

Take $\epsilon=\frac{1}{4C_1}$, we can deduce that $\int_{\mathbb{R}^N}|U_{k,q_n}|^2|\nabla U_{k,q_n}|^2$ is uniformly bounded for q_n nearby 4_+. Thus the claim follows. As a consequence of the claim, $\{(U_{k,q_n})_i\}$ is also bounded in X for any i and q_n nearby 4_+. So up to a subsequence, there is a sequence q_n with $q_n\searrow 4_+$ and $(U_{k,4})_i\in X$ such that

$$\begin{aligned}&(U_{k,q_n})_i \rightharpoonup (U_{k,4})_i \text{ weakly in } X,\\ &(U_{k,q_n})_i \to (U_{k,4})_i \text{ strongly in } L^s(\mathbb{R}^N),\quad s\in(2,2^*).\end{aligned}$$

Note that $\int_{\mathbb{R}^N}|\nabla(U_{k,q_n})_i^2|^2$ is bounded, which implies $|(U_{k,q_n})_i|^2\in D^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$. Then $|(U_{k,q_n})_i|\in L^{22^*}(\mathbb{R}^N)$ and so $|(U_{k,4})_i|\in L^{22^*}(\mathbb{R}^N)$.

Let $q\in(2,22^*)$, then there exists $s\in(2,2^*)$ and $\tau\in(0,1)$ such that $q=\tau s+(1-\tau)22^*$. By Hölder inequality and Minkowski inequality we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N}|(U_{k,q_n})_i-(U_{k,4})_i|^q\\ &\le\Big(\int_{\mathbb{R}^N}|(U_{k,q_n})_i-(U_{k,4})_i|^s\Big)^{\tau}\Big(\int_{\mathbb{R}^N}|(U_{k,q_n})_i-(U_{k,4})_i|^{22^*}\Big)^{1-\tau}\\ &\le\Big(\int_{\mathbb{R}^N}|(U_{k,q_n})_i-(U_{k,4})_i|^s\Big)^{\tau}\Big(\Big(\int_{\mathbb{R}^N}|(U_{k,q_n})_i|^{22^*}\Big)^{\frac{1}{22^*}}+\Big(\int_{\mathbb{R}^N}|(U_{k,4})_i|^{22^*}\Big)^{\frac{1}{22^*}}\Big)^{22^*(1-\tau)}\\ &\to 0.\end{aligned}\tag{3.11}$$

Hence $(U_{k,q_n})_i\to(U_{k,4})_i$ strongly $L^s(\mathbb{R}^N)$ as $s\in(2,22^*)$.

In order to obtain the desired results, we prove each component

$$(U_{k,4})_i\neq 0,\quad i=1,\dots,k+1.\tag{3.12}$$

Note from $\langle I'_{\lambda,q_n}((U_{k,q_n})_i),(U_{k,q_n})_i\rangle=0$. By using Lemma 2.4, there is a number $\eta>0$ such that

$$\liminf_{n\to\infty}\|(U_{k,q_n})_i\|\ge\eta>0.$$

This combined with Lemma 2.2, gives that

$$\eta^2\le\|(U_{k,q_n})_i\|^2\le\lambda\int_{\mathbb{R}^N}|(U_{k,q_n})_i|^{q_n}\to\lambda\int_{\mathbb{R}^N}|(U_{k,4})_i|^4.$$

So (3.12) follows.

We assert that $U_{k,q_n}\to U_{k,4}$ strongly in X as $q_n\searrow 4_+$. Let $h_{k,q_n}=U_{k,q_n}-U_{k,4}$. Then by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}0&=\lim_{n\to\infty}\big(\|U_{k,q_n}\|^2+2|U_{k,q_n}\nabla U_{k,q_n}|^2_{L^2}-\lambda|U_{k,q_n}|^{q_n}_{L^{q_n}}\big)\\ &\ge\lim_{n\to\infty}\|h_{k,q_n}\|^2+\|U_{k,4}\|^2+2|U_{k,4}\nabla U_{k,4}|^2_{L^2}-\lambda|U_{k,4}|^4_{L^4}\\ &=\lim_{n\to\infty}\|h_{k,q_n}\|^2+\langle I'_{\lambda,4}(U_{k,4}),U_{k,4}\rangle.\end{aligned}$$

Hence $\langle I'_{\lambda,4}(U_{k,4}), U_{k,4}\rangle \le 0$. From Lemma 2.4, if $\langle I'_{\lambda,4}(U_{k,4}), U_{k,4}\rangle < 0$, then there exists $t \in (0,1)$ such that $tU_{k,4} \in \mathcal{N}_{k,4}$. Therefore, we get

