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# Eigenvalue-Free Interval for Seidel Matrices of Cographs

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**Abstract** The distribution of Seidel eigenvalues of cographs is investigated in this paper. We prove that there is no Seidel eigenvalue of nontrivial cographs in the interval $(-1, 1)$. We also show the optimality of the interval $(-1, 1)$ in the sense that for any $\epsilon > 0$ either of the intervals $(1, 1 + \epsilon)$ and $(-1 - \epsilon, -1)$ contains a Seidel eigenvalue of some cograph of order $n$ when $n$ is sufficiently large.

**Keywords** cograph; Seidel matrix; eigenvalue-free interval

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## 1. Introduction

Let $G$ be a simple graph with $n$ vertices. We use $A(G)$ and $S(G) = J - I - 2A(G)$ to denote the adjacency matrix and the Seidel matrix of $G$, respectively, where $J$ is the all-ones matrix and $I$ is the identity matrix of order $n$. The eigenvalues of $A(G)$ (resp., $S(G)$) will be referred to as eigenvalues (resp., Seidel eigenvalues) of $G$.

A cograph is a graph containing no $P_4$ as an induced subgraph. A distinguished subclass of cographs is the family of threshold graphs which are graphs containing none of $\{2K_2, P_4, C_4\}$ as an induced subgraph. Spectral properties of threshold graphs have received considerable attentions in recent years, see, for example, [1–6]. Jacobs et al. [7] showed that no threshold graphs have eigenvalues in $(-1, 0)$. This result has been strengthened by Ghorbani [8] and Mohammadian-Trevisan [9] in two different directions. Ghorbani [8] proved that, except possibly the two trivial eigenvalues 0 and $-1$, the larger interval $[(-1-\sqrt{2})/2, (-1+\sqrt{2})/2]$ does not contain an eigenvalue of any threshold graph, a fact first conjectured by Aguilar et al. [10]. We mention that a new proof of Ghorbani's result with further generalization was reported by Allem et al. [11]. For the other direction, Mohammadian and Trevisan [9] extended the aforementioned result of Jacobs et al. from threshold graphs to cographs, i.e., no cographs have eigenvalues in $(-1, 0)$.

Besides the adjacency matrix, the problem of finding eigenvalue-free intervals for threshold graphs and cographs have also been investigated with respect to other kinds of matrices associated with a graph. For example, Lu, Huang and Lou [12] considered the distance matrix; they showed that no connected threshold graphs have distance eigenvalues in $(-2, -1)$. As an extension of this

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result, Alazemi et al. [13] proved that any connected threshold graph has no distance eigenvalues in $(\frac{-3-\sqrt{2}}{2}, -1)$, except possibly $-2$. Another extension was obtained by Lou and Lin [14] who showed that no connected cographs have distance eigenvalues in the interval $(-2, -1)$.

In a recent paper, Xiong and Hou [15] investigated the Seidel eigenvalues of threshold graphs. It was proved that no threshold graphs have Seidel eigenvalues in the interval $(-\sqrt{2}, \sqrt{2})$ except possibly the trivial eigenvalues $\pm 1$. It is natural to consider the corresponding problem for cographs. The main result of this paper is the following theorem.

**Theorem 1.1** *No cographs with at least two vertices have Seidel eigenvalues in the interval* $(-1, 1)$.

We also prove the optimality of the $(-1, 1)$. Precisely, we prove:

**Theorem 1.2** *For any* $\epsilon > 0$*, either of the intervals* $(1, 1+\epsilon)$ *and* $(-1-\epsilon, -1)$ *contains a Seidel eigenvalue of some cograph of sufficiently large order.*

The proof of Theorem 1.1 is given in Section 2. Indeed, we shall prove a stronger result (Theorem 2.2 in Section 2) from which Theorem 1.1 is a direct consequence. Theorem 1.2 is proved in Section 3 where a desired family of cographs is constructed. The main part of the proof is to compute explicitly the corresponding Seidel characteristic polynomials. Some tools from elementary calculus are successfully employed to estimate the eigenvalue nearest to $\pm 1$.

