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Advancements in Fixed Point Theorems for α-Geraghty Contractions in Complete Metric Space

V. C. BORKAR[1], **Mohammed M. A. TALEB**[2,3], **Saeed A. A. AL-SALEHI**[2,4,*]

1. *School of Mathematical at Yeshwant Mahavidyalaya, Swami Ramanand Teerth Marathwada University, Nanded-431606, India;*
2. *School of Mathematical at Science College, Swami Ramanand Teerth Marathwada University, Nanded-431606, India;*
3. *Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen;*
4. *Department of Mathematics, Aden University, Aden, Yemen*

Abstract This article improves and advances the results achieved by Vishal and Naveen, leveraging the use of Lebesgue integrable functions. Going beyond, we investigate fixed point theorems associated with a concept introduced by Ovidiu Popescu. Our research not only bolsters the theoretical framework but also unveils practical applications in the domain of Lebesgue integrals as in Example 3.6. Furthermore, our contribution enhances the understanding of fixed point theorems in metric spaces and introduces novel perspectives in the study of generalized α-Geraghty contractive mappings on Lebesgue integrals.

Keywords fixed point theorems; contractive mappings; α-Geraghty contractions; complete metric spaces; Lebesgue integrals; α-admissible; α-orbital; α-orbital attractive

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1. Introduction

Fixed point theorems in the realm of contraction mappings have found widespread applications across a multitude of disciplines, including computer science, biology, physics, optimization, dynamical systems, and chemistry. Moreover, they have established a substantial foothold within various mathematical domains, including differential equations, functional analysis, and integral calculus. This article focuses on a particular class of contractions known as α-Geraghty contractions, which have garnered significant attention due to their remarkable ability to ensure the existence and uniqueness of fixed points in metric spaces [1]. A distinctive feature of generalized α-Geraghty contractions lies in their affiliation with Lebesgue integrable functions. When the function $\Omega(\rho)$ in the contraction condition is Lebesgue integrable, it signifies that the contraction criterion accounts for the global behavior of the function across the entire metric space. This property bestows us more robust convergence results, especially when dealing with functions exhibiting varying degrees of smoothness and regularity. Building on previous research on fixed point theorems for generalized weak contractions [2–7] and in the field of integrals [8–12],

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* Corresponding author
E-mail address: alsalehi.saeed72@gmail.com (Saeed A. A. AL-SALEHI)

this article delves into the concept of generalized α-Geraghty contractions and their intriguing Lebesgue integrability property [13, 14]. Development of this field traces its roots back to Geraghty's pioneering work on contractive mappings in 1973 (see [15]). Subsequently, researchers have expanded this foundation at various junctures. Hussain et al. presented an article in 2013 (see [16]). In 1994, Matthews studied partial metric topology [17]. In 2011, Abduljawad published an article on generalized weakly contractive mappings [18], and in the same year, Abduljawad et al. explored the existence and uniqueness of fixed point [19]. Some researchers investigated the results of multi-valued mappings and coupled fixed point theorems [20–22]. In 2021–2023, some researchers studied Geraghty-weak contraction [23–25]. This article improves and advances the results achieved by Vishal and Naveen in their 2013 publication [26], leveraging the use of Lebesgue integrable functions. Moreover, we investigate fixed point theorems associated with a concept introduced by Ovidiu Popescu in 2014 (see [27]). Here we will present the most important mathematical concepts on which our study was based.

Definition 1.1 ([28]) *A self-map $\Omega : X \to X$ is called α-admissible if there exists a function $\alpha : X \times X \to \mathbb{R}^+$, such that*

$$\forall \rho, \sigma \in X,\ \alpha(\rho, \sigma) \geq 1 \Rightarrow \alpha(\Omega(\rho), \Omega(\sigma)) \geq 1. \tag{1.1}$$

Definition 1.2 ([29]) *A self-map $\Omega : X \to X$ is called a triangular α-admissible, if there exists a function $\alpha : X \times X \to \mathbb{R}^+$, such that*

(a) $\forall \rho, \sigma \in X, \alpha(\rho, \sigma) \geq 1 \Rightarrow \alpha(\Omega(\rho), \Omega(\sigma)) \geq 1$;

(b) $\forall \rho, \sigma, \zeta \in X, \alpha(\rho, \sigma) \geq 1, \alpha(\sigma, \zeta) \geq 1 \Rightarrow \alpha(\rho, \zeta) \geq 1$.

Definition 1.3 ([30]) *A self-map $\Omega : X \to X$ is called α-orbital admissible, if there exists $\alpha : X \times X \to \mathbb{R}$, such that*

$$\alpha(\rho, \Omega(\rho)) \geq 1 \Rightarrow \alpha(\Omega(\rho), \Omega^2(\rho)) \geq 1. \tag{1.2}$$

Definition 1.4 ([30]) *A self-map $\Omega : X \to X$ is called a triangular α-orbital admissible, if there exists $\alpha : X \times X \to \mathbb{R}$, such that the following conditions must be satisfied:*

(a) $\alpha(\rho, \Omega(\rho)) \geq 1 \Rightarrow \alpha(\Omega(\rho), \Omega^2(\rho)) \geq 1$;

(b) $\alpha(\rho, \sigma) \geq 1 \wedge \alpha(\sigma, \Omega(\sigma)) \geq 1 \Rightarrow \alpha(\rho, \Omega(\sigma)) \geq 1$.

Definition 1.5 ([30]) *A self-map $\Omega : X \to X$ is called α-orbital attractive, if there exists $\alpha : X \times X \to \mathbb{R}$, such that*

$$\alpha(\rho, \Omega(\rho)) \geq 1 \Rightarrow \alpha(\rho, \sigma) \geq 1 \vee \alpha(\sigma, \Omega(\rho)) \geq 1, \quad \forall \rho, \sigma \in X. \tag{1.3}$$

Lemma 1.6 ([30]) *Consider a self-map $\Omega : X \to X$ that is a triangular α-orbital admissible. Suppose that there exists $\rho_1 \in X$, such that $\alpha(\rho_1, \Omega(\rho_1)) \geq 1$. If we define a sequence $\{\rho_n\}$ by $\Omega(\rho_n) = \rho_{n+1}$, then for all natural numbers n and m, where $n < m$, we have $\alpha(\rho_n, \rho_m) \geq 1$.*

2. Preliminaries

We will introduce the pivotal definition and theorem that have significantly contributed to the development of our study.

