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Extensions and Deformations of 3-Lie Algebras with Higher Derivations

Shuangjian GUO
School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guizhou 550025*, P. R. China*

Abstract In this paper, we call a tuple consisting of 3-Lie algebra and a higher derivation on it a 3-LieHDer pair. We introduce a cohomology theory of 3-LieHDer pairs. Next, we interpret the second cohomology group as the space of all isomorphism classes of abelian extensions. Finally, we consider formal deformations of 3-LieHDer pairs that are governed by the cohomology with self-coefficient.

Keywords higher derivations; 3-LieHDer pairs; cohomology; extensions; formal deformations

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1. Introduction

Higher derivations were introduced by Hasse and Schmidt [1], therefore, they are also called Hasse-Schmidt derivations. Not all higher derivations arise from derivations in the above way. See [2, 3] for more results about higher derivations. Higher derivations are useful in the theory of automorphisms of complete local rings and Galois theory of fields. n-Lie algebras (also called Filippov algebras) have attracted attention from both mathematics and physics [4, 5]. As a special case of n-Lie algebras, 3-Lie algebras play important roles in the study of the Bagger-Lambert-Gustavsson theory of multiple M2-branes. Moreover, the metric 3-Lie algebras, or more generally, the 3-Lie algebras with invariant symmetric bilinear forms attract even more attentions in physics. 3-Lie algebras got a lot of attention [6–9].

Recently, the authors introduced the notion of LieDer pairs and stuided extensions and deformations of LieDer pairs [10]. Later, the author studied associative algebras with higher derivations in [11]. Our aim in this paper is to consider 3-LieHDer pairs, which includes 3-Lie algebras and higher derivations. More precisely, we call a tuple consisting of 3-Lie algebra and a higher derivation on it a 3-LieHDer pair. We introduce a cohomology theory of 3-LieHDer pairs. Next, we study abelian extensions of a 3-LieHDer pair and relate them with the second cohomology group of the 3-LieHDer pair. Finally, we consider formal deformations of 3-LieHDer pairs that are governed by the cohomology with self-coefficient.

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E-mail address: shuangjianguo@126.com

The paper is organized as follows. In Section 2, we define representations of 3-LieHDer pairs and construct the semi-direct product. In Section 3, we intoduce a cohomology theory of 3-LieHDer pairs. In Section 4, we interpret the second cohomology group as the space of all isomorphism classes of abelian extensions. In Section 5, we consider formal deformations of 3-LieHDer pairs that are governed by the cohomology with self-coefficient.

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$ and finite-dimensional.

2. 3-LieHDer pairs and their representations

In this section, we define representations of 3-LieHDer pairs and construct the semi-direct product.

Definition 2.1 *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra. A higher derivation (of rank N) on $\mathfrak{g}$ is a tuple $\boldsymbol{d} = (d_1, \ldots, d_N)$ of linear maps on $\mathfrak{g}$ satisfying the following identities*

$$d_l([a,b,c]) = \sum_{i+j+k=l} [d_i(a), d_j(b), d_k(c)] \tag{2.1}$$

for $l = 1, \ldots, N$ and $a, b, c \in \mathfrak{g}$, with the convention that $d_0 = \mathrm{id}_{\mathfrak{g}}$.

It follows from (2.1) that d_1 is a derivation on $\mathfrak{g}$. In particular, if d is a derivation on $\mathfrak{g}$, then $\mathbf{d} = (d)$ is a higher derivation of rank 1.

Example 2.2 If d is a derivation on $\mathfrak{g}$, then $\mathbf{d} = (d_1 = 0, \ldots, d_i = d, \ldots, d_N = 0)$ is a higher derivation of rank N.

Example 2.3 Let d be a derivation on $\mathfrak{g}$. Then, $\mathbf{d} = (d, \frac{d^2}{2!}, \ldots, \frac{d^N}{N!})$ is a higher derivation of rank N.

Example 2.4 Let $\mathbf{d} = (d_1, \ldots, d_N)$ be a higher derivation on $\mathfrak{g}$. For any $1 \le q \le N$, we define a new tuple $\mathbf{d}' = (d'_1, \ldots, d'_N)$ of linear maps by

$$d'_l = \begin{cases} 0, & \text{if } q \dagger l, \\ d_s, & \text{if } l = sq. \end{cases}$$

Then, $\mathbf{d}' = (d'_1, \ldots, d'_N)$ is a higher derivation of rank N.