$$\begin{aligned} c_{k,4} &\ge \limsup_{q_n \searrow 4+} \big(I_{\lambda,q_n}(U_{k,q_n}) - \frac{1}{q_n}\langle I'_{\lambda,q_n}(U_{k,q_n}), U_{k,q_n}\rangle\big) \\ &= \limsup_{q_n \searrow 4+} \big((\frac{1}{2} - \frac{1}{q_n})\|U_{k,q_n}\|^2 + (\frac{1}{2} - \frac{2}{q_n})|U_{k,q_n}\nabla U_{k,q_n}|_{L^2}^2\big) = \frac{1}{4}\|U_{k,4}\|^2 \\ &> \frac{1}{4}t^2\|U_{k,4}\|^2 = I_{\lambda,4}(tU_{k,4}) - \frac{1}{4}\langle I'_{\lambda,4}(tU_{k,4}), tU_{k,4}\rangle \ge c_{k,4}, \end{aligned}$$

which leads to a contradiction. Then $\langle I'_{\lambda,4}(U_{k,4}), U_{k,4}\rangle = 0$. Notice that $U_{k,q_n} \rightharpoonup U_{k,4}$ in X. Therefore, $U_{k,q_n} \to U_{k,4}$ strongly X.

So (3.5) follows and Step 2 is completed.

Step 3. We show that $I_{\lambda,4}(U_{k,4}) = c_{k,4}$.

To this end, we first prove $U_{k,4}$ is a nontrivial weak solution of (1.4). For any direction $v \in C_0^\infty(\mathbb{R}^N)$, it suffices to prove that

$$\langle I'_{\lambda,4}(U_{k,4}), v\rangle = 0. \tag{3.13}$$

It follows from Proposition 3.1 that

$$\begin{aligned} \langle I'_{\lambda,q_n}(U_{k,q_n}), v\rangle = &\int_{\mathbb{R}^N} \nabla U_{k,q_n}\nabla v + \int_{\mathbb{R}^N} U_{k,q_n}^2 \nabla U_{k,q_n}\nabla v + \int_{\mathbb{R}^N} |\nabla U_{k,q_n}|^2 U_{k,q_n} v + \\ &\int_{\mathbb{R}^N} V(|x|)U_{k,q_n} v - \int_{\mathbb{R}^N} \lambda |U_{k,q_n}|^2 U_{k,q_n} v = 0. \end{aligned} \tag{3.14}$$

Due to (3.5) and Lemma 2.2, we obtain that

$$\begin{aligned} &\lim_{n\to\infty} \langle I'_{\lambda,q_n}(U_{k,q_n}) - I'_{\lambda,4}(U_{k,4}), v\rangle \\ &= \lim_{n\to\infty} \Big(\int_{\mathbb{R}^N} \nabla(U_{k,q_n} - U_{k,4})\nabla v + \int_{\mathbb{R}^N} (U_{k,q_n}^2\nabla U_{k,q_n} - U_{k,4}^2\nabla U_{k,4})\nabla v + \\ &\int_{\mathbb{R}^N} V(|x|)(U_{k,q_n} - U_{k,4})v + \int_{\mathbb{R}^N} (|\nabla U_{k,q_n}|^2 U_{k,q_n} - |\nabla U_{k,4}|^2 U_{k,4})v - \\ &\int_{\mathbb{R}^N} \lambda(|U_{k,q_n}|^{q_n-2}U_{k,q_n} - |U_{k,4}|^2 U_{k,4})v\Big) \\ &= 0. \end{aligned}$$

This together with (3.14), yields (3.13).

We know that each $(U_{k,4})_i$ satisfies

$$\begin{cases} -\Delta(U_{k,4})_i + V(|x|)(U_{k,4})_i - \frac{1}{2}\Delta(|(U_{k,4})_i|^2)(U_{k,4})_i = \lambda|(U_{k,4})_i|^2(U_{k,4})_i, & \text{in } B_i^{\mathbf{r}_{k,4}}, \\ (U_{k,4})_i = 0, & \text{on } \partial B_i^{\mathbf{r}_{k,4}}. \end{cases}$$

Then the classical elliptic regularity argument and the strong maximum principle yield that either $(U_{k,4})_i < 0$ or $(U_{k,4})_i > 0$ in $B_i^{\mathbf{r}_{k,4}}$. Thus $U_{k,4}$ has exactly $k+1$ nodal domains. In addition, it follows from (3.1) that

$$c_{k,4} \ge \limsup_{q_n\to 4} \big(I_{\lambda,q_n}(U_{k,q_n}) - \frac{1}{q_n}\langle I'_{\lambda,q_n}(U_{k,q_n}), U_{k,q_n}\rangle\big)$$

$$= \limsup_{q_n \to 4} \left((\frac{1}{2} - \frac{1}{q_n}) \|U_{k,q_n}\|^2 + (\frac{1}{2} - \frac{2}{q_n}) |U_{k,q_n} \nabla U_{k,q_n}|_{L^2}^2 \right)$$
$$= \frac{1}{4}\|U_{k,4}\|^2 = I_{\lambda,4}(U_{k,4}) - \frac{1}{4}\langle I'_{\lambda,4}(U_{k,4}), U_{k,4}\rangle = I_{\lambda,4}(U_{k,4})$$
$$\geq c_{k,4}.$$

Then $I_{\lambda,4}(U_{k,4}) = c_{k,4}$. Hence $U_{k,4}$ is a radial nodal solution of (1.4) with exactly k nodes and $I_{\lambda,4}(U_{k,4}) = c_{k,4}$. The proof is completed. □