Definition 2.1 ([30]) *Let a self-map $\Omega : X \to X$ be defined on a metric space (X, d) and let $\alpha : X \times X \to \mathbb{R}$ be a function. Then Ω is called a generalized α-Geraghty contraction, if there exists $\beta \in \mathcal{F}$, such that*

$$\forall\, \rho, \sigma \in X \Rightarrow \alpha(\rho, \sigma) d(\Omega(\rho), \Omega(\sigma)) \leq \beta(M_\Omega(\rho, \sigma)) M_\Omega(\rho, \sigma), \tag{2.1}$$

where

$$M_\Omega(\rho, \sigma) = \mathfrak{Max}\{d(\rho, \sigma), d(\rho, \Omega(\rho)), d(\sigma, \Omega(\sigma)), \frac{d(\rho, \Omega(\sigma)) + d(\sigma, \Omega(\rho))}{2}\}. \tag{2.2}$$

Theorem 2.2 ([31]) *Let us consider a self-map $\Omega : X \to X$ which is defined on a complete metric space (X, d), such that*

$$\forall\, \rho, \sigma \in X \Rightarrow \int_0^{d(\Omega(\rho), \Omega(\sigma))} \Theta(t) \mathrm{d}t \leq \beta(d(\rho, \sigma)) \int_0^{M_\Omega(\rho, \sigma)} \Theta(t) \mathrm{d}t, \tag{2.3}$$

where

$$M_\Omega(\rho, \sigma) = \mathfrak{Max}\{\frac{d(\rho, \Omega(\rho)) d(\sigma, \Omega(\sigma))}{d(\rho, \sigma)}, d(\rho, \sigma)\}. \tag{2.4}$$

Consider a function $\Theta : [0, +\infty) \to [0, +\infty)$ that is Lebesgue-integrable and summable on every compact subset of the positive real numbers, ensuring it is non-negative. Additionally, suppose that $\epsilon > 0$, $\int_0^\epsilon \Theta(t) \mathrm{d}t > 0$. Furthermore, let $\beta : \mathbb{R}^+ \to [0, 1)$ be a function with $\lim_{t \to r} \sup \beta(t) < 1$, $\forall\, t > 0$. In this case, Ω has a unique fixed point.

3. Main results

Within the confines of this paper, we have meticulously expounded the advancement of generalized α-Geraghty contraction mapping theorems. Our approach harnesses the power of Lebesgue-integrable functions as a methodological cornerstone, facilitating the elegant and intuitive proof of our theorems via the following streamlined pathways:

Theorem 3.1 *Let us consider a self-map $\Omega : X \to X$ which is defined on a complete metric space (X, d). Additionally, there exists a function $\alpha : X \times X \to \mathbb{R}$, if Ω satisfies the following conditions:*

(a) Ω is a generalized α-Geraghty contraction;
(b) Ω is triangular α-orbital admissible;
(c) $\exists\, \rho_1 \in X$, such that $\alpha(\rho_1, \Omega(\rho_1)) \geq 1$;
(d) Ω is continuous,

then Ω has a fixed point $\rho^* \in X$ and

$$\int_0^{\alpha(\rho, \sigma) d(\Omega(\rho), \Omega(\sigma))} \Theta(t) \mathrm{d}t \leq \beta(M_\Omega(\rho, \sigma)) \int_0^{M_\Omega(\rho, \sigma)} \Theta(t) \mathrm{d}t, \tag{3.1}$$

where

$$M_\Omega(\rho,\sigma) = \mathfrak{Max}\{d(\rho,\sigma), d(\rho,\Omega(\rho)), d(\sigma,\Omega(\sigma)), \frac{d(\rho,\Omega(\sigma)) + d(\sigma,\Omega(\rho))}{2}\}. \tag{3.2}$$

Let $\beta \in \mathcal{F}$, $\beta : \mathbb{R}^+ \cup \{0\} \to [0,1)$ and $\Theta : [0,+\infty) \to [0,+\infty)$ be Lebesgue function which is summable on $\mathbb{R}^+$, non-negative and $\forall\, \epsilon > 0$, $\int_0^\epsilon \Theta(t)\mathrm{d}t > 0$.

Proof Let $\rho_1 \in X$, such that $\alpha(\rho_1, \Omega(\rho_1)) \geq 1$. We define a sequence $\{\rho_n\}$ by $\Omega(\rho_n) = \rho_{n+1}$, $\forall\, n \geq 1$. If $\rho_{n(0)} = \rho_{n(0)+1}$ for some $n(0) \geq 1$, then Ω has a fixed point and (1.1) holds.

Step 1. We must claim that $\lim_{n\to\infty} d(\rho_n, \rho_{n+1}) = 0$. Now, if we suppose that $\rho_n \neq \rho_{n+1}$, $\forall\, n \geq 1$, by Lemma 1.6, we get $\alpha(\rho_n, \rho_{n+1}) \geq 1$, $\forall\, n \geq 1$, so that we obtain

$$\begin{aligned}\int_0^{d(\rho_{n+1},\rho_{n+2})} \Theta(t)\mathrm{d}t &= \int_0^{d(\Omega(\rho_n),\Omega(\rho_{n+1}))} \Theta(t)\mathrm{d}t \\ &\leq \int_0^{\alpha(\rho_n,\rho_{n+1})d(\Omega(\rho_n),\Omega(\rho_{n+1}))} \Theta(t)\mathrm{d}t \\ &\leq \beta(M_\Omega(\rho_n,\rho_{n+1})) \int_0^{M_\Omega(\rho_n,\rho_{n+1})} \Theta(t)\mathrm{d}t,\end{aligned} \tag{3.3}$$

where,

$$\begin{aligned}M_\Omega(\rho_n,\rho_{n+1}) =& \mathfrak{Max}\{d(\rho_n,\rho_{n+1}), d(\rho_n,\Omega(\rho_n)), d(\rho_{n+1},\Omega(\rho_{n+1})), \\ & \frac{d(\rho_n,\Omega(\rho_{n+1})) + d(\rho_{n+1},\Omega(\rho_n))}{2}\}, \quad \forall\, n \geq 1 \\ =& \mathfrak{Max}\{d(\rho_n,\rho_{n+1}), d(\rho_{n+1},\rho_{n+2}), \frac{d(\rho_n,\rho_{n+2})}{2}\} \\ =& \mathfrak{Max}\{d(\rho_n,\rho_{n+1}), d(\rho_{n+1},\rho_{n+2}), \frac{d(\rho_n,\rho_{n+1}) + d(\rho_{n+1},\rho_{n+2})}{2}\} \\ =& \mathfrak{Max}\{d(\rho_n,\rho_{n+1}), d(\rho_{n+1},\rho_{n+2})\}.\end{aligned} \tag{3.4}$$

