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Outer-Independent Roman Domination on Cartesian Product of Paths

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Abstract Outer-independent Roman domination on graphs originates from the defensive strategy of Ancient Rome, which is that if any city without an army is attacked, a neighboring city with two armies could mobilize an army to support it and any two cities that have no army cannot be adjacent. The outer-independent Roman domination on graphs is an attractive topic in graph theory, and the definition is described as follows. Given a graph $G=(V,E)$, a function $f: V(G)\rightarrow\{0,1,2\}$ is an outer-independent Roman dominating function (OIRDF) if f satisfies that every vertex $v\in V$ with $f(v)=0$ has at least one adjacent vertex $u\in N(v)$ with $f(u)=2$, where $N(v)$ is the open neighborhood of v, and the set $V_0=\{v|f(v)=0\}$ is an independent set. The weight of an OIRDF f is $w(f)=\sum_{v\in V}f(v)$. The value of $\min_f w(f)$ is the outer-independent Roman domination number of G, denoted as $\gamma_{oiR}(G)$. This paper is devoted to the study of the outer-independent Roman domination number of the Cartesian product of paths $P_n\Box P_m$. With the help of computer, we find some recursive OIRDFs and then we present an upper bound of $\gamma_{oiR}(P_n\Box P_m)$. Furthermore, we prove the lower bound of $\gamma_{oiR}(P_n\Box P_m)$ $(n\le 3)$ is equal to the upper bound. Hence, we achieve the exact value of $\gamma_{oiR}(P_n\Box P_m)$ for $n\le 3$ and the upper bound of $\gamma_{oiR}(P_n\Box P_m)$ for $n\ge 4$.

Keywords Roman domination; outer-independent Roman domination; Cartesian product graphs; paths

MR(2020) Subject Classification 05C69

1. Introduction

Outer-independent Roman domination on a graph is an imaginative problem in graph theory which is related to the defence issue of the Roman Empire. It was decreed that no more than two armies could be stationed in a city. If at least one army was stationed in a city, it was considered safe. If there is no army stationed in a city, there must be at least one city with two armies stationed around the city, so that one army can be deployed at any time to guard the city. In addition, a city is considered more vulnerable if it does not have an army around it to protect it. Therefore, the best case for a location without troops is to be completely surrounded by the locations with troops stationed, that is, it has no neighbor without troops.

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The definition of outer-independent Roman domination was given by Ahangar et al. [1] in 2017. For a graph $G=(V,E)$, let $f: V \to \{0,1,2\}$ be a function, and $V_i = \{v \in V(G) | f(v) = i\}$ for $i \in \{0,1,2\}$. We can write $f=(V_0,V_1,V_2)$. A function $f: V \to \{0,1,2\}$ is an outer-independent Roman dominating function (OIRDF) on G if every vertex $u \in V$ for which $f(u)=0$ is adjacent to at least one vertex v satisfying $f(v)=2$ and V_0 is an independent set. The weight of an OIRDF is $w(f)=\sum_{v\in V(G)} f(v)$. The outer-independent Roman domination number $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on G. An OIRDF of weight $\gamma_{oiR}(G)$ is called a $\gamma_{oiR}(G)$-function.

Since the definition of outer-independent Roman domination was presented, many researchers have studied this topic. Poureidi et al. [2] proposed an algorithm to compute $\gamma_{oiR}(G)$ in $O(|V|)$ time. Martínez et al. [3] obtained some bounds on $\gamma_{oiR}(G)$ in terms of other parameters. Nazari-Moghaddam et al. [4] provided a constructive characterization of trees T whose outer-independent Roman domination number is equal to its Roman domination number. Gao et al. [5] studied outer independent Roman domination number of torus graphs. There are also some researches related to outer-independent Roman domination, such as outer-independent total Roman domination [6,7], outer-independent double Roman domination [8,9], outer-independent signed double Roman domination [10], and so on.