4. Energy comparison and the convergence properties of nodal solutions

In this section, we shall utilize subtle energy estimates to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2 According to Theorem 1.1, for any fixed positive integer $k \geq 1$, Eq. (1.4) admits a radial nodal solution $U_{k,4} := \sum_{i=1}^{k+1}(U_{k,4})_i$ with exactly k nodes. Since $\sum_{i=1}^{k}(U_{k,4})_i \in \mathcal{N}_{k,4}$ and $I_{\lambda,4}((U_{k,4})_i) > 0$, it follows immediately that

$$I_{\lambda,4}(U_{k,4}) = \sum_{i=1}^{k+1} I_{\lambda,4}((U_{k,4})_i) > \sum_{i=1}^{k} I_{\lambda,4}((U_{k,4})_i) = I_{\lambda,4}(U_{k-1,4}).$$

Thus $I_{\lambda,4}(U_{k,4})$ is strictly increasing in k.

Next, since $I_{\lambda,4}((U_{k,4})_i) \geq I_{\lambda,4}(U_0) > 0$ for all $i \in \{1, \ldots, k+1\}$, it follows easily that

$$I_{\lambda,4}(U_{k+1,4}) = \sum_{i=1}^{k+2} I_{\lambda,4}((U_{k,4})_i) \geq (k+2)I_{\lambda,4}(U_0) > (k+1)I_{\lambda,4}(U_0).$$

The proof is completed. □

In the following, in order to prove Theorem 1.3, we need some notations and give a useful lemma. We denote $\mathcal{N}_{k,4}$ by $\mathcal{N}_{k,4}^{\lambda}$, and the nodal solution of (1.4) obtained in Theorem 1.1 by $U_{k,4}^{\lambda} = \sum_{i=1}^{k+1}(U_{k,4}^{\lambda})_i \in X$ to emphasize the dependence on λ.

As we know, if u solves (1.4), then $\bar{u} = \lambda^{\frac{1}{2}} u$ satisfies the following equation

$$-\Delta \bar{u} + V(|x|)\bar{u} - \frac{1}{2\lambda}\Delta(|\bar{u}|^2)\bar{u} = |\bar{u}|^2\bar{u}. \tag{4.1}$$

We define a functional $\bar{I}_{\lambda,4} : X \to \mathbb{R}$ associated with (4.1) by

$$\bar{I}_{\lambda,4}(u) := \frac{1}{2}\|u\|^2 + \frac{1}{2\lambda}\int_{\mathbb{R}^N} u^2|\nabla u|^2 - \frac{1}{4}\int_{\mathbb{R}^N} |u|^4. \tag{4.2}$$

Then

$$\langle \bar{I}'_{\lambda,4}(u), u\rangle = \|u\|^2 + \frac{2}{\lambda}\int_{\mathbb{R}^N} u^2|\nabla u|^2 - \int_{\mathbb{R}^N} |u|^4.$$

Define the Nehari type set

$$\bar{\mathcal{N}}_{k,4}^{\lambda} = \big\{u \in X : \text{there exists } \mathbf{r}_k \in \Gamma_k \text{ such that } u_i \neq 0 \text{ in } B_i^{\mathbf{r}_k}, \quad \langle \bar{I}'_{\lambda,4}(u), u_i\rangle = 0, \text{ for all } i \in \{1, \ldots, k+1\}\big\}, \tag{4.3}$$

where $u_i = u$ in $B_i^{\mathbf{r}_k}$ and $u_i = 0$ on $\partial B_i^{\mathbf{r}_k}$. Consider the infimum level

$$\bar{c}_{k,4} = \inf_{u \in \bar{\mathcal{N}}_{k,4}^{\lambda}} \bar{I}_{\lambda,4}(u). \tag{4.4}$$

Lemma 4.1 *The following statements are true.*

(i) $u \in \mathcal{N}_{k,4}$ *is equivalent to* $\bar{u} \in \bar{\mathcal{N}}_{k,4}^{\lambda}$.

(ii) *If* $I_{\lambda,4}(U_{k,4}^{\lambda}) = c_{k,4}$, *then* $\bar{I}_{\lambda,4}(\bar{U}_{k,4}^{\lambda}) = \bar{c}_{k,4}$.

Proof It is easy to check that

$$\langle \bar{I}'_{\lambda,4}(\lambda^{\frac{1}{2}}u), \lambda^{\frac{1}{2}}u\rangle = \lambda\|u\|^2 + 2\lambda\int_{\mathbb{R}^N} u^2|\nabla u|^2 - \lambda^2\int_{\mathbb{R}^N}|u|^4 = \lambda\langle I'_{\lambda,4}(u), u\rangle. \tag{4.5}$$