If we take $M_\Omega(\rho_n,\rho_{n+1}) = d(\rho_{n+1},\rho_{n+2})$, then $d(\rho_n,\rho_{n+1}) \leq d(\rho_{n+1},\rho_{n+2})$ in (2.2), we get

$$\begin{aligned}\int_0^{d(\rho_{n+1},\rho_{n+2})} \Theta(t)\mathrm{d}t &= \int_0^{d(\Omega(\rho_n),\Omega(\rho_{n+1}))} \Theta(t)\mathrm{d}t \\ &\leq \int_0^{\alpha(\rho_n,\rho_{n+1})d(\Omega(\rho_n),\Omega(\rho_{n+1}))} \Theta(t)\mathrm{d}t \\ &\leq \beta(d(\rho_{n+1},\rho_{n+2})) \int_0^{d(\rho_{n+1},\rho_{n+2})} \Theta(t)\mathrm{d}t \\ &\leq \int_0^{d(\rho_{n+1},\rho_{n+2})} \Theta(t)\mathrm{d}t,\end{aligned}$$

which is a contradiction. We obtain

$$M_\Omega(\rho_n,\rho_{n+1}) = d(\rho_n,\rho_{n+1}) \text{ and } d(\rho_n,\rho_{n+1}) > d(\rho_{n+1},\rho_{n+2}),$$

so that the sequence $\{\int_0^{d(\rho_n,\rho_{n+1})} \Theta(t)\mathrm{d}t\}$ is positive and decreasing.

So, $\exists\, \lambda \geq 0$, such that $\lim_{n\to\infty} \int_0^{d(\rho_n,\rho_{n+1})} \Theta(t)\mathrm{d}t = \lambda$. Now, we prove that $\lambda = 0$. On the contrary suppose that $\lambda > 0$, then we get

$$\int_0^{d(\rho_{n+1},\rho_{n+2})} \Theta(t)\mathrm{d}t \le \beta(d(\rho_n,\rho_{n+1})) \int_0^{d(\rho_n,\rho_{n+1})} \Theta(t)\mathrm{d}t$$
$$\Rightarrow \frac{\int_0^{d(\rho_{n+1},\rho_{n+2})} \Theta(t)\mathrm{d}t}{\int_0^{d(\rho_n,\rho_{n+1})} \Theta(t)\mathrm{d}t} \le \beta(d(\rho_n,\rho_{n+1})) < 1$$
$$\Rightarrow \lim_{n\to\infty} \beta(d(\rho_n,\rho_{n+1})) = 1. \tag{3.5}$$

Since $\beta \in \mathcal{F}$ we get

$$\lim_{n\to\infty} \int_0^{d(\rho_n,\rho_{n+1})} \Theta(t)\mathrm{d}t = 0 \Rightarrow \lim_{n\to\infty} d(\rho_n,\rho_{n+1}) = 0,$$

which is a contradiction. Hence, our supposition is wrong so that $\lambda = 0$.

Step 2. We must claim that the sequence $\{\rho_n\}$ is a Cauchy sequence. On the other hand, we suppose that $\exists\, \epsilon > 0$, such that $\forall\, k \ge 1$, $\exists\, m(k) > n(k) > k$ with $d(\rho_{n(k)},\rho_{m(k)}) \ge \epsilon$. Suppose that $m(k)$ is the smallest number which satisfies the above condition. Hence, $d(\rho_{n(k)},\rho_{m(k)}) < \epsilon$.

So, $\epsilon \le d(\rho_{n(k)},\rho_{m(k)}) \le d(\rho_{n(k)},\rho_{m(k)-1}) + d(\rho_{m(k)-1},\rho_{m(k)})$. Now, let $k \to \infty$. We obtain $\lim_{k\to\infty} d(\rho_{n(k)},\rho_{m(k)}) = \epsilon$. Since $|d(\rho_{n(k)},\rho_{m(k)-1}) - d(\rho_{n(k)},\rho_{m(k)})| \le d(\rho_{m(k)},\rho_{m(k)-1})$, we get $d(\rho_{n(k)},\rho_{m(k)}) = \epsilon$. Similarly, $\lim_{k\to\infty} d(\rho_{m(k)},\rho_{n(k)-1}) = \lim_{k\to\infty} d(\rho_{m(k)-1},\rho_{n(k)-1}) = \epsilon$ by Lemma 1.6, we obtain $\alpha(\rho_{n(k)-1},\rho_{m(k)-1}) \ge 1$, thus

$$\int_0^{d(\rho_{n(k)},\rho_{m(k)})} \Theta(t)\mathrm{d}t = \int_0^{d(\Omega(\rho_{n(k)-1}),\Omega(\rho_{m(k)-1}))} \Theta(t)\mathrm{d}t$$
$$\le \int_0^{\alpha(\rho_{n(k)-1},\rho_{m(k)-1})d(\Omega(\rho_{n(k)-1}),\Omega(\rho_{m(k)-1}))} \Theta(t)\mathrm{d}t$$
$$\le \beta(M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})) \int_0^{M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})} \Theta(t)\mathrm{d}t, \tag{3.6}$$

where

$$\begin{aligned} M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1}) =& \mathfrak{Max}\{d(\rho_{n(k)-1},\rho_{m(k)-1}), d(\rho_{n(k)-1},\Omega(\rho_{n(k)-1})),\\ & d(\rho_{m(k)-1},\Omega(\rho_{m(k)-1})),\\ & \frac{d(\rho_{n(k)-1},\Omega(\rho_{m(k)-1})) + d(\rho_{m(k)-1},\Omega(\rho_{n(k)-1}))}{2}\}, \quad \forall\, n \ge 1\\ =& \mathfrak{Max}\{d(\rho_{n(k)-1},\rho_{m(k)-1}), d(\rho_{n(k)-1},\rho_{n(k)}),\\ & d(\rho_{m(k)-1},\rho_{m(k)}), \frac{d(\rho_{n(k)-1},\rho_{m(k)}) + d(\rho_{m(k)-1},\rho_{n(k)})}{2}\}. \end{aligned} \tag{3.7}$$

Clearly, we obtain $\lim_{k\to\infty} d(\rho_{n(k)-1},\rho_{m(k)-1}) = \epsilon$. Hence, we get

$$\frac{\int_0^{d(\rho_{n(k)},\rho_{m(k)})} \Theta(t)\mathrm{d}t}{\int_0^{M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})} \Theta(t)\mathrm{d}t} \le \beta(M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})).$$

Let $k \to \infty$. We obtain

$$\lim_{k\to\infty} \beta(M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})) = 1 \Rightarrow \lim_{k\to\infty} \int_0^{M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})} \Theta(t)\mathrm{d}t = 0$$
$$\Rightarrow \lim_{k\to\infty} M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})) = 0.$$

Hence, $\epsilon = 0$ which is a contradiction. So, our supposition is wrong. Thus the sequence $\{\rho_n\}$ is a Cauchy sequence.

Step 3. We must claim that Ω has a fixed point. Since (X, d) is complete metric space.