**Remark 1.3** During the revision of this paper, the authors learned that Li et al. [16] independently obtained Theorem 1.1 using a different method.

## 2. Rank of a vertex-weighted cograph

The main aim of this section is to prove Theorem 1.1. The basic technique comes from Chang-Huang-Yeh [17], where the authors extended a result of Royle [18] using an elementary short proof.

A vertex-weighted graph $(G, w)$ is a simple graph $G$ equipped with a weight function $w$ from $V(G)$ to $[-1, 1]$. Naturally, we call $(G, w)$ a vertex-weighted cograph if $G$ is a cograph. The Seidel matrix of a vertex-weighted graph $(G, w)$, denoted by $S((G, w))$ or simply $S(G, w)$, is defined to be the sum of $S(G)$ and the diagonal matrix $\operatorname{diag}(w(v_1), \dots, w(v_n))$. In other words, if $s_{i,j}$ is the $(i, j)$-th entry of $S(G, w)$, then

$$s_{ij} = \begin{cases} -1, & \text{if } v_i \sim v_j, \\ 1, & \text{if } v_i \nsim v_j,\ i \neq j, \\ w(v_i), & \text{if } i = j. \end{cases}$$

For a vertex-weighted graph $(G, w)$, the complement of $(G, w)$, denoted by $\overline{(G, w)}$, is defined to be $(\overline{G}, -w)$, where $\overline{G}$ is the usual complement of the $G$ and $-w$ is the opposite weight function, i.e., $(-w)(v) = -w(v)$ for $v \in V(\overline{G}) = V(G)$. It is clear that $S(\overline{(G, w)}) = -S(G, w)$.

Let $M$ be a matrix and $\alpha, \beta$ be two row vectors of $M$. We say $\alpha$ and $\beta$ are essentially the same if $\alpha = \pm\beta$; and are essentially different if $\alpha \neq \pm\beta$. Let $\operatorname{edr} M$ denote the number of essentially different rows of $M$. Clearly, $\operatorname{edr} M \geq \operatorname{rank} M$ for any matrix $M$.

**Example 2.1** Consider the matrix

$$M = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & \frac{1}{2} & -1 \\ -1 & 1 & -1 & \frac{1}{2} \end{pmatrix}.$$

Let $\xi_i$ denote the $i$-th row of $M$. It is easy to see that $\operatorname{edr} M = 3$ as $\xi_i \neq \pm\xi_j$ for any distinct $i, j \in \{1, 2, 3, 4\}$ unless $\{i, j\} = \{1, 2\}$.

We note that the above matrix $M$ is the Seidel matrix of $C_4$ with some evident weight function. Direct calculation shows that $\operatorname{rank} M = 3$ and hence $\operatorname{rank} M = \operatorname{edr} M$ for this case. The key result of this section is to show that this is not an accident.

**Theorem 2.2** *Let $(G, w)$ be any vertex-weighted cograph with at least two vertices. Then*

$$\operatorname{rank} S(G, w) = \operatorname{edr} S(G, w).$$

Before proving Theorem 2.2, we show that Theorem 1.1 is an easy consequence of Theorem 2.2.

**Proof of Theorem 1.1** Let $G$ be a cograph of order $n$ with $n \geq 2$. Pick any $x_0 \in (-1, 1)$ and define $w(\cdot) = -x_0$ to be the constant weight function of $G$. Then we have $S(G, w) = S(G) - x_0 I$. Note that each off-diagonal entry of $S(G, w)$ is $\pm 1$ whereas each diagonal entry is neither $+1$ nor $-1$. Thus, all rows of $S(G, w)$ are essentially different, i.e., $\operatorname{edr} S(G, w) = n$. It follows from Theorem 2.2 that $\operatorname{rank} S(G, w) = n$. This means $x_0$ is not an eigenvalue of $S(G)$. Thus, $G$ has no eigenvalue in $(-1, 1)$ by the arbitrariness of $x_0$. □