So (i) follows immediately. Moreover, we have $\bar{I}_{\lambda,4}(\lambda^{\frac{1}{2}}u) = \lambda I_{\lambda,4}(u)$. Thus

$$\inf_{u\in\mathcal{N}_{k,4}} I_{\lambda,4}(u) = \inf_{u\in\mathcal{N}_{k,4}}\frac{1}{\lambda}\bar{I}_{\lambda,4}(\lambda^{\frac{1}{2}}u) = \inf_{\bar{u}\in\bar{\mathcal{N}}_{k,4}^{\lambda}}\frac{1}{\lambda}\bar{I}_{\lambda,4}(\bar{u}). \tag{4.6}$$

Since $U_{k,4}^{\lambda}$ is the solution of (1.4), $\bar{U}_{k,4}^{\lambda} = \lambda^{\frac{1}{2}}U_{k,4}^{\lambda}$ satisfies (4.1) which is

$$-\Delta\bar{U}_{k,4}^{\lambda} + V(|x|)\bar{U}_{k,4}^{\lambda} - \frac{1}{2\lambda}\Delta(|\bar{U}_{k,4}^{\lambda}|^2)\bar{U}_{k,4}^{\lambda} = |\bar{U}_{k,4}^{\lambda}|^2\bar{U}_{k,4}^{\lambda}. \tag{4.7}$$

Due to (4.5) and (4.6), there holds

$$\bar{I}_{\lambda,4}(\bar{U}_{k,4}^{\lambda}) = \bar{c}_{k,4}. \tag{4.8}$$

Thus (ii) holds. The proof is completed. □

Proof of Theorem 1.3 We divide the whole proof into three steps.

Step 1. We claim that for any sequence $\{\lambda_n\}$ with $\lambda_n \to +\infty$ as $n\to\infty$, $\bar{U}_{k,4}^{\lambda_n} = \lambda_n^{\frac{1}{2}}U_{k,4}^{\lambda_n}$ is bounded in X.

Indeed, by taking $\mathbf{r}_k \in \Gamma_k$ and nonzero radial functions $\varphi_i \in C_c^{\infty}(B_i^{\mathbf{r}_k})$, $i = 1,\dots,k+1$. We denote $\tilde{b} > 0$ and $\bar{\varphi}_i(x)$ by

$$\tilde{b}^2 = \max_{1\le i\le n}\frac{2\int_{B_i^{\mathbf{r}_k}}|\varphi_i|^2|\nabla\varphi_i|^2}{\int_{B_i^{\mathbf{r}_k}}|\varphi_i|^4},\quad \bar{\varphi}_i(x) = b\varphi_i(\frac{x}{\tilde{b}}),\quad i = 1,\dots,k+1,$$

where b is defined later. Note that

$$\begin{aligned}\Big\langle \bar{I}'_{\lambda,4}\Big(\sum_{i=1}^{k+1}\bar{\varphi}_i(x)\Big), \bar{\varphi}_i(x)\Big\rangle &= \|\bar{\varphi}_i(x)\|^2 + \frac{2}{\lambda}\int_{\mathbb{R}^N}|\bar{\varphi}_i(x)|^2|\nabla\bar{\varphi}_i(x)|^2 - \int_{\mathbb{R}^N}|\bar{\varphi}_i(x)|^4\\ &= b^2\tilde{b}^{N-2}\|\nabla\varphi_i\|^2_{L^2(B_i^{\mathbf{r}_k})} + b^2\tilde{b}^N\int_{B_i^{\mathbf{r}_k}}V(\tilde{b}|x|){\varphi_i}^2 +\\ &\quad \frac{2}{\lambda}b^4\tilde{b}^{N-2}\int_{B_i^{\mathbf{r}_k}}|\varphi_i|^2|\nabla\varphi_i|^2 - b^4\tilde{b}^N\int_{B_i^{\mathbf{r}_k}}|\varphi_i|^4\\ &=: F_i(b).\end{aligned} \tag{4.9}$$

Some direct computations show that there exists a small number $\delta > 0$ and a large number $L > 0$ such that for given $\lambda > 0$, we have $F_i(\delta) > 0$ and $F_i(L) < 0$, $i = 1,\dots,k+1$. Then there is $a_i(\lambda) \in (0,1]$ such that $a_i(\lambda)L \in [\delta, L]$ and

$$F_i(a_i(\bar{\lambda})L) = 0,\quad i = 1,\dots,k+1. \tag{4.10}$$

Let $b = a_i(\bar{\lambda})$. Then we obtain $\sum_{i=1}^{k+1}\bar{\varphi}_i(x) \in \bar{\mathcal{N}}_{k,4}^{\bar{\lambda}}$. Therefore, by using Lemma 4.1, we can find a $C_0 > 0$ such that for n large enough,