So, $\exists\, \rho_* = \lim_{n\to\infty}\{\rho_n\} \in X$. Since Ω is continuous, we have $\Omega(\rho_n) = \Omega(\rho_*) = \rho_* \Rightarrow \Omega(\rho_*) = \rho_*$ i.e., ρ_* is a fixed point of Ω. □

Theorem 3.2 *Let us consider a self-map $\Omega : X \to X$ which is defined on a complete metric space (X, d). Additionally, there exists a function $\alpha : X \times X \to \mathbb{R}$, if Ω satisfies the following conditions:*

(a) Ω is a generalized α-Geraghty contraction;

(b) Ω is triangular α-orbital admissible;

(c) $\exists\, \rho_1 \in X$, such that $\alpha(\rho_1, \Omega(\rho_1)) \geq 1$;

(d) $\exists\, \{\rho_n\} \in X$, such that $\alpha(\rho_n, \Omega(\rho_n)) \geq 1$, $\forall\, n \in \mathbb{N}$ and $\lim_{n\to\infty} \rho_n = \rho$, where $\rho \in X$. Moreover, there exists a subsequence $\{\rho_{n(k)}\}$ of $\{\rho_n\}$, such that $\alpha(\rho_{n(k)}, \rho^*) \geq 1$, $\forall\, k \in \mathbb{N}$. Then Ω has a fixed point $\rho^* \in X$, therefore,

$$\begin{aligned}\int_0^{d(\rho,\sigma)} \Theta(t)\mathrm{d}t &= \int_0^{d(\Omega(\rho),\Omega(\sigma))} \Theta(t)\mathrm{d}t \\ &\leq \int_0^{\alpha(\rho,\sigma)d(\Omega(\rho),\Omega(\sigma))} \Theta(t)\mathrm{d}t \\ &\leq \beta(M_\Omega(\rho,\sigma)) \int_0^{M_\Omega(\rho,\sigma)} \Theta(t)\mathrm{d}t, \end{aligned} \tag{3.8}$$

where

$$M_\Omega(\rho,\sigma) = \mathfrak{Max}\{d(\rho,\sigma), d(\rho,\Omega(\rho)), d(\sigma,\Omega(\sigma)), \frac{d(\rho,\Omega(\sigma)) + d(\sigma,\Omega(\rho))}{2}\}. \tag{3.9}$$

Let $\beta \in \mathcal{F}$, $\beta : \mathbb{R}^+ \cup \{0\} \to [0,1)$ and $\Theta : [0,+\infty) \to [0,+\infty)$ be Lebesgue function which is summable on $\mathbb{R}^+$, non-negative and for all $\epsilon > 0$ $\int_0^\epsilon \Theta(t)\mathrm{d}t > 0$.

Proof Let the sequence $\{\rho_n\}$ be defined by $\Omega(\rho_n) = \rho_{n+1}$, $\forall\, n \geq 1$ and $\lim_{n\to\infty}\{\rho_n\} \to \rho_* \in X$. Now by condition (d), there is a subsequence $\{\rho_{n(k)}\}$ of $\{\rho_n\}$, such that $\alpha(\rho_{n(k)}, \rho_*) \geq 1$, $\forall\, k \in \mathbb{N}$ and

$$\begin{aligned}\int_0^{d(\rho_{n(k)+1},\Omega(\rho_*))} \Theta(t)\mathrm{d}t &= \int_0^{d(\Omega(\rho_{n(k)}),\Omega(\rho_*))} \Theta(t)\mathrm{d}t \\ &\leq \int_0^{\alpha(\rho_{n(k)},\rho_*)d(\Omega(\rho_{n(k)}),\Omega(\rho_*))} \Theta(t)\mathrm{d}t \\ &\leq \beta(M_\Omega(\rho_{n(k)},\rho_*)) \int_0^{M_\Omega(\rho_{n(k)},\rho_*)} \Theta(t)\mathrm{d}t, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} M_\Omega(\rho_{n(k)},\rho_*) =& \mathfrak{Max}\{d(\rho_{n(k)},\rho_*), d(\rho_{n(k)},\Omega(\rho_{n(k)})), d(\rho_*,\Omega(\rho_*)), \\ & \frac{d(\rho_{n(k)},\Omega(\rho_*)) + d(\rho_*,\Omega(\rho_{n(k)}))}{2}\} \\ =& \mathfrak{Max}\{d(\rho_{n(k)},\rho_*), d(\rho_{n(k)},\rho_{n(k)+1}), d(\rho_*,\Omega(\rho_*)), \end{aligned}$$

$$\frac{d(\rho_{n(k)}, \Omega(\rho_*)) + d(\rho_*, \rho_{n(k)+1})}{2}\}. \tag{3.11}$$

Now, we suppose that $\Omega(\rho_*) \neq \rho_*$, i.e., $d(\rho_*, \Omega(\rho_*)) > 0$. Let $k \to \infty$ and by the equality (3.11), we obtain $\lim_{k\to\infty} M_\Omega(\rho_{n(k)}, \rho_*) = d(\rho_*, \Omega(\rho_*))$. Since

$$\frac{\int_0^{d(\rho_{n(k)+1}, \Omega(\rho_*))} \Theta(t)\mathrm{d}t}{\int_0^{M_\Omega(\rho_{n(k)}, \rho_*)} \Theta(t)\mathrm{d}t} \leq \beta(M_\Omega(\rho_{n(k)}, \rho_*)), \quad \forall\, k \in \mathbb{N}.$$

Let $k \to \infty$. We can obtain

$$\lim_{k\to\infty} \beta(M_\Omega(\rho_{n(k)}, \rho_*)) = 1 \Rightarrow \lim_{k\to\infty} \int_0^{M_\Omega(\rho_{n(k)}, \rho_*)} \Theta(t)\mathrm{d}t = 0 \Rightarrow \lim_{k\to\infty} M_\Omega(\rho_{n(k)}, \rho_*) = 0.$$

Hence, $d(\rho_*, \Omega(\rho_*)) = 0$ which is a contradiction, so that $\Omega(\rho_*) = \rho_*$, i.e., ρ_* is a fixed point of Ω. □

Note. We consider this condition:

(c$_*$) $\forall\, \rho \neq \sigma \in X$, $\exists\, \zeta \in X$, such that $\alpha(\rho, \zeta) \geq 1$, $\alpha(\sigma, \zeta) \geq 1$ and $\alpha(\zeta, \Omega(\zeta)) \geq 1$.

Remark 3.3 If we replace the condition (c) in Theorems 3.1 and 3.2 by condition (c$_*$), we have ρ_* is a unique fixed point of Ω.