Now we turn to the proof of Theorem 2.2. We begin with some necessary notions. For a vertex $v$ in $G$, we use $N(v)$ (resp., $N[v]$) to denote the open (resp., closed) neighbor of $v$ in $G$, i.e., $N(v) = \{u \in V(G) \colon u \sim v\}$ and $N[v] = N(v) \cup \{v\}$. Two vertices $u$ and $v$ in a graph $G$ are duplicates (resp., coduplicates) if $N(u) = N(v)$ (resp., $N[u] = N[v]$). We call $u$ and $v$ siblings if they are either duplicates or coduplicates. We note that two vertices are duplicates in $G$ if and only if they are coduplicates in $\overline{G}$.

The proof of Theorem 2.2 relies on the following characterization of cographs.

**Proposition 2.3** ([19, Corollary 4.2]) *A graph $G$ is a cograph if and only if every induced subgraph of $G$ with at least two vertices has siblings.*

We also need a simple inequality.

**Lemma 2.4** *Let $a, b \in [-1, 1]$, not both equal to $-1$. Then $-1 \leq (ab - 1)/(a + b + 2) < 1$ with equality holding if and only if $a = -1$ or $b = -1$.*

**Proof** Clearly, if $a = -1$ or $b = -1$, then $(ab - 1)/(a + b + 2) = -1$. Next we assume $a, b \in (-1, 1]$. Then $(ab - 1) + (a + b + 2) = (1 + a)(1 + b) > 0$ and $(ab - 1) - (a + b + 2) = (1 - a)(1 - b) - 4 < 0$.

This means $-(a+b+2) < ab-1 < a+b+2$, i.e., $-1 < (ab-1)/(a+b+2) < 1$. This proves the lemma. $\square$

Note that for any matrix $M$, we clearly have $\operatorname{rank}(-M) = \operatorname{rank}(M)$ and $\operatorname{edr}(-M) = \operatorname{edr}(M)$. Since $S(\overline{(G,w)}) = -S(G,w)$, the following lemma is immediate.

**Lemma 2.5** *For any vertex-weighted graph* $(G,w)$, *we have* $\operatorname{rank} S(\overline{(G,w)}) = \operatorname{rank} S(G,w)$ *and* $\operatorname{edr} S(\overline{(G,w)}) = \operatorname{edr} S(G,w)$.

**Proof of Theorem 2.2** Let $n = |V(G)|$. We prove the theorem by induction on $n$. If $n = 2$, then $G$ is either the complete graph $K_2$ or the empty graph $2K_1$. According to Lemma 2.5, we may assume $G = K_2$ since otherwise we can deal with the complement of $(G,w)$. Then, we have

$$S(K_2,w) = \begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix},$$

where the weights $a, b \in [-1,1]$. It is easy to see

$$\operatorname{rank} S(K_2,w) = \operatorname{edr} S(K_2,w) = \begin{cases} 1, & \text{if } a=b=1 \text{ or } a=b=-1, \\ 2, & \text{otherwise.} \end{cases}$$

This indicates that Theorem 2.2 holds for $n = 2$. Now we assume that Theorem 2.2 holds for $n = k$, where $k \geq 2$, and we proceed to check it for $n = k+1$. By Proposition 2.3, $G$ has a pair of siblings, say, $u$ and $v$. By Lemma 2.5, we may safely assume that $u$ and $v$ are coduplicates by taking the complement of $(G,w)$ if necessary. We assume further that $w(u) \leq w(v)$. Let $u$ and $v$ correspond to the last two rows of the Seidel matrix of $(G,w)$. Then $S(G,w)$ has the following form:

$$S(G,w) = \left(\begin{array}{c|cc} T & \xi & \xi \\ \hline \xi^{\mathrm{T}} & w(u) & -1 \\ \xi^{\mathrm{T}} & -1 & w(v) \end{array}\right)_{(k+1)\times(k+1)},$$

where $T$ is a matrix of order $k-1$, whereas $\xi$ is a $(k-1)$-dimensional column vector with entries in $\{-1,1\}$. We consider the following two cases:

Case 1. $w(u) = w(v) = -1$.