$$\begin{aligned}\frac{1}{4}\|\bar{U}_{k,4}^{\lambda_n}\|^2 &\leq \bar{I}_{\lambda_n,4}\Big(\sum_{i=1}^{k+1}\bar{\varphi}_i(x)\Big)\\ &= \sum_{i=1}^{k+1}(\bar{I}_{\lambda_n,4}(\bar{\varphi}_i(x)) - \frac{1}{4}\langle\bar{I}'_{\lambda_n,4}(\bar{\varphi}_i(x)), \bar{\varphi}_i(x)\rangle)\\ &= \frac{1}{4}\sum_{i=1}^{k+1}\Big(a_i^2(\bar{\lambda})L^2\tilde{b}^{N-2}|\nabla\varphi_i|^2_{L^2(B_i^{\mathbf{r}_k})} + a_i^2(\bar{\lambda})L^2\tilde{b}^N\int_{B_i^{\mathbf{r}_k}} V(\tilde{b}|x|)\varphi_i^2 \mathrm{d}x\Big)\\ &\leq \frac{1}{4}\sum_{i=1}^{k+1}\Big(L^2\tilde{b}^{N-2}|\nabla\varphi_i|^2_{L^2(B_i^{\mathbf{r}_k})} + L^2\tilde{b}^N\int_{B_i^{\mathbf{r}_k}} V(\tilde{b}|x|)\varphi_i^2 \mathrm{d}x\Big)\\ &=: C_0. \end{aligned} \tag{4.11}$$

Note from (4.8) and (4.11) that

$$\bar{c}_{k,4} = \bar{I}_{\lambda_n,4}(\bar{U}_{k,4}^{\lambda_n}) - \frac{1}{4}\langle\bar{I}'_{\lambda_n,4}(\bar{U}_{k,4}^{\lambda_n}), \bar{U}_{k,4}^{\lambda_n}\rangle = \frac{1}{4}\|\bar{U}_{k,4}^{\lambda_n}\|^2. \tag{4.12}$$

Thus $\bar{c}_{k,4}$ and $\|\bar{U}_{k,4}^{\lambda_n}\|$ are bounded. By applying similar arguments as in (3.10), we conclude from (4.8) that

$$\begin{aligned}\frac{1}{2\lambda_n}\int_{\mathbb{R}^N}|\bar{U}_{k,4}^{\lambda_n}|^2|\nabla\bar{U}_{k,4}^{\lambda_n}|^2 &\leq 3\bar{c}_{k,4} + \frac{1}{2}\|\bar{U}_{k,4}^{\lambda_n}\|^2 + \frac{1}{4}\Big(\int_{\mathbb{R}^N}|\bar{U}_{k,4}^{\lambda_n}|^{22^*}\Big)^{\frac{N-2}{N+2}}\Big(\int_{\mathbb{R}^N}|\bar{U}_{k,4}^{\lambda_n}|^2\Big)^{\frac{4}{N+2}},\\ &\leq 3\bar{c}_{k,4} + C(\int_{\mathbb{R}^N}|\nabla(\bar{U}_{k,4}^{\lambda_n})^2|^2)^{\frac{N}{N+2}}\Big(\int_{\mathbb{R}^N}|\bar{U}_{k,4}^{\lambda_n}|^2\Big)^{\frac{4}{N+2}}\\ &\leq 3\bar{c}_{k,4} + \epsilon\int_{\mathbb{R}^N}|\bar{U}_{k,4}^{\lambda_n}|^2|\nabla\bar{U}_{k,4}^{\lambda_n}|^2 + C(\epsilon)\Big(\int_{\mathbb{R}^N}|(\bar{U}_{k,4}^{\lambda_n})|^2\Big)^2. \end{aligned} \tag{4.13}$$

Hence,

$$(\frac{1}{2\lambda_n} - \epsilon C_2)\int_{\mathbb{R}^N}|\bar{U}_{k,4}^{\lambda_n}|^2|\nabla\bar{U}_{k,4}^{\lambda_n}|^2\mathrm{d}x \leq C_1 + C_4C(\epsilon)\|\bar{U}_{k,4}^{\lambda_n}\|^4.$$

By taking $\epsilon = \frac{1}{4\lambda_n C_2}$, it follows immediately that $\frac{1}{2\lambda_n}\int_{\mathbb{R}^N}|\bar{U}_{k,4}^{\lambda_n}|^2|\nabla(\bar{U}_{k,4}^{\lambda_n})|^2$ is bounded. Thus $\{\bar{U}_{k,4}^{\lambda_n}\}$ and $\{(\bar{U}_{k,4}^{\lambda_n})_i\}$ are bounded in X. So the claim follows immediately.

Step 2. We prove that $\bar{U}_{k,4}^0$ is a radial nodal solution of (1.11) with exactly $k+1$ nodal domains.

Notice that $\{(\bar{U}_{k,4}^{\lambda_n})_i\}$ is bounded in X. Then there is a subsequence $\{(\bar{U}_{k,4}^{\lambda_{n_j}})_i\}$ and $(\bar{U}_{k,4}^0)_i \in X$ such that

$$\begin{aligned}&(\bar{U}_{k,4}^{\lambda_{n_j}})_i \rightharpoonup (\bar{U}_{k,4}^0)_i \text{ weakly in } X,\\ &(\bar{U}_{k,4}^{\lambda_{n_j}})_i \to (\bar{U}_{k,4}^0)_i \text{ strongly in } L^s(\mathbb{R}^N), \quad s \in (2, 2^*).\end{aligned}$$