Theorem 3.4 *Let us consider a self-map $\Omega : X \to X$ which is defined on a complete metric space (X, d). Additionally, there exists a function $\alpha : X \times X \to \mathbb{R}$, if Ω satisfies the following conditions:*

(a) Ω is a generalized α-Geraghty contraction;

(b) Ω is triangular α-orbital admissible;

(c) $\exists\, \rho_1 \in X$, such that $\alpha(\rho_1, \Omega(\rho_1)) \geq 1$;

(d) Ω is α-orbital attractive,

then, Ω has a unique fixed point $\rho^* \in X$ and

$$\begin{aligned}\int_0^{d(\rho,\sigma)} \Theta(t)\mathrm{d}t &= \int_0^{d(\Omega(\rho),\Omega(\sigma))} \Theta(t)\mathrm{d}t \\ &\leq \int_0^{\alpha(\rho,\sigma)d(\Omega(\rho),\Omega(\sigma))} \Theta(t)\mathrm{d}t \\ &\leq \beta(M_\Omega(\rho,\sigma)) \int_0^{M_\Omega(\rho,\sigma)} \Theta(t)\mathrm{d}t, \end{aligned} \tag{3.12}$$

where

$$M_\Omega(\rho, \sigma) = \mathfrak{Max}\{d(\rho, \sigma), d(\rho, \Omega(\rho)), d(\sigma, \Omega(\sigma)), \frac{d(\rho, \Omega(\sigma)) + d(\sigma, \Omega(\rho))}{2}\}. \tag{3.13}$$

Let $\beta \in \mathcal{F}$, $\beta : \mathbb{R}^+ \cup \{0\} \to [0, 1)$ and $\Theta : [0, +\infty) \to [0, +\infty)$ be Lebesgue function which is summable on $\mathbb{R}^+$, non-negative and for all $\epsilon > 0$, $\int_0^\epsilon \Theta(t)\mathrm{d}t > 0$.

Proof Let $\rho_1 \in X$, such that $\alpha(\rho_1, \Omega(\rho_1)) \geq 1$. We define a sequence $\{\rho_n\}$ by $\Omega(\rho_n) = \rho_{n+1}$, $\forall\, n \geq 1$. If $\rho_{n(0)} = \rho_{n(0)+1}$ for some $n(0) \geq 1$, then Ω has a fixed point and (3.12) holds.

Step 1. We must claim that $\lim_{n\to\infty} d(\rho_n, \rho_{n+1}) = 0$. Now, suppose that $\rho_n \neq \rho_{n+1}$, $\forall\, n \geq 1$.

Since, Ω is α-orbital admissible, we get

$$\alpha(\rho_1,\rho_2)=\alpha(\rho_1,\Omega(\rho_1))\geq 1\Rightarrow \alpha(\Omega(\rho_1),\Omega(\rho_2))=\alpha(\rho_2,\rho_3)\geq 1. \tag{3.14}$$

When we repeat the similar step in (3.14), we obtain $\alpha(\rho_n,\rho_{n+1})\geq 1$, $\forall\, n\geq 1$. Since,

$$\begin{aligned}\int_0^{d(\rho_{n+1},\rho_{n+2})}\Theta(t)\mathrm{d}t &= \int_0^{d(\Omega(\rho_n),\Omega(\rho_{n+1}))}\Theta(t)\mathrm{d}t\\ &\leq \int_0^{\alpha(\rho_n,\rho_{n+1})d(\Omega(\rho_n),\Omega(\rho_{n+1}))}\Theta(t)\mathrm{d}t\\ &\leq \beta(M_\Omega(\rho_n,\rho_{n+1}))\int_0^{M_\Omega(\rho_n,\rho_{n+1})}\Theta(t)\mathrm{d}t, \end{aligned}\tag{3.15}$$

where

$$\begin{aligned}M_\Omega(\rho_n,\rho_{n+1}) =&\,\mathfrak{Max}\{d(\rho_n,\rho_{n+1}),d(\rho_n,\Omega(\rho_n)),d(\rho_{n+1},\Omega(\rho_{n+1})),\\ &\frac{d(\rho_n,\Omega(\rho_{n+1}))+d(\rho_{n+1},\Omega(\rho_n))}{2}\},\quad \forall\, n\geq 1\\ =&\,\mathfrak{Max}\{d(\rho_n,\rho_{n+1}),d(\rho_{n+1},\rho_{n+2}),\frac{d(\rho_n,\rho_{n+2})}{2}\}\\ =&\,\mathfrak{Max}\{d(\rho_n,\rho_{n+1}),d(\rho_{n+1},\rho_{n+2}),\frac{d(\rho_n,\rho_{n+1})+d(\rho_{n+1},\rho_{n+2})}{2}\}\\ =&\,\mathfrak{Max}\{d(\rho_n,\rho_{n+1}),d(\rho_{n+1},\rho_{n+2})\}.\end{aligned}\tag{3.16}$$

Now, if we take $M_\Omega(\rho_n,\rho_{n+1})=d(\rho_{n+1},\rho_{n+2})$ then $d(\rho_n,\rho_{n+1})\leq d(\rho_{n+1},\rho_{n+2})$. By (3.15), we get

$$\int_0^{d(\rho_{n+1},\rho_{n+2})}\Theta(t)\mathrm{d}t\leq \beta(d(\rho_{n+1},\rho_{n+2}))\int_0^{d(\rho_{n+1},\rho_{n+2})}\Theta(t)\mathrm{d}t\leq \int_0^{d(\rho_{n+1},\rho_{n+2})}\Theta(t)\mathrm{d}t,$$

which is a contradiction. So we get

$$M_\Omega(\rho_n,\rho_{n+1})=d(\rho_n,\rho_{n+1})\text{ and }d(\rho_n,\rho_{n+1})>d(\rho_{n+1},\rho_{n+2}).$$

By (3.15), we have

$$\int_0^{d(\rho_{n+1},\rho_{n+2})}\Theta(t)\mathrm{d}t\leq \beta(d(\rho_n,\rho_{n+1}))\int_0^{d(\rho_n,\rho_{n+1})}\Theta(t)\mathrm{d}t\leq \int_0^{d(\rho_n,\rho_{n+1})}\Theta(t)\mathrm{d}t. \tag{3.17}$$

Thus, the sequence $\{\int_0^{d(\rho_n,\rho_{n+1})}\Theta(t)\mathrm{d}t\}$ is positive and non-increasing so, $\exists\,\lambda\geq 0$, such that $\lim_{n\to\infty}\int_0^{d(\rho_n,\rho_{n+1})}\Theta(t)\mathrm{d}t=\lambda$, we must prove that $\lambda=0$. By the contrary we suppose that $\lambda>0$, then from (3.17) we have

$$\frac{\int_0^{d(\rho_{n+1},\rho_{n+2})}\Theta(t)\mathrm{d}t}{\int_0^{d(\rho_n,\rho_{n+1})}\Theta(t)\mathrm{d}t}\leq \beta(d(\rho_n,\rho_{n+1}))\leq 1.$$

When $\beta\in\mathcal{F}$, we get $\lim_{n\to\infty}\beta(d(\rho_n,\rho_{n+1}))=1$ and $\lim_{n\to\infty}\int_0^{d(\rho_n,\rho_{n+1})}\Theta(t)\mathrm{d}t=0$, which is a contradiction. Hence, $\lambda=0$ and

$$\lim_{n\to\infty}\int_0^{d(\rho_n,\rho_{n+1})}\Theta(t)\mathrm{d}t=0\Rightarrow \lim_{n\to\infty}d(\rho_n,\rho_{n+1})=0.$$

So, our supposition is wrong.