In this case, we find that the last two rows (and hence the last two columns) of $S(G,w)$ are equal. Let

$$M = \left(\begin{array}{c|c} T & \xi \\ \hline \xi^{\mathrm{T}} & w(u) \end{array}\right)_{k\times k}$$

be the matrix obtained from $S(G,w)$ by removing the last row and the last column. It is easy to see that $\operatorname{rank} S(G,w) = \operatorname{rank} M$ and $\operatorname{edr} S(G,w) = \operatorname{edr} M$. Let $G_v$ be the graph obtained from $G$ by deleting the vertex $v$, and $w_v$ be the restriction of $w$ on $V(G_v)$. Note that $S(G_v,w_v) = M$. By induction hypothesis, we have $\operatorname{rank} S(G_v,w_v) = \operatorname{edr} S(G_v,w_v)$. Thus, $\operatorname{rank} S(G,w) = \operatorname{edr} S(G,w)$, as desired.

Case 2. $w(u) \neq -1$ or $w(v) \neq -1$.

Recall that $w(u), w(v) \in [-1,1]$. Noting that we have assumed that $w(u) \le w(v)$, we must have $w(v) \neq -1$ since otherwise we would have $w(u) = w(v) = -1$, a contradiction. As $w(v) \neq -1$, it is easy to see that

$$\operatorname{edr}\left(\begin{array}{c|cc} T & \xi & \xi \\ \hline \xi^{\mathrm{T}} & w(u) & -1 \\ \xi^{\mathrm{T}} & -1 & w(v) \end{array}\right) = \operatorname{edr}\left(\begin{array}{c|cc} T & \xi & \xi \\ \hline \xi^{\mathrm{T}} & w(u) & -1 \end{array}\right) + 1. \tag{2.1}$$

For $x \in [-1,1]$, define matrices

$$N_1 = \left(\begin{array}{c|c} T & \xi \end{array}\right), N_2 = \left(\begin{array}{c|cc} T & \xi & \xi \end{array}\right), M_x = \left(\begin{array}{c|c} T & \xi \\ \hline \xi^{\mathrm{T}} & x \end{array}\right) \text{ and } \widetilde{M}_x = \left(\begin{array}{c|cc} T & \xi & \xi \\ \hline \xi^{\mathrm{T}} & x & -1 \end{array}\right).$$

We claim further that

$$\operatorname{edr} \widetilde{M}_x = \begin{cases} \operatorname{edr} M_{-1}, & \text{if } x = -1, \\ \operatorname{edr} N_1 + 1, & \text{if } x \neq -1. \end{cases} \tag{2.2}$$

Indeed, if $x = -1$, then the last two columns of $\widetilde{M}_x$ are the same and hence $\operatorname{edr} \widetilde{M}_x = \operatorname{edr} M_x$. If $x \neq -1$, then one easily sees that in $\widetilde{M}_x$ the last row is essentially different from any other row. Thus,

$$\operatorname{edr} \widetilde{M}_x = \operatorname{edr} N_2 + 1 = \operatorname{edr} N_1 + 1.$$

This proves (2.2). In order to find the rank of $S(G,w)$, we use a series of elementary transformations:

$$\begin{aligned} S(G,w) &= \left(\begin{array}{c|cc} T & \xi & \xi \\ \hline \xi^{\mathrm{T}} & w(u) & -1 \\ \xi^{\mathrm{T}} & -1 & w(v) \end{array}\right) \xrightarrow{r_n - r_{n-1}} \left(\begin{array}{c|cc} T & \xi & \xi \\ \hline \xi^{\mathrm{T}} & w(u) & -1 \\ 0 & -w(u)-1 & w(v)+1 \end{array}\right) \\ &\xrightarrow{c_n - c_{n-1}} \left(\begin{array}{c|cc} T & \xi & 0 \\ \hline \xi^{\mathrm{T}} & w(u) & -w(u)-1 \\ 0 & -w(u)-1 & w(u)+w(v)+2 \end{array}\right) \\ &\xrightarrow{c_{n-1} + \frac{w(u)+1}{w(u)+w(v)+2} c_n} \left(\begin{array}{c|cc} T & \xi & 0 \\ \hline \xi^{\mathrm{T}} & \frac{w(u)w(v)-1}{w(u)+w(v)+2} & -w(u)-1 \\ 0 & 0 & w(u)+w(v)+2 \end{array}\right). \end{aligned} \tag{2.3}$$