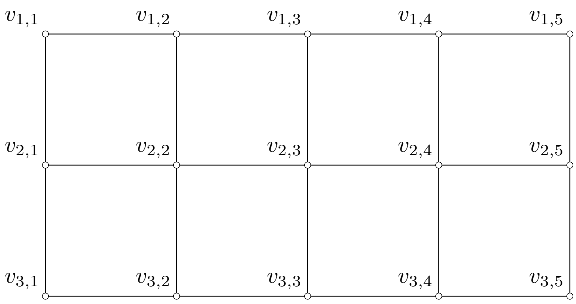

Figure 1 Graph $P_3 \Box P_5$

In this paper, we study the outer-independent Roman domination on the Cartesian product of paths $P_n \Box P_m$. We design an effective algorithm to help us construct some OIRDFs. Upon these functions, we get some sharp upper bounds on $\gamma_{oiR}(P_n \Box P_m)$. For $P_1 \Box P_m$, $P_2 \Box P_m$ and $P_3 \Box P_m$, we prove the lower bound is equal to the upper bound. Thus, we achieve the exact values of $\gamma_{oiR}(P_1 \Box P_m)$, $\gamma_{oiR}(P_2 \Box P_m)$ and $\gamma_{oiR}(P_3 \Box P_m)$ and present an upper bound on $\gamma_{oiR}(P_n \Box P_m)$ for $n \geq 4$.

The Cartesian product graph of paths is denoted as $G = P_n \Box P_m$, and the vertex set is $V(G) = \{v_{i,j} | 1 \leq i \leq n, 1 \leq j \leq m\}$. $G = P_n \Box P_m$ is a grid graph and is an important network topology structure graph. Figure 1 shows graph $P_3 \Box P_5$. Let $f: V \to \{0,1,2\}$ be an OIRDF on

$P_n\Box P_m$. Then we denote f as follows

$$f=\begin{pmatrix} f(v_{1,1}) & f(v_{1,2}) & \cdots & f(v_{1,m}) \\ f(v_{2,1}) & f(v_{2,2}) & \cdots & f(v_{2,m}) \\ \vdots & \vdots & \ddots & \vdots \\ f(v_{n,1}) & f(v_{n,2}) & \cdots & f(v_{n,m}) \end{pmatrix}.$$

2. The outer-independent Roman domination number of $P_n\Box P_m$ for $n\leq 3$

In this section, we achieve the exact values of the outer-independent Roman domination number of $P_1\Box P_m$, $P_2\Box P_m$ and $P_3\Box P_m$.

Theorem 2.1 *For any integers $m\geq 2$, $\gamma_{oiR}(P_1\Box P_m)=\lceil\frac{3m-1}{4}\rceil$.*

Proof Since $P_1\Box P_m\cong P_m$, readers can read the proof of Observation 2.1 in reference [1]. □

Theorem 2.2 *For any integers $m\geq 2$, $\gamma_{oiR}(P_2\Box P_m)=\lceil\frac{4m}{3}\rceil$.*

Proof With the help of computer, we construct some recursive OIRDFs for $P_2\Box P_m$ with the desired weights as shown below. Then, we get $\gamma_{oiR}(P_2\Box P_m)\leq\lceil\frac{4m}{3}\rceil$.

$$m\equiv 0 \pmod 6),\quad f=\begin{pmatrix} 020101\cdots 020101 \\ 101020\cdots 101020 \end{pmatrix};$$

$$m\equiv 1 \pmod 6),\quad f=\begin{pmatrix} 020101\cdots 020101 & 1 \\ 101020\cdots 101020 & 1 \end{pmatrix};$$

$$m\equiv 2 \pmod 6),\quad f=\begin{pmatrix} 020101\cdots 020101 & 02 \\ 101020\cdots 101020 & 10 \end{pmatrix};$$

$$m\equiv 3 \pmod 6),\quad f=\begin{pmatrix} 020101\cdots 020101 & 020 \\ 101020\cdots 101020 & 101 \end{pmatrix};$$

$$m\equiv 4 \pmod 6),\quad f=\begin{pmatrix} 020101\cdots 020101 & 0201 \\ 101020\cdots 101020 & 1011 \end{pmatrix};$$

$$m\equiv 5 \pmod 6),\quad f=\begin{pmatrix} 020101\cdots 020101 & 02010 \\ 101020\cdots 101020 & 10102 \end{pmatrix}.$$

Next, we prove $\gamma_{oiR}(P_2\Box P_m)\geq\lceil\frac{4m}{3}\rceil$.