Step 2. We must claim the sequence $\{\rho_n\}$ is a Cauchy sequence. On contrary, we suppose that $\{\rho_n\}$ is not a Cauchy, then $\exists\, \epsilon > 0$, such that $\forall\, n \geq 1$, there exist subsequence $m(k)$, $n(k)$ such that $m(k) > n(k) > k$ with $d(n(k), m(k)) \geq \epsilon$. Now, suppose that $m(k)$ is the smallest number which satisfies the above conditions. Hence, we get $d(\rho_{n(k)}, \rho_{m(k)}) < \epsilon$, and

$$\begin{aligned}\epsilon \leq d(\rho_{n(k)}, \rho_{m(k)}) &\leq d(\rho_{n(k)}, \rho_{m(k)-1}) + d(\rho_{m(k)-1}, \rho_{m(k)})\\ &< \epsilon + d(\rho_{m(k)-1}, \rho_{m(k)}).\end{aligned}$$

Let $k \to \infty$. We have $\lim_{k\to\infty} d(\rho_{n(k)}, \rho_{m(k)}) = \epsilon$. Since

$$|d(\rho_{n(k)}, \rho_{m(k)-1}) - d(\rho_{n(k)}, \rho_{m(k)})| \leq d(\rho_{m(k)}, \rho_{m(k)-1}),$$

we obtain $\lim_{k\to\infty} d(\rho_{n(k)}, \rho_{m(k)-1}) = \epsilon$. Similarly, we get

$$\begin{aligned}\lim_{k\to\infty} d(\rho_{m(k)}, \rho_{n(k)-1}) &= \lim_{k\to\infty} d(\rho_{m(k)-1}, \rho_{n(k)-1})\\ &= \lim_{k\to\infty} d(\rho_{m(k)-1}, \rho_{n(k)+1}) = \epsilon.\end{aligned}$$

Since, $\alpha(\rho_{n(k)-1}, \rho_{n(k)}) \geq 1$ and Ω is α-orbital attractive by Lemma 1.6, we obtain

$$\alpha(\rho_{n(k)-1}, \rho_{m(k)-1}) \geq 1 \text{ or } \alpha(\rho_{m(k)-1}, \rho_{n(k)}) \geq 1.$$

Hence, we get two cases as follows:

Case 1. There is an infinite subset P of $\mathbb{N}$ such that $\alpha(\rho_{n(k)-1}, \rho_{m(k)-1}) \geq 1$, $\forall\, k \in P$.

Case 2. There is an infinite subset Q of $\mathbb{N}$ such that $\alpha(\rho_{m(k)-1}, \rho_{n(k)}) \geq 1$, $\forall\, k \in Q$.

From the first case, we obtain

$$\begin{aligned}\int_0^{d(\rho_{n(k)},\rho_{m(k)})} \Theta(t)\mathrm{d}t &= \int_0^{d(\Omega(\rho_{n(k)-1}),\Omega(\rho_{m(k)-1}))} \Theta(t)\mathrm{d}t\\ &\leq \int_0^{\alpha(\rho_{n(k)-1},\rho_{m(k)-1})d(\Omega(\rho_{n(k)-1}),\Omega(\rho_{m(k)-1}))} \Theta(t)\mathrm{d}t\\ &\leq \beta(M_\Omega(\rho_{n(k)-1}, \rho_{m(k)-1})) \int_0^{M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})} \Theta(t)\mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}M_\Omega(\rho_{n(k)-1}, \rho_{m(k)-1}) =& \mathfrak{Max}\{d(\rho_{n(k)-1}, \rho_{m(k)-1}), d(\rho_{n(k)-1}, \Omega(\rho_{n(k)-1})), d(\rho_{m(k)-1}, \Omega(\rho_{m(k)-1})),\\ &\frac{d(\rho_{n(k)-1}, \Omega(\rho_{m(k)-1})) + d(\rho_{m(k)-1}, \Omega(\rho_{n(k)-1}))}{2}\}\\ =& \mathfrak{Max}\{d(\rho_{n(k)-1}, \rho_{m(k)-1}), d(\rho_{n(k)-1}, \rho_{n(k)}), d(\rho_{m(k)-1}, \rho_{m(k)}),\\ &\frac{d(\rho_{n(k)-1}, \rho_{m(k)}) + d(\rho_{m(k)-1}, \rho_{n(k)})}{2}\}.\end{aligned}$$

Now, let $k \to \infty$, $k \in P$. We must get $\lim_{k\to\infty,k\in P} M_\Omega(\rho_{n(k)-1}, \rho_{m(k)-1}) = \epsilon$ and since

$$\frac{\int_0^{d(\rho_{n(k)},\rho_{m(k)})} \Theta(t)\mathrm{d}t}{\int_0^{M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})} \Theta(t)\mathrm{d}t} \leq \beta(M_\Omega(\rho_{n(k)-1}, \rho_{m(k)-1})).$$

So that we get $\lim_{k\to\infty,k\in P} \beta(M_\Omega(\rho_{n(k)-1}, \rho_{m(k)-1})) = 1$. Since $\beta \in \mathcal{F}$, we get

$$\lim_{k\to\infty,k\in P} \int_0^{M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})} \Theta(t)\mathrm{d}t = 0,$$

which is a contradiction. Hence, $\epsilon = 0$ and

$$\lim_{k\to\infty, k\in P} \int_0^{M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1})} \Theta(t)\mathrm{d}t = 0$$
$$\Rightarrow \lim_{k\to\infty, k\in P} M_\Omega(\rho_{n(k)-1},\rho_{m(k)-1}) = 0.$$