Let

$$c = \frac{w(u)w(v)-1}{w(u)+w(v)+2}.$$

By Lemma 2.4, we have $c \in [-1,1)$. As $w(u)+w(v)+2$ is the only nonzero entry in the last row of the last matrix in Eq. (2.3), we obtain

$$\operatorname{rank} S(G,w) = \operatorname{rank} M_c + 1. \tag{2.4}$$

Note that $M_c$ is a Seidel matrix of some vertex-weighted cograph of order $k$. Therefore, by induction hypothesis, we have $\operatorname{rank} M_c = \operatorname{edr} M_c$ and hence we can rewrite Eq. (2.4) as

$$\operatorname{rank} S(G,w) = \operatorname{edr} M_c + 1. \tag{2.5}$$

Comparing Eq. (2.1) with Eq. (2.5), in order to establish the equality $\operatorname{rank} S(G,w) = \operatorname{edr} S(G,w)$, it suffices to show that $\operatorname{edr} \widetilde{M}_{w(u)} = \operatorname{edr} M_c$. By Eq. (2.2), we are done if we can show

$$\operatorname{edr} M_c = \begin{cases} \operatorname{edr} M_{-1}, & \text{if } w(u) = -1, \\ \operatorname{edr} N_1 + 1, & \text{if } w(u) \neq -1. \end{cases} \tag{2.6}$$

Indeed, if $w(u) = -1$, then $c = -1$ and hence Eq. (2.6) trivially holds for this case. Now assume $w(u) \neq -1$. Recall that $w(v) \neq -1$. Thus, by Lemma 2.4, we obtain $c \in (-1,1)$. Noting that each entry of $\xi$ is $\pm 1$, it is easy to see that in the matrix

$$M_c = \left(\begin{array}{c|c} T & \xi \\ \hline \xi^{\mathrm{T}} & c \end{array}\right),$$

the last row is essentially different from any other row. It follows that

$$\operatorname{edr} M_c = \operatorname{edr} N_1 + 1.$$

This proves Eq. (2.6).

Since in either case we have established the equality $\operatorname{rank} S(G,w) = \operatorname{edr} S(G,w)$, the proof of Theorem 2.2 is completed by induction. □

## 3. Optimality of the interval $(-1, 1)$

Let $n \geq 3$ and we define a family of cographs

$$H_n = \begin{cases} \ell P_2 \cup P_1, & \text{if } n = 2\ell + 1, \\ P_3 \cup (\ell - 1) P_2 \cup P_1, & \text{if } n = 2\ell + 2. \end{cases}$$

For a cograph $G$, we use $\mu_{\max}^{-}(G)$ to denote the maximal negative Seidel eigenvalue of $G$, in the interval $(-\infty, -1)$. Similarly, we use $\mu_{\min}^{+}(G)$ to denote the minimal positive Seidel eigenvalue of $G$ in $(1, +\infty)$. We make the convention that $\mu_{\max}^{-}(G) = -\infty$ (resp., $\mu_{\min}^{+}(G) = +\infty$) if $G$ has no Seidel eigenvalue in $(-\infty, -1)$ (resp., in $(1, +\infty)$). The main result of this section is to show the following.