Let f be an arbitrary γ_{oiR}-function of $P_2\Box P_m$. The vertex set of $P_2\Box P_m$ is $V=\{v_{1,1},v_{1,2},\ldots,v_{1,m},v_{2,1},v_{2,2},\ldots,v_{2,m}\}$. We denote $V_j=\{v_{1,j},v_{2,j}\}$ and $f_j=f(v_{1,j})+f(v_{2,j})$ $(1\leq j\leq m)$.

Since every vertex $v\in V$ with $f(v)=0$ has at least one neighbor u with $f(u)=2$ and V_0 is an independent set, we can get the following facts.

(i) $f_j=f(v_{1,j})+f(v_{2,j})\geq 1$ for $1\leq j\leq m$.

(ii) If $f_1=1$ $(f_m=1)$, then $f_2\geq 2$ $(f_{m-1}\geq 2)$.

(iii) If $f_j=1$ and $f_{j+1}=1$, then $f_{j-1}\geq 2$ and $f_{j+2}\geq 2$.

We use the following algorithm-like process to group f_j $(1 \le j \le m)$ into three categories.

Algorithm 1 Grouping for $P_2 \Box P_m$

Let $D[j] = 0$ for $1 \le j \le m$, and $t_i = 0$ for $0 \le i \le 2$.

For j from 1 to m with $f_j \ge 2 \wedge D[j] = 0$, do:

  If $f_{j-1} = 1 \wedge f_{j+1} = 1$,

    $t_0 = t_0 + 1$, $B_{0,t_0} = \{V_{j-1}, V_j, V_{j+1}\}$, $D[j-1] = D[j] = D[j+1] = 1$.

    Note: $f(B_{0,t_0}) = f_{j-1} + f_j + f_{j+1} = 4 \ge |B_{0,t_0}| \times \frac{4}{3}$.

  Else if $f_{j-1} = 1$,

    $t_1 = t_1 + 1$, $B_{1,t_1} = \{V_{j-1}, V_j\}$, $D[j-1] = D[j] = 1$.

    Note: $f(B_{1,t_1}) = f_{j-1} + f_j = 3 \ge |B_{1,t_1}| \times \frac{4}{3}$.

  Else if $f_{j+1} = 1$,

    $t_1 = t_1 + 1$, $B_{1,t_1} = \{V_j, V_{j+1}\}$, $D[j] = D[j+1] = 1$.

    Note: $f(B_{1,t_1}) = f_j + f_{j+1} = 3 \ge |B_{1,t_1}| \times \frac{4}{3}$.

  Else,

    $t_2 = t_2 + 1$, $B_{2,t_2} = \{V_j\}$, $D[j] = 1$.

    Note: $f(B_{2,t_2}) = f_j = 2 \ge |B_{2,t_2}| \times \frac{4}{3}$.

By the grouping process Algorithm 1, we have

$$\begin{aligned}\sum_{j=1}^{m} f_j &\ge \sum_{j=1}^{t_0} f(B_{0,j}) + \sum_{j=1}^{t_1} f(B_{1,j}) + \sum_{j=1}^{t_2} f(B_{2,j}) \\ &\ge \frac{4}{3}\Big(\sum_{j=1}^{t_0} |B_{0,j}| + \sum_{j=1}^{t_1} |B_{1,j}| + \sum_{j=1}^{t_2} |B_{2,j}|\Big) = \frac{4m}{3}.\end{aligned}$$

Thus, $\gamma_{oiR}(P_2 \Box P_m) \ge \lceil \frac{4m}{3} \rceil$. $\Box$

Theorem 2.3 *For any integers $m \ge 2$, $\gamma_{oiR}(P_3 \Box P_m) = 2m$.*

Proof We construct OIRDFs for $P_3 \Box P_m$ as follows. Then, $\gamma_{oiR}(P_3 \Box P_m) \le 2m$.

$$m \equiv 0 \pmod 2,\quad f = \begin{pmatrix} 01\cdots01 \\ 20\cdots20 \\ 01\cdots01 \end{pmatrix};\quad m \equiv 1 \pmod 2,\quad f = \begin{pmatrix} 01\cdots01 & 0 \\ 20\cdots20 & 2 \\ 01\cdots01 & 0 \end{pmatrix}.$$

Next, we prove $\gamma_{oiR}(P_3 \Box P_m) \ge 2m$.