Since $\int_{\mathbb{R}^N}|\nabla(\bar{U}_{k,4}^{\lambda_n})_i^2|^2$ is bounded, we deduce from Sobolev inequality that $|(\bar{U}_{k,4}^{\lambda_n})_i|_{22^*}$ is bounded. Moreover, by similar arguments as in (3.11), we have

$$(\bar{U}_{k,4}^{\lambda_{n_j}})_i \to (\bar{U}_{k,4}^0)_i \text{ strongly in } L^s(\mathbb{R}^N), \quad s \in (2, 22^*).$$

Next, we prove $(\bar{U}^0_{k,4})_i \neq 0$. It follows from $\langle \bar{I}'_{\lambda_{n_j}}((\bar{U}^{\lambda_{n_j}}_{k,4})_i), (\bar{U}^{\lambda_{n_j}}_{k,4})_i\rangle = 0$ that

$$\liminf_{\lambda_{n_j}\to\infty} \|(\bar{U}^{\lambda_{n_j}}_{k,4})_i\| > 0.$$

This together with the compactly embedding $X \hookrightarrow L^s(\mathbb{R}^N)$, yields that

$$0 < \liminf_{\lambda_{n_j}\to\infty} \|(\bar{U}^{\lambda_{n_j}}_{k,4})_i\|^2 \leq \liminf_{\lambda_{n_j}\to\infty} \int_{\mathbb{R}^N} |(\bar{U}^{\lambda_{n_j}}_{k,4})_i|^4 = \int_{\mathbb{R}^N} |(\bar{U}^0_{k,4})_i|^4,$$

which shows that $(\bar{U}^0_{k,4})_i \neq 0$. We claim $\bar{U}^{\lambda_{n_j}}_{k,4} \to \bar{U}^0_{k,4}$ in X. In fact, let $\bar{h}^{\lambda_{n_j}}_{k,4} = \bar{U}^{\lambda_{n_j}}_{k,4} - \bar{U}^0_{k,4}$, there holds

$$\begin{aligned} 0 &= \lim_{\lambda_{n_j}\to\infty} \Big(\|\bar{U}^{\lambda_{n_j}}_{k,4}\|^2 + \frac{2}{\lambda_{n_j}} |\bar{U}^{\lambda_{n_j}}_{k,4} \nabla \bar{U}^{\lambda_{n_j}}_{k,4}|^2_{L^2} - |\bar{U}^{\lambda_{n_j}}_{k,4}|^4_{L^4}\Big) \\ &\geq \lim_{\lambda_{n_j}\to\infty} \|\bar{h}^{\lambda_{n_j}}_{k,4}\|^2 + \|\bar{U}^0_{k,4}\|^2 + \frac{2}{\lambda_{n_j}} |\bar{U}^{\lambda_{n_j}}_{k,4} \nabla \bar{U}^{\lambda_{n_j}}_{k,4}|^2_{L^2} - |\bar{U}^0_{k,4}|^4_{L^4} \\ &= \lim_{\lambda_{n_j}\to\infty} \|\bar{h}^{\lambda_{n_j}}_{k,4}\|^2 + \langle \bar{I}'_{0,4}(\bar{U}^0_{k,4}), \bar{U}^0_{k,4}\rangle. \end{aligned}$$

Therefore, $\langle \bar{I}'_{0,4}(\bar{U}^0_{k,4}), \bar{U}^0_{k,4}\rangle \leq 0$. If $\langle \bar{I}'_{0,4}(\bar{U}^0_{k,4}), \bar{U}^0_{k,4}\rangle < 0$, then there exists $t \in (0,1)$ such that $t\bar{U}^0_{k,4} \in \bar{\mathcal{N}}_{k,4}$. Then we have

$$\begin{aligned} \bar{c}_{k,4} &\geq \lim_{\lambda_{n_j}\to+\infty} \Big(\bar{I}_{\lambda_{n_j},4}(\bar{U}^{\lambda_{n_j}}_{k,4}) - \frac{1}{4}\langle \bar{I}'_{\lambda_{n_j},4}(\bar{U}^{\lambda_{n_j}}_{k,4}), \bar{U}^{\lambda_{n_j}}_{k,4}\rangle\Big) \\ &= \lim_{\lambda_{n_j}\to+\infty} \frac{1}{4}\|\bar{U}^{\lambda_{n_j}}_{k,4}\|^2 = \frac{1}{4}\|\bar{U}^0_{k,4}\|^2 \\ &> \bar{I}_{0,4}(t\bar{U}^0_{k,4}) - \frac{1}{4}\langle \bar{I}'_{0,4}(t\bar{U}^0_{k,4}), t\bar{U}^0_{k,4}\rangle \geq \bar{c}_{k,4}, \end{aligned}$$

which is a contradiction. So $\langle \bar{I}'_{0,4}(\bar{U}^0_{k,4}), \bar{U}^0_{k,4}\rangle = 0$ and

$$\bar{U}^{\lambda_{n_j}}_{k,4} \to \bar{U}^0_{k,4} \text{ strongly in } X. \tag{4.14}$$

Therefore, $\bar{U}^0_{k,4}$ is a radial nodal solution of the limit equation (1.11) with exactly $k+1$ nodal domains.