**Proposition 3.1** *We have* $\lim_{n\to\infty} \mu_{\max}^{-}(H_n) = -1$.

Let $G$ be a graph of order $n$ and $A = A(G)$ be its adjacency matrix. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_r$ be all the distinct eigenvalues of $A$ with multiplicity $m_1, m_2, \ldots, m_r$, respectively. Let

$$P_i = (\eta_1^{(i)})(\eta_1^{(i)})^{\mathrm{T}} + \cdots + (\eta_{m_i}^{(i)})(\eta_{m_i}^{(i)})^{\mathrm{T}}, \quad i = 1, 2, \ldots, r,$$

where $\eta_1^{(i)}, \ldots, \eta_{m_i}^{(i)}$ constitute an orthonormal basis of the eigenspace corresponding to $\lambda_i$. The main angle [20] of $G$ associated with $\lambda_i$ is the number

$$\beta_i = \frac{||P_i e||}{\sqrt{n}}, \tag{3.1}$$

where $e$ is the all-ones vector in $\mathbb{R}^n$. For any square matrix $M$, we write $\chi(M;x) = \det(xI - M)$, the characteristic polynomial of $M$. The following relation between the Seidel characteristic polynomial $\chi(S(G);x)$ and the ordinary characteristic polynomial $\chi(A(G);x)$ is useful.

**Lemma 3.2** ([20, Proposition 2.1.4]) *For any graph $G$ with $n$ vertices,*

$$\chi(S(G);x) = (-2)^n \chi(A(G); -\frac{x+1}{2})\Big(1 - n\sum_{i=1}^{r} \frac{\beta_i^2}{x+1+2\lambda_i}\Big).$$

We are ready to show Proposition 3.1. We first consider the odd case which we rewrite as a new proposition.

**Proposition 3.3** *We have* $\lim_{\ell\to\infty} \mu^-_{\max}(\ell P_2 \cup P_1) = -1$.

**Proof** Let $A_{2\ell+1} = A(\ell P_2 \cup P_1)$. Clearly, $A_{2\ell+1}$ is the block diagonal matrix

$$A_{2\ell+1} = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0\right).$$

Thus $\chi(A_{2\ell+1}; x) = (x^2-1)^\ell x$. By routine calculation, we obtain all the eigenvalues and the corresponding eigenvectors as illustrated in Table 1. Normalizing each vector in Table 1, we obtain an orthonormal basis for each eigenvalue $\lambda_i$. Then the main angles are easily computed using Eq. (3.1) and are recorded in Table 2.

| $i$ | Eigenvalue $\lambda_i$ | Multiplicity $m_i$ | Basis for eigenspace |
|---|---|---|---|
| 1 | 1 | $\ell$ | $(\underbrace{0,\ldots,0}_{2k-2},1,1,\underbrace{0,\ldots,0}_{2\ell-2k+1})^{\mathrm{T}}, k=1,2,\ldots,\ell$ |
| 2 | $-1$ | $\ell$ | $(\underbrace{0,\ldots,0}_{2k-2},1,-1,\underbrace{0,\ldots,0}_{2\ell-2k+1})^{\mathrm{T}}, k=1,2,\ldots,\ell$ |
| 3 | 0 | 1 | $(\underbrace{0,\ldots,0}_{2\ell},1)^{\mathrm{T}}$ |

Table 1 Eigensystems of $A(\ell P_2 \cup P_1)$

| $i$ | Eigenvalue $\lambda_i$ | Orthonormal basis for eigenspace | Main angle $\beta_i$ |
|---|---|---|---|
| 1 | 1 | $(\underbrace{0,\ldots,0}_{2k-2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\underbrace{0,\ldots,0}_{2\ell-2k+1})^{\mathrm{T}}, k=1,2,\ldots,\ell$ | $\frac{\sqrt{2\ell}}{\sqrt{2\ell+1}}$ |
| 2 | $-1$ | $(\underbrace{0,\ldots,0}_{2k-2},\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},\underbrace{0,\ldots,0}_{2\ell-2k+1})^{\mathrm{T}}, k=1,2,\ldots,\ell$ | 0 |
| 3 | 0 | $(\underbrace{0,\ldots,0}_{2\ell},1)^{\mathrm{T}}$ | $\frac{1}{\sqrt{2\ell+1}}$ |