Let f be an arbitrary γ_{oiR}-function of $P_3 \Box P_m$. The vertex set of $P_3 \Box P_m$ is

$$V = \{v_{1,1}, v_{1,2}, \ldots, v_{1,m}, v_{2,1}, v_{2,2}, \ldots, v_{2,m}, v_{3,1}, v_{3,2}, \ldots, v_{3,m}\}.$$

We denote $V_j = \{v_{1,j}, v_{2,j}, v_{3,j}\}$ and $f_j = f(v_{1,j}) + f(v_{2,j}) + f(v_{3,j})$ $(1 \le j \le m)$.

Since every vertex $v \in V$ with $f(v) = 0$ has at least one neighbor u with $f(u) = 2$ and V_0 is an independent set, we can get the following facts.

(i) $f_j = f(v_{1,j}) + f(v_{2,j}) + f(v_{3,j}) \ge 1$ for $1 \le j \le m$.

(ii) If $f_1 = 1$ $(f_m = 1)$, then $f_2 \ge 4$ $(f_{m-1} \ge 4)$, as shown in Figure 2 (a).

(iii) If $f_j = 1$ ($j \geq 2$), then $f_{j-1} \geq 2$ and $f_{j+2} \geq 2$, and $f_{j-1} + f_{j+2} \geq 6$, as shown in Figure 2 (b) and (c).

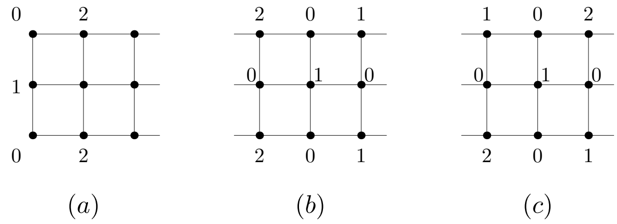

Figure 2 Skatch for $f_1 = 1$ and $f_j = 1$ ($j \geq 2$)

We use the following algorithm to group f_j ($1 \leq j \leq m$) into three categories.

Algorithm 2 Grouping for $P_3 \square P_m$

Let $D[j] = 0$ for $1 \leq j \leq m$, and $t_i = 0$ for $0 \leq i \leq 2$.

Step 1. For every j from 1 to m with $f_j \geq 4 \wedge D[j] = 0$, do:

$t_0 = t_0 + 1$, $B_{0,t_0} = \{V_j\}$, $D[j] = 1$.

If $f_{j-1} = 1 \wedge D[j-1] = 0$, then $D[j-1] = 1$, $B_{0,t_0} = B_{0,t_0} \cup \{V_{j-1}\}$.

If $f_{j+1} = 1 \wedge D[j+1] = 0$, then $D[j+1] = 1$, $B_{0,t_0} = B_{0,t_0} \cup \{V_{j+1}\}$.

Note: $f(B_{0,t_0}) \geq |B_{0,t_0}| \times 2$.

Step 2. For every j from 1 to m with $f_j = 3 \wedge D[j] = 0$, do:

$t_1 = t_1 + 1$, $B_{1,t_1} = \{V_j\}$, $D[j] = 1$.

If $f_{j+1} = 1 \wedge D[j+1] = 0$, then $D[j+1] = 1$, $B_{1,t_1} = B_{1,t_1} \cup \{V_{j+1}\}$.

Note: $f(B_{1,t_1}) \geq |B_{1,t_1}| \times 2$.

Step 3. For every j from 1 to m with $f_j = 2 \wedge D[j] = 0$, do:

$t_2 = t_2 + 1$, $B_{2,t_2} = \{V_j\}$, $D[j] = 1$.

Note: $f(B_{2,t_2}) \geq |B_{2,t_2}| \times 2$.