From the second case, we obtain

$$\begin{aligned}\int_0^{d(\rho_{m(k)},\rho_{n(k)+1})} \Theta(t)\mathrm{d}t &= \int_0^{d(\Omega(\rho_{m(k)-1}),\Omega(\rho_{n(k)}))} \Theta(t)\mathrm{d}t \\ &\le \int_0^{\alpha(\rho_{m(k)-1},\rho_{n(k)})d(\Omega(\rho_{m(k)-1}),\Omega(\rho_{n(k)}))} \Theta(t)\mathrm{d}t \\ &\le \beta(M_\Omega(\rho_{m(k)-1},\rho_{n(k)})) \int_0^{M_\Omega(\rho_{m(k)-1},\rho_{n(k)})} \Theta(t)\mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}M_\Omega(\rho_{m(k)-1},\rho_{n(k)}) =& \mathfrak{Max}\{d(\rho_{m(k)-1},\rho_{n(k)}), d(\rho_{m(k)-1},\Omega(\rho_{m(k)-1})), \mathrm{d}(\rho_{n(k)},\Omega(\rho_{n(k)})), \\ &\frac{d(\rho_{m(k)-1},\Omega(\rho_{n(k)})) + d(\rho_{n(k)},\Omega(\rho_{m(k)-1}))}{2}\} \\ =& \mathfrak{Max}\{d(\rho_{m(k)-1},\rho_{n(k)}), d(\rho_{m(k)-1},\rho_{m(k)}), d(\rho_{n(k)},\rho_{n(k)+1}), \\ &\frac{d(\rho_{m(k)-1},\rho_{n(k)+1}) + d(\rho_{n(k)},\rho_{m(k)})}{2}\}.\end{aligned}$$

Now, let $k \to \infty$, $k \in Q$. We must get $\lim_{k\to\infty,k\in Q} M_\Omega(\rho_{m(k)-1},\rho_{n(k)}) = \epsilon$ and since,

$$\frac{\int_0^{d(\rho_{m(k)},\rho_{n(k)+1})} \Theta(t)\mathrm{d}t}{\int_0^{M_\Omega(\rho_{m(k)-1},\rho_{n(k)})} \Theta(t)\mathrm{d}t} \le \beta(M_\Omega(\rho_{m(k)-1},\rho_{n(k)})).$$

So that we get $\lim_{k\to\infty,k\in Q} \beta(M_\Omega(\rho_{m(k)-1},\rho_{n(k)})) = 1$. Since $\beta \in \mathcal{F}$, we get

$$\lim_{k\to\infty, k\in Q} \int_0^{M_\Omega(\rho_{m(k)-1},\rho_{n(k)})} \Theta(t)\mathrm{d}t = 0,$$

which is a contradiction. Hence, $\epsilon = 0$ and we get

$$\lim_{k\to\infty, k\in Q} \int_0^{M_\Omega(\rho_{m(k)-1},\rho_{n(k)})} \Theta(t)\mathrm{d}t = 0$$
$$\Rightarrow \lim_{k\to\infty, k\in Q} M_\Omega(\rho_{m(k)-1},\rho_{n(k)}) = 0.$$

So that our supposition is wrong.

Therefore, $\{\rho_n\}$ is a Cauchy sequence. Since (X, d) is complete metric space and $\rho_* \in X$.

So, $\exists\, \rho_* = \lim_{n\to\infty} \rho_n \in X$.

Step 3. We must claim that $\rho_* = \Omega(\rho_*)$ is a fixed point of Ω. On the contrary, suppose that $\rho_* \ne \Omega(\rho_*)$. Since Ω is α-orbital attractive, we have for all $n \ge 1$ and $\alpha(\rho_n, \rho_*) \ge 1$ or $\alpha(\rho_*, \rho_{n+1}) \ge 1$. So there exists sub-sequence $\{\rho_{n(k)}\}$ of $\{\rho_n\}$, such that $\alpha(\rho_{n(k)}, \rho_*) \ge 1$ or $\alpha(\rho_*, \rho_{n(k)}) \ge 1$, $\forall\, k \ge 1$. In the first case, we obtain

$$\int_0^{d(\rho_{n(k)+1},\Omega(\rho_*))} \Theta(t)\mathrm{d}t \le \int_0^{\alpha(\rho_{n(k)},\rho_*)d(\rho_{n(k)+1},\Omega(\rho_*))} \Theta(t)\mathrm{d}t$$

$$\leq \beta(M_\Omega(\rho_{n(k)}, \rho_*)) \int_0^{M_\Omega(\rho_{n(k)},\rho_*)} \Theta(t)\mathrm{d}t, \quad \forall\, k \geq 1,$$

where

$$\begin{aligned} M_\Omega(\rho_{n(k)}, \rho_*) =& \mathfrak{Max}\{d(\rho_{n(k)}, \rho_*), d(\rho_{n(k)}, \Omega(\rho_{n(k)})), d(\rho_*, \Omega(\rho_*)), \\ & \frac{d(\rho_{n(k)}, \Omega(\rho_*)) + d(\rho_*, \Omega(\rho_{n(k)}))}{2}\} \\ =& \mathfrak{Max}\{d(\rho_{n(k)}, \rho_*), d(\rho_{n(k)}, \rho_{n(k)+1}), d(\rho_*, \Omega(\rho_*)), \\ & \frac{d(\rho_{n(k)}, \Omega(\rho_*)) + d(\rho_*, \rho_{n(k)+1})}{2}\}. \end{aligned}$$

Now, let $k \to \infty$. We obtain $\lim_{k\to\infty} M_\Omega(\rho_{n(k)}, \rho_*) = d(\rho_*, \Omega(\rho_*))$ and since

$$\frac{\int_0^{d(\rho_{n(k)+1},\Omega(\rho_*))} \Theta(t)\mathrm{d}t}{\int_0^{M_\Omega(\rho_{n(k)},\rho_*)} \Theta(t)\mathrm{d}t} \leq \beta(M_\Omega(\rho_{n(k)}, \rho_*)),$$

we have $\lim_{k\to\infty} \beta(M_\Omega(\rho_{n(k)}, \rho_*)) = 1$. Since $\beta \in \mathcal{F}$ on the first case, we have

$$\lim_{k\to\infty} \int_0^{M_\Omega(\rho_{n(k)},\rho_*)} \Theta(t)\mathrm{d}t = 0 \Rightarrow \lim_{k\to\infty} M_\Omega(\rho_{n(k)}, \rho_*) = 0,$$

which is a contradiction so that

$$\lim_{k\to\infty} M_\Omega(\rho_{n(k)}, \rho_*) = d(\rho_*, \Omega(\rho_*)) = 0 \Rightarrow \Omega(\rho_*) = \rho_*.$$

Similarly, on the second case, we get $\Omega(\rho_*) = \rho_* \Rightarrow \rho_*$ is the fixed point of Ω.