Table 2 Main angles of $\ell P_2 \cup P_1$

Let $S_{2\ell+1} = S(\ell P_2 \cup P_1)$. Then it follows from Lemma 3.2 that

$$\begin{aligned}\chi(S_{2\ell+1};x) &= (-2)^{2\ell+1}\chi(A_{2\ell+1}; -\frac{x+1}{2})\Big(1-(2\ell+1)\sum_{i=1}^{3}\frac{\beta_i^2}{x+1+2\lambda_i}\Big) \\ &= (-2)^{2\ell+1}((-\frac{x+1}{2})^2-1)^\ell(-\frac{x+1}{2})(1-(2\ell+1)(\frac{\frac{2\ell}{2\ell+1}}{x+3}+\frac{\frac{1}{2\ell+1}}{x+1}))\end{aligned}$$

$$= (x+3)^{\ell-1}(x-1)^{\ell}(x^2-(2\ell-3)x-2\ell). \tag{3.2}$$

Using Eq. (3.2), all distinct eigenvalues of $S_{2\ell+1}$ are precisely

$$\mu_1=-3,\ \mu_2=1,\ \mu_3=\frac{(2\ell-3)+\sqrt{(2\ell-3)^2+8\ell}}{2},\ \mu_4=\frac{(2\ell-3)-\sqrt{(2\ell-3)^2+8\ell}}{2}.$$

Let $\phi(x)=x^2-(2\ell-3)x-2\ell$. Noting that $\phi(-1)=-2<0$ and $\phi(-3)=4\ell>0$, we know that $\phi(x)$ has a root in $(-3,-1)$ by the zero point theorem. This means $\mu_4\in(-3,-1)$ and hence $\mu^-_{\max}(\ell P_2\cup P_1)=\mu_4$. It follows that

$$\begin{aligned}\lim_{\ell\to\infty}\mu^-_{\max}(\ell P_2\cup P_1)&=\lim_{\ell\to\infty}\frac{(2\ell-3)-\sqrt{(2\ell-3)^2+8\ell}}{2}\\ &=\lim_{\ell\to\infty}\frac{(2\ell-3)^2-((2\ell-3)^2+8\ell)}{2((2\ell-3)+\sqrt{(2\ell-3)^2+8\ell})}\\ &=\lim_{\ell\to\infty}\frac{-4\ell}{(2\ell-3)+\sqrt{(2\ell-3)^2+8\ell}}\\ &=\lim_{\ell\to\infty}\frac{-4}{(2-\frac{3}{\ell})+\sqrt{4-\frac{4}{\ell}+\frac{9}{\ell^2}}}\\ &=-1.\end{aligned}$$

This completes the proof. □

Next we consider the even case for Proposition 3.1.

**Proposition 3.4** *We have* $\lim_{\ell\to\infty}\mu^-_{\max}(P_3\cup(\ell-1)P_2\cup P_1)=-1$.

**Proof** Let $A_{2\ell+2}=A(P_3\cup(\ell-1)P_2\cup P_1)$. Then

$$A_{2\ell+2}=\mathrm{diag}\left(\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\dots,\begin{pmatrix}0&1\\1&0\end{pmatrix},0\right).$$

Thus

$$\chi(A_{2\ell+2};x)=x^2(x^2-1)^{\ell-1}(x^2-2).$$

Using similar computations as in the proof of Proposition 3.3, we can determine the main angles for all eigenvalues; see Table 3.

Let $S_{2\ell+2}$ be the Seidel matrix of $P_3\cup(\ell-1)P_2\cup P_1$. By Lemma 3.2 and using routine calculation, we obtain

$$\chi(S_{2\ell+2};x)=(x+1)(x-1)^{\ell-1}(x+3)^{\ell-2}(x^4+(4-2\ell)x^3-(6\ell+2)x^2+(10\ell-20)x+14\ell+1).$$