By the grouping process Algorithm 2, we have

$$\begin{aligned}\sum_{j=1}^{m} f_j \geq& \sum_{j=1}^{t_0} f(B_{0,j}) + \sum_{j=1}^{t_1} f(B_{1,j}) + \sum_{j=1}^{t_2} f(B_{2,j}) \\ \geq& 2(\sum_{j=1}^{t_0} |B_{0,j}| + \sum_{j=1}^{t_1} |B_{1,j}| + \sum_{j=1}^{t_2} |B_{2,j}|) = 2m.\end{aligned}$$

Thus, $\gamma_{oiR}(P_3 \square P_m) \geq 2m$. □

3. The outer-independent Roman domination number of $P_n \square P_m$ ($n, m \geq 4$)

In this section, we present an upper bound on the outer-independent Roman domination number of $P_n \square P_m$ for $n, m \geq 4$.

Theorem 3.1 *For any integers $n, m \geq 4$, $\gamma_{oiR}(P_n \square P_m) \leq \lfloor \frac{5mn+2m+2n}{8} \rfloor$.*

Proof We construct some OIRDFs for $P_n \Box P_m$ and get the upper bound.

(1) For $n \equiv 0 \pmod 4$, OIRDFs are defined as f_i^0 for $m \equiv i \pmod 4$ $(i = 0,1,2,3)$.

$$f_0^0 = \begin{pmatrix} 1020 & \cdots & 1020 \\ 0101 & \cdots & 0101 \\ 2010 & \cdots & 2010 \\ 0101 & \cdots & 0102 \\ \vdots & \ddots & \vdots \\ 1020 & \cdots & 1020 \\ 0101 & \cdots & 0101 \\ 2010 & \cdots & 2010 \\ 0102 & \cdots & 0102 \end{pmatrix}, \quad f_1^0 = \begin{pmatrix} 1020 & \cdots & 1020 & 1 \\ 0101 & \cdots & 0101 & 0 \\ 2010 & \cdots & 2010 & 2 \\ 0101 & \cdots & 0101 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 1 \\ 0101 & \cdots & 0101 & 0 \\ 2010 & \cdots & 2010 & 2 \\ 0102 & \cdots & 0102 & 0 \end{pmatrix},$$

$$f_2^0 = \begin{pmatrix} 1020 & \cdots & 1020 & 10 \\ 0101 & \cdots & 0101 & 02 \\ 2010 & \cdots & 2010 & 20 \\ 0101 & \cdots & 0101 & 01 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 10 \\ 0101 & \cdots & 0101 & 02 \\ 2010 & \cdots & 2010 & 20 \\ 0102 & \cdots & 0102 & 01 \end{pmatrix}, \quad f_3^0 = \begin{pmatrix} 1020 & \cdots & 1020 & 102 \\ 0101 & \cdots & 0101 & 010 \\ 2010 & \cdots & 2010 & 201 \\ 0101 & \cdots & 0101 & 010 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 102 \\ 0101 & \cdots & 0101 & 010 \\ 2010 & \cdots & 2010 & 201 \\ 0102 & \cdots & 0102 & 011 \end{pmatrix}.$$

Thus, $n \equiv 0 \pmod 4$,

$$\gamma_{oiR}(P_n \Box P_m) \leq \begin{cases} \frac{5mn+2m+2n-8}{8}, & m \equiv 0 \pmod 4, \\ \frac{5mn+2m+n-2}{8}, & m \equiv 1 \pmod 4, \\ \frac{5mn+2m+2n-4}{8}, & m \equiv 2 \pmod 4, \\ \frac{5mn+2m+n+2}{8}, & m \equiv 3 \pmod 4. \end{cases}$$

(2) For $n \equiv 1 \pmod 4$, OIRDFs are defined as f_i^1 for $m \equiv i \pmod 4$ $(i = 0,1,2,3)$.