Step 4. We must claim that the fixed point is unique. Suppose that there is another fixed point $\sigma_* \neq \rho_*$ of Ω such that $\Omega(\rho_*) = \rho_*$, $\Omega(\sigma_*) = \sigma_*$. Now from the hypothesis, we get

$$\alpha(\rho_n, \sigma_*) \geq 1 \text{ or } \alpha(\sigma_*, \rho_{n+1}) \geq 1.$$

Hence, there exists sub-sequence $\{\rho_{n(k)}\}$ of $\{\rho_n\}$ such that $\alpha(\rho_{n(k)}, \sigma_*) \geq 1$ or $\alpha(\sigma_*, \rho_{n(k)}) \geq 1$, $\forall\, k \geq 1$, so that in the first case, we get

$$\begin{aligned} \int_0^{d(\rho_{n(k)+1},\Omega(\sigma_*))} \Theta(t)\mathrm{d}t \leq& \int_0^{\alpha(\rho_{n(k)},\sigma_*)d(\rho_{n(k)+1},\Omega(\sigma_*))} \Theta(t)\mathrm{d}t \\ \leq& \beta(M_\Omega(\rho_{n(k)}, \sigma_*)) \int_0^{M_\Omega(\rho_{n(k)},\sigma_*)} \Theta(t)\mathrm{d}t, \quad \forall\, k \geq 1, \end{aligned}$$

where

$$\begin{aligned} M_\Omega(\rho_{n(k)}, \sigma_*) =& \mathfrak{Max}\{d(\rho_{n(k)}, \sigma*), d(\rho_{n(k)}, \Omega(\rho_{n(k)})), d(\sigma*, \Omega(\sigma*)), \\ & \frac{d(\rho_{n(k)}, \Omega(\sigma*)) + d(\sigma*, \Omega(\rho_{n(k)}))}{2}\} \\ =& \mathfrak{Max}\{d(\rho_{n(k)}, \sigma*), d(\rho_{n(k)}, \rho_{n(k)+1}), d(\sigma*, \Omega(\sigma*)), \\ & \frac{d(\rho_{n(k)}, \Omega(\sigma*)) + d(\sigma*, \rho_{n(k)+1})}{2}\}. \end{aligned}$$

Now, let $k \to \infty$. We obtain $\lim_{k\to\infty} M_\Omega(\rho_{n(k)}, \sigma_*) = d(\rho_*, \sigma_*)$ and since

$$\frac{\int_0^{d(\rho_{n(k)+1},\Omega(\sigma_*))} \Theta(t)\mathrm{d}t}{\int_0^{M_\Omega(\rho_{n(k)},\sigma_*)} \Theta(t)\mathrm{d}t} \leq \beta(M_\Omega(\rho_{n(k)}, \sigma_*)),$$

we get $\lim_{k\to\infty}\beta(M_\Omega(\rho_{n(k)},\sigma_*))=1$. Since $\beta\in\mathcal{F}$, we get

$$\lim_{k\to\infty}\int_0^{M_\Omega(\rho_{n(k)},\sigma_*)}\Theta(t)\mathrm{d}t=0\Rightarrow\lim_{k\to\infty}M_\Omega(\rho_{n(k)},\sigma_*)=0,$$

so $d(\rho_*,\sigma_*)=0$, which is a contradiction. i.e., $\rho_*=\sigma_*$. Similarly, in the second case, therefore, ρ_* is a unique fixed point of Ω. □

Remark 3.5 If we put $\Theta(t)=1$ in Theorem 3.4, then we have

$$\alpha(\rho,\sigma)d(\Omega(\rho),\Omega(\sigma))\le\beta(M_\Omega(\rho,\sigma))M_\Omega(\rho,\sigma),$$

which is a generalized α-Geraghty contraction type in metric space [1].

Example 3.6 ([32]) Consider the metric space (X,d), where $X=[0,\infty)$ and $d(\rho_1,\rho_2)=|\rho_1-\rho_2|$. Define the function $\Omega:X\to X$ as $\Omega(\rho)=\sqrt{\rho}$, for all $\rho\in\mathbb{R}^+$. Let $\Theta:X\to X$ be defined as $\Theta(\rho)=\frac{\rho}{2}$ for all $\rho\in\mathbb{R}^+$. Define $\alpha:X\times X\to\mathbb{R}$ as follows:

$$\alpha(\rho_1,\rho_2)=\begin{cases}1, & \text{if } \rho_1\rho_2>0,\\ 0, & \text{otherwise.}\end{cases}$$

Also let $\beta:[0,\infty)\to[0,1)$ be defined as $\beta(t)=\frac{1}{2}$ for all $t\in\mathbb{R}^+$ and (X,d) be a complete metric space. We will analyze whether Ω satisfies the conditions of Theorem 3.4, for two cases.

Case 1. If we choose $\alpha(\rho_1,\rho_2)=1$, then

$$\begin{aligned}\int_0^{d(\Omega(\rho_1),\Omega(\rho_2))}\Theta(t)\mathrm{d}t&=\int_0^{\alpha(\rho_1,\rho_2)d(\Omega(\rho_1),\Omega(\rho_2))}\Theta(t)\mathrm{d}t\\&=\int_0^{|\rho_1^{1/2}-\rho_2^{1/2}|}\frac{t}{2}\mathrm{d}t=\frac{1}{2}\int_0^{|\rho_1^{1/2}-\rho_2^{1/2}|}t\mathrm{d}t\\&=\frac{1}{4}|\rho_1^{1/2}-\rho_2^{1/2}|^2=\frac{1}{4}|\sqrt{\rho_1}-\sqrt{\rho_2}|^2\\&\le\frac{1}{4}|\rho_1-\rho_2|<\frac{1}{2}|\rho_1-\rho_2|.\end{aligned}$$

We can show that this is bounded by $\beta(d(\rho_1,\rho_2))\int_0^{M_\Omega(\rho_1,\rho_2)}\Theta(t)\mathrm{d}t$ by noting that $\beta(t)=\frac{1}{2}$ and $\frac{1}{4}|\rho_1-\rho_2|\le\frac{1}{2}|\rho_1-\rho_2|$.

Case 2. If we choose $\alpha(\rho_1,\rho_2)=0$, then

$$\begin{aligned}\int_0^{d(\Omega(\rho_1),\Omega(y\rho_2))}\Theta(t)\mathrm{d}t&=\int_0^{\alpha(\rho_1,\rho_2)d(\Omega(\rho_1),\Omega(\rho_2))}\Theta(t)\mathrm{d}t\\&=\int_0^0\Theta(t)\mathrm{d}t=0,\end{aligned}$$

which trivially satisfies the inequality $\le\beta(d(\rho_1,\rho_2))\int_0^{M_\Omega(\rho_1,\rho_2)}\Theta(t)\mathrm{d}t$. In both cases, since $\Omega(0)=\sqrt{0}=0$, we can conclude that "0" is a fixed point of Ω in this metric space (X,d).

4. Conclusion

In conclusion, this research significantly extends the scope of a generalized α-Geraghty contraction type within a complete metric space, employing a Lebesgue integrable function. The foundation for this extension is built upon the concept of α-Geraghty contractions introduced by Ovidiu Popescu in 2014. The practical significance of these advancements becomes particularly evident when applied to Lebesgue integrals, as demonstrated in Example 3.6.

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