Step 3. We prove that $\bar{U}^0_{k,4}$ is a least energy radial solution of (1.11) among all the radial solutions changing sign exactly k times.

According to [30, Theorem 2.1], we assume that there is $\bar{\mathbf{r}}_k \in \Gamma_k$ such that $V_{k,4} := v_1 + \ldots + v_{k+1}$ with $v_i \neq 0$ is a least energy radial solution of (1.11) among all the nodal solutions changing sign exactly k times. Clearly, for each $\lambda_n > 0$, $\sum_{i=1}^{k+1} a_{i,n} v_i \in \bar{\mathcal{N}}^{\lambda_n}_{k,4}$ if and only if

$$\begin{aligned} f_i^n(a_{i,n}) :&= a_{i,n}^2\|v_i\|^2 + \frac{2}{\lambda_n} a_{i,n}^4 \int_{B_i^{\bar{\mathbf{r}}_k}} v_i^2|\nabla v_i|^2 - a_{i,n}^4 \int_{B_i^{\bar{\mathbf{r}}_k}} |v_i|^4 \\ &= 0, \quad i = 1, \ldots, k+1. \end{aligned} \tag{4.15}$$

Notice that

$$f_i^n(1) > \|v_i\|^2 - \int_{B_i^{\bar{\mathbf{r}}_k}} |v_i|^4 = 0, \quad i = 1, \ldots, k+1 \tag{4.16}$$

and

$$L^2\|v_i\|^2 - L^4\int_{B_i^{\bar{r}_k}} |v_i|^4 = L^4\Big(\frac{\|v_i\|_i^2}{L^2} - \int_{B_i^{\bar{r}_k}} |v_i|^4\Big) < 0 \text{ for any } L > 1.$$

Take $L_j := 1 + \frac{1}{j}$ with $j \geq 1$. Then for each $j \geq 1$, there exists $N(j) > 0$ such that for each $m \geq N(j)$, there holds

$$f_i^m(L_j) = L_j^2\|v_i\|^2 + \frac{2}{\lambda_m}L_j^4\int_{B_i^{\bar{r}_k}} |v_i|^2|\nabla v_i|^2 - L_j^4\int_{B_i^{\bar{r}_k}} |v_i|^4 < 0. \tag{4.17}$$

This combined with (4.16), yields that for each m, there exists $a_{i,m} \in (1, L_j)$ such that

$$f_i^m(a_{i,m}) = 0, \quad i = 1, \ldots, k+1.$$

Therefore, $\sum_{i=1}^{k+1} a_{i,m}v_i \in \bar{\mathcal{N}}_{k,4}^{\lambda_m}$ and $a_{i,m} \to 1$ as $j \to \infty$. Then

$$\begin{aligned}\bar{I}_{0,4}(V_{k,4}) \leq \bar{I}_{0,4}(\bar{U}_{k,4}^0) = \lim_{\lambda_m\to\infty} \bar{I}_{\lambda_m,4}(\bar{U}_{k,4}^{\lambda_m}) \leq \lim_{\lambda_m\to\infty} \bar{I}_{\lambda_m,4}\Big(\sum_{i=1}^{k+1} a_{i,m}v_i\Big)\\ = \bar{I}_{0,4}\Big(\sum_{i=1}^{k+1} v_i\Big) = \bar{I}_{0,4}(V_{k,4}).\end{aligned}$$

Thus, $\bar{U}_{k,4}^0$ is a least energy radial solution of (1.11) which changes sign exactly k times. The proof is completed. □

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References

[1] W. A. STRAUSS. *Existence of solitary waves in higher dimensions*. Comm. Math. Phys. 1977, **55**(2): 149–162.

[2] L. BRIILL, H. LANGE. *Solitary waves for quasilinear Schrödinger equations*. Exposition. Math., 1986, **4**(3): 279–288.

[3] H. LANGE, M. POPPENBERG, H. TEISMANN. *Nash-Moser methods for the solution of quasilinear Schrödinger equations*. Comm. Partial Differential Equations, 1999, **24**(7-8): 1399–1418.

[4] A. NAKAMURA. *Damping and modification of exciton solitary waves*. J. Phys. Soc. Japan, 1977, **42**(6): 1824–1835.

[5] M. PORKOLAB, M. V. GOLDMAN. *Upper-hybrid solitons and oscillating two-stream instabilities*. Phys. Fluids, 1976, **19**(6): 872–881.

[6] M. POPPENBERG. *On the local well posedness of quasilinear Schrödinger equations in arbitrary space dimension*. J. Differential Equations, 2001, **172**(1): 83–115.

[7] J. BEZERRA DO Ó, O. MIYAGAKI, S. SOARES. *Soliton solutions for quasilinear Schrödinger equations with critical growth*. J. Differential Equations, 2010, **248**(4): 722–744.

[8] A. BAHROUNI, H. OUNAIES, V. RĂDULESCU. *Infinitely many solutions for a class of sublinear Schrödinger equations with indefinite potentials*. Proc. Roy. Soc. Edinburgh Sect. A., 2015, **145**(3): 445–465.