$$f_0^1 = \left(\begin{array}{ccc} 1020 & \cdots & 1020 \\ 0101 & \cdots & 0101 \\ 2010 & \cdots & 2010 \\ 0101 & \cdots & 0102 \\ \vdots & \ddots & \vdots \\ 1020 & \cdots & 1020 \\ 0101 & \cdots & 0101 \\ 2010 & \cdots & 2010 \\ 0101 & \cdots & 0102 \\ \hdashline 1020 & \cdots & 1020 \end{array}\right), \quad f_1^1 = \left(\begin{array}{cccc} 1020 & \cdots & 1020 & 1 \\ 0101 & \cdots & 0101 & 0 \\ 2010 & \cdots & 2010 & 2 \\ 0101 & \cdots & 0101 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 1 \\ 0101 & \cdots & 0101 & 0 \\ 2010 & \cdots & 2010 & 2 \\ 0101 & \cdots & 0101 & 0 \\ \hdashline 1020 & \cdots & 1020 & 1 \end{array}\right),$$

$$f_2^1 = \begin{pmatrix} 1020 & \cdots & 1020 & 10 \\ 0101 & \cdots & 0101 & 02 \\ 2010 & \cdots & 2010 & 20 \\ 0101 & \cdots & 0101 & 01 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 10 \\ 0101 & \cdots & 0101 & 02 \\ 2010 & \cdots & 2010 & 20 \\ 0101 & \cdots & 0101 & 01 \\ \hdashline 1020 & \cdots & 1020 & 11 \end{pmatrix}, \quad f_3^1 = \begin{pmatrix} 1020 & \cdots & 1020 & 102 \\ 0101 & \cdots & 0101 & 010 \\ 2010 & \cdots & 2010 & 201 \\ 0101 & \cdots & 0101 & 010 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 102 \\ 0101 & \cdots & 0101 & 010 \\ 2010 & \cdots & 2010 & 201 \\ 0101 & \cdots & 0101 & 010 \\ \hdashline 1020 & \cdots & 1020 & 102 \end{pmatrix}.$$

Thus, $n \equiv 1 \pmod 4$,

$$\gamma_{oiR}(P_n \Box P_m) \leq \begin{cases} \frac{5mn+m+2n-2}{8}, & m \equiv 0 \pmod 4, \\ \frac{5mn+m+n+1}{8}, & m \equiv 1 \pmod 4, \\ \frac{5mn+m+2n+2}{8}, & m \equiv 2 \pmod 4, \\ \frac{5mn+m+n+5}{8}, & m \equiv 3 \pmod 4. \end{cases}$$

(3) For $n \equiv 2 \pmod 4$, OIRDFs are defined as f_i^2 for $m \equiv i \pmod 4$ $(i = 0, 1, 2, 3)$.

$$f_0^2 = \begin{pmatrix} 1020 & \cdots & 1020 \\ 0101 & \cdots & 0101 \\ 2010 & \cdots & 2010 \\ 0101 & \cdots & 0102 \\ \vdots & \ddots & \vdots \\ 1020 & \cdots & 1020 \\ 0101 & \cdots & 0101 \\ 2010 & \cdots & 2010 \\ 0101 & \cdots & 0102 \\ \hdashline 1020 & \cdots & 1020 \\ 0201 & \cdots & 0201 \end{pmatrix}, \qquad f_1^2 = \begin{pmatrix} 1020 & \cdots & 1020 & 1 \\ 0101 & \cdots & 0101 & 0 \\ 2010 & \cdots & 2010 & 2 \\ 0101 & \cdots & 0101 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 1 \\ 0101 & \cdots & 0101 & 0 \\ 2010 & \cdots & 2010 & 2 \\ 0101 & \cdots & 0101 & 0 \\ \hdashline 1020 & \cdots & 1020 & 1 \\ 0201 & \cdots & 0201 & 1 \end{pmatrix},$$

$$f_2^2 = \begin{pmatrix} 1020 & \cdots & 1020 & 10 \\ 0101 & \cdots & 0101 & 02 \\ 2010 & \cdots & 2010 & 20 \\ 0101 & \cdots & 0101 & 01 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 10 \\ 0101 & \cdots & 0101 & 02 \\ 2010 & \cdots & 2010 & 20 \\ 0101 & \cdots & 0101 & 01 \\ \hdashline 1020 & \cdots & 1020 & 10 \\ 0201 & \cdots & 0201 & 02 \end{pmatrix}, \quad f_3^2 = \begin{pmatrix} 1020 & \cdots & 1020 & 102 \\ 0101 & \cdots & 0101 & 010 \\ 2010 & \cdots & 2010 & 201 \\ 0101 & \cdots & 0101 & 010 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 102 \\ 0101 & \cdots & 0101 & 010 \\ 2010 & \cdots & 2010 & 201 \\ 0101 & \cdots & 0101 & 010 \\ \hdashline 1020 & \cdots & 1020 & 102 \\ 0201 & \cdots & 0201 & 020 \end{pmatrix}.$$