Let $\phi(x)=x^4+(4-2\ell)x^3-(6\ell+2)x^2+(10\ell-20)x+14\ell+1$. We claim that $\phi(x)$ has a root in $(-1-\frac{1}{\sqrt{\ell}},-1)$ when $\ell\geq 2$. Note that $\phi(-1)=16>0$ and

$$\phi(-1-\frac{1}{\sqrt{\ell}})=16+\frac{1}{\ell^2}-\frac{8}{\ell}+\frac{10}{\sqrt{\ell}}-16\sqrt{\ell}.$$

If $\ell = 2$, then $\phi(-1 - \frac{1}{\sqrt{\ell}}) = -3.306\cdots < 0$. If $\ell \geq 3$, then

$$\phi(-1 - \frac{1}{\sqrt{\ell}}) < 16 + \frac{1}{\ell^2} + \frac{10}{\sqrt{\ell}} - 16\sqrt{\ell} \leq 16 + \frac{1}{3^2} + \frac{10}{\sqrt{3}} - 16\sqrt{3} = -5.828\cdots < 0.$$

This means $\phi(-1 - \frac{1}{\sqrt{\ell}}) < 0$ holds for $\ell \geq 2$. Thus, $\phi(x)$ has a root in $(-1 - \frac{1}{\sqrt{\ell}}, -1)$ and hence

$$-1 - \frac{1}{\sqrt{\ell}} < \mu^-_{\max}(P_3 \cup (\ell - 1)P_2 \cup P_1) < -1 \quad \text{for} \quad \ell \geq 2.$$

| $i$ | Eigenvalue | Orthonormal basis for eigenspace | Main angle $\beta_i$ |
|---|---|---|---|
| 1 | 1 | $(\underbrace{0,\ldots,0}_{2k+1}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \underbrace{0,\ldots,0}_{2\ell-2k-1})^{\mathrm{T}}, k = 1, 2, \ldots, \ell - 1$ | $\frac{\sqrt{2\ell-2}}{\sqrt{2\ell+2}}$ |
| 2 | $-1$ | $(\underbrace{0,\ldots,0}_{2k+1}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \underbrace{0,\ldots,0}_{2\ell-2k-1})^{\mathrm{T}}, k = 1, 2, \ldots, \ell - 1$ | 0 |
| 3 | 0 | $(\underbrace{0,\ldots,0}_{2\ell+1}, 1)^{\mathrm{T}}, (\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, \underbrace{0,\ldots,0}_{2\ell-1})^{\mathrm{T}}$ | $\frac{1}{\sqrt{2\ell+2}}$ |
| 4 | $-\sqrt{2}$ | $(\frac{1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}, \underbrace{0,\ldots,0}_{2\ell-1})^{\mathrm{T}}$ | $\frac{\sqrt{1.5-\sqrt{2}}}{\sqrt{2\ell+2}}$ |
| 5 | $\sqrt{2}$ | $(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, \underbrace{0,\ldots,0}_{2\ell-1})^{\mathrm{T}}$ | $\frac{\sqrt{1.5+\sqrt{2}}}{\sqrt{2\ell+2}}$ |

Table 3 Main angles of $P_3 \cup (\ell - 1)P_2 \cup P_1$

Noting that $\lim_{\ell\to\infty}(-1 - \frac{1}{\sqrt{\ell}}) = -1$, Proposition 3.4 follows by the squeeze theorem. □

**Proof of Theorem 1.2** By Proposition 3.1, we know that $H_n$ has an eigenvalue in $(-1-\epsilon, -1)$ for sufficiently large $n$. Note that any graph $G$ and its complement $\overline{G}$ have opposite Seidel eigenvalues. We see that the graph $\overline{H_n}$ has an eigenvalue in $(1, 1 + \epsilon)$ for sufficiently large $n$. This completes the proof of Theorem 1.2. □

We end this paper by proposing the following conjecture.

**Conjecture 3.5** *Let $n \geq 3$. Then $\mu^-_{\max}(H_n) \geq \mu^-_{\max}(G)$ (or equivalently $\mu^+_{\min}(\overline{H_n}) \leq \mu^+_{\min}(G)$) for any cograph $G$ of order $n$.*

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