[9] M. COLIN, L. JEANJEAN. *Solutions for a quasilinear Schrödinger equation: a dual approach*. Nonlinear Anal., 2004, **56**(2): 213–226.

[10] M. COLIN, J. JEANJEAN, M. SQUASSINA. *Stability and instability results for standing waves of quasilinear Schrödinger equations*. Nonlinearity, 2010, **23**(6): 1353–1385.

[11] S. CHEN, V. RĂDULESCU, Xianhua TANG, et al. *Ground state solutions for quasilinear Schrödinger equations with variable potential and superlinear reaction*. Rev. Mat. Iberoam., 2020, **36**(5): 1549–1570.

[12] Jianhua CHEN, Xianhua TANG, Bitao CHENG. *Existence of ground state solutions for quasilinear Schrödinger equations with super-quadratic condition*. Appl. Math. Lett., 2018, **79**: 27–33.

[13] Yinbin DENG, Shuangjie PENG, Shusen YAN. *Critical exponents and solitary wave solutions for generalized quasilinear Schrödinger equations*. J. Differential Equations, 2016, **260**(2): 1228–1262.

[14] Jiaquan LIU, Zhiqiang WANG. *Soliton solutions for quasilinear Schrödinger equations. I.* Proc. Amer. Math. Soc., 2003, **131**(2): 441–448.

[15] Jiaquan LIU, Yaqi WANG, Zhiqiang WANG. *Soliton solutions for quasilinear Schrödinger equations. II.* J. Differential Equations, 2003, **187**(2): 473–493.

[16] Jiaquan LIU, Yaqi WANG, Zhiqiang WANG. *Solutions for quasilinear Schrödinger equations via the Nehari method.* Comm. Partial Differential Equations, 2004, **29**(5-6): 879–901.

[17] M. POPPENBERG, K. SCHMITT, Zhiqiang WANG. *On the existence of soliton solutions to quasilinear Schrödinger equations*. Calc. Var. Partial Differential Equations, 2002, **14**(3): 329–344.

[18] D. RUIZ, G. SICILIANO. *Existence of ground states for a modified nonlinear Schrödinger equation*. Nonlinearity, 2010, **23**(5): 1221–1233.

[19] U. B. SEVERO, E. GLOSS, E. D. SILVA. *On a class of quasilinear Schrödinger equations with superlinear or asymptotically linear terms*. J. Differential Equations, 2017, **263**(6): 3550–3580.

[20] Li WANG, Binlin ZHANG, Zhangkun CHENG. *Ground state sign-changing solutions for the Schrödinger-Kirchhoff equation in* $\mathbb{R}^3$. J. Math. Anal. Appl., 2018, **466**(2): 1545–1569.

[21] Yanfang XUE, Chunlei TANG. *Existence of a bound state solution for quasilinear Schrödinger equations*. Adv. Nonlinear Anal., 2019, **8**(1): 323–338.

[22] Minbo YANG, Yanheng DING. *Existence of semiclassical states for a quasilinear Schrödinger equation with critical exponent in* $\mathbb{R}^N$. Ann. Mat. Pura Appl., 2013, **192**(5): 783–804.

[23] Xianyong YANG, Wenbo WANG, Fukun ZHAO. *Infinitely many radial and non-radial solutions to a quasilinear Schrödinger equation*. Nonlinear Anal., 2015, **114**: 158–168.

[24] Jian ZHANG, Xianhua TANG, Wen ZHANG. *Existence of infinitely many solutions for a quasilinear elliptic equation*. Appl. Math. Lett., 2014, **37**: 131–135.

[25] Yinbin DENG, Shuangjie PENG, Jixiu WANG. *Infinitely many sign-changing solutions for quasilinear Schrödinger equations in* $\mathbb{R}^N$. Commun. Math. Sci., 2011, **9**(3): 859–878.

[26] Yinbin DENG, Shuangjie PENG, Jixiu WANG. *Nodal soliton solutions for quasilinear Schrödinger equations with critical exponent*. J. Math. Phys., 2013, **54**(1): 011504, 27 pp.

[27] Minbo YANG, C. A. SANTOS, Jiazheng ZHOU. *Least energy nodal solutions for a defocusing Schrödinger equation with supercritical exponent*. Proc. Edinb. Math. Soc. (2), 2019, **62**(1): 1–23.

[28] Wei ZHANG, Xiangqing LIU. *Infinitely many sign-changing solutions for a quasilinear elliptic equation in* $\mathbb{R}^N$. J. Math. Anal. Appl., 2015, **427**(2): 722–740.

[29] Hui ZHANG, Zhisu LIU, Chunlei TANG, et al. *Existence and multiplicity of sign-changing solutions for quasilinear Schrödinger equations with sub-cubic nonlinearity*. J. Differential Equations, 2023, **365**: 199–234.

[30] T. BARTSCH, M. WILLEM. *Infinitely many radial solutions of a semilinear elliptic problem on* $\mathbb{R}^N$. Arch. Ration. Mech. Anal., 1993, **124**(3): 261–276.