Thus, $n \equiv 2 \pmod 4$,

$$\gamma_{oiR}(P_n \Box P_m) \le \begin{cases} \frac{5mn+2m+2n-4}{8}, & m \equiv 0 \pmod 2, \\ \frac{5mn+2m+n+2}{8}, & m \equiv 1 \pmod 4, \\ \frac{5mn+2m+n+2}{8}, & m \equiv 3 \pmod 4. \end{cases}$$

(4) For $n \equiv 3 \pmod 4$, OIRDFs are defined as f_i^3 for $m \equiv i \pmod 4$ $(i = 0, 1, 2, 3)$.

$$f_0^3 = \begin{pmatrix} 1020 & \cdots & 1020 \\ 0101 & \cdots & 0101 \\ 2010 & \cdots & 2010 \\ 0101 & \cdots & 0102 \\ \vdots & \ddots & \vdots \\ 1020 & \cdots & 1020 \\ 0101 & \cdots & 0101 \\ 2010 & \cdots & 2010 \\ 0101 & \cdots & 0102 \\ \hdashline 1020 & \cdots & 1020 \\ 0101 & \cdots & 0101 \\ 2010 & \cdots & 2011 \end{pmatrix}, \qquad f_1^3 = \begin{pmatrix} 1020 & \cdots & 1020 & 1 \\ 0101 & \cdots & 0101 & 0 \\ 2010 & \cdots & 2010 & 2 \\ 0101 & \cdots & 0101 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 1 \\ 0101 & \cdots & 0101 & 0 \\ 2010 & \cdots & 2010 & 2 \\ 0101 & \cdots & 0101 & 0 \\ \hdashline 1020 & \cdots & 1020 & 1 \\ 0101 & \cdots & 0101 & 0 \\ 2010 & \cdots & 2010 & 2 \end{pmatrix},$$

$$f_2^3 = \begin{pmatrix} 1020 & \cdots & 1020 & 10 \\ 0101 & \cdots & 0101 & 02 \\ 2010 & \cdots & 2010 & 20 \\ 0101 & \cdots & 0101 & 01 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 10 \\ 0101 & \cdots & 0101 & 02 \\ 2010 & \cdots & 2010 & 20 \\ 0101 & \cdots & 0101 & 01 \\ \hdashline 1020 & \cdots & 1020 & 10 \\ 0101 & \cdots & 0101 & 02 \\ 2010 & \cdots & 2010 & 20 \end{pmatrix}, \quad f_3^3 = \begin{pmatrix} 1020 & \cdots & 1020 & 102 \\ 0101 & \cdots & 0101 & 010 \\ 2010 & \cdots & 2010 & 201 \\ 0101 & \cdots & 0101 & 010 \\ \vdots & \ddots & \vdots & \vdots \\ 1020 & \cdots & 1020 & 102 \\ 0101 & \cdots & 0101 & 010 \\ 2010 & \cdots & 2010 & 201 \\ 0101 & \cdots & 0101 & 010 \\ \hdashline 1020 & \cdots & 1020 & 102 \\ 0101 & \cdots & 0101 & 010 \\ 2010 & \cdots & 2010 & 201 \end{pmatrix}.$$

Thus, $n \equiv 3 \pmod 4$,

$$\gamma_{oiR}(P_n \Box P_m) \le \begin{cases} \frac{5mn+m+2n+2}{8}, & m \equiv 0 \pmod 4, \\ \frac{5mn+m+n+5}{8}, & m \equiv 1 \pmod 4, \\ \frac{5mn+m+2n+2}{8}, & m \equiv 2 \pmod 4, \\ \frac{5mn+m+n+5}{8}, & m \equiv 3 \pmod 4. \end{cases}$$

By cases (1)–(4), we can get $\gamma_{oiR}(P_n \Box P_m) \le \lfloor \frac{5mn+2m+2n}{8} \rfloor$. $\Box$

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Conflict of Interest The authors declare no conflict of interest.

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