Throughout the paper, by a higher derivation, we shall always mean a higher derivation of fixed rank N.

Definition 2.5 *Given 3-LieHDer pairs $(\mathfrak{g}, \boldsymbol{d})$, $(\mathfrak{g}', \boldsymbol{d}')$, a homomorphism of 3-LieHDer pairs from $(\mathfrak{g}, \boldsymbol{d})$ to $(\mathfrak{g}', \boldsymbol{d}')$ is a 3-Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ such that $\varphi \circ d_l = d'_l \circ \varphi$, $l = 1, \ldots, N$.*

Recall that a representation of a 3-Lie algebra $\mathfrak{g}$ is a pair (V, ρ), where V is a vector space, $\rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(V),\ a \wedge b \mapsto (v \mapsto \rho(a,b)v)$ is a homomorphism of 3-Lie algebras for all $a, b \in \mathfrak{g}$ and $v \in V$.

Definition 2.6 *Let $(\mathfrak{g}, \boldsymbol{d})$ be a 3-LieHDer pair.*

(i) A representation over $(\mathfrak{g}, \boldsymbol{d})$ *is given by a pair* (V, ρ) *and linear maps* $\boldsymbol{d}^V = (d_1^V, \ldots, d_N^V)$ *satisfying*

$$d_l^V(\rho(a,b)v) = \sum_{i+j+k=l} \rho(d_i(a), d_j(b)) d_k^V(v), \quad l = 1, \ldots, N$$

with convention that $d_0 = \mathrm{id}_{\mathfrak{g}}$ *and* $d_0^V = \mathrm{id}_V$*, for all* $x, y \in \mathfrak{g}, v \in V$.

(ii) Let $(U, \rho_U, \boldsymbol{d}^U)$ *and* $(V, \rho_V, \boldsymbol{d}^V)$ *be representations of* $(\mathfrak{g}, \boldsymbol{d})$*. If a linear map* $f : U \to V$ *satisfies* $f \circ d_l^U = d_l^V \circ f, l = 1, \ldots, N$ *and*

$$f \circ \rho_U(a,b) = \rho_V(a,b) \circ f, \quad \forall\, a, b \in \mathfrak{g},$$

then f is called a homomorphism of representations from $(U, \rho_U, \mathbf{d}^U)$ to $(V, \rho_V, \mathbf{d}^V)$.

Example 2.7 Let $(\mathfrak{g}, \mathbf{d})$ be a 3-LieHDer pair. Then there is a natural representations

$$\rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ a \wedge b \mapsto (c \mapsto [a,b,c]).$$

This is called the adjoint representation over the 3-LieHDer pair $(\mathfrak{g}, \mathbf{d})$.

Example 2.8 Let $\mathfrak{g}$ be a 3-Lie algebra and (V, ρ) be a representation of it. Then the pair $(V, \mathbf{id}^V)$ is a representation of the 3-LieHDer pair $(\mathfrak{g}, \mathbf{id})$.

It is easy to check the following proposition and we omit it.

Proposition 2.9 *Let* $(\mathfrak{g}, \boldsymbol{d})$ *be a 3-LieHDer pair and* $(V, \boldsymbol{d}^V)$ *be a representation of* $(\mathfrak{g}, \boldsymbol{d})$*. Define the multiplication* $[\cdot,\cdot,\cdot]$ *on* $\mathfrak{g} \oplus V$ *by*

$$[a+u, b+v, c+w] = [a,b,c] + \rho(a,b)w + \rho(b,c)u + \rho(c,a)v, \quad \forall\, a,b,c \in \mathfrak{g},\ u,v,w \in V$$

and linear maps $d_l \oplus d_l^V : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ *by*

$$(d_l \oplus d_l^V)(x+u) = d_l(x) + d_l^V(u), \quad l = 1, \ldots, N.$$

Then $(\mathfrak{g} \oplus V, \boldsymbol{d} \oplus \boldsymbol{d}^V)$ *is a 3-LieHDer pair.*

3. Cohomology of 3-LieHDer pairs

In this section, we introduce the cohomology theory of 3-LieHDer pairs $(\mathfrak{g}, \mathbf{d})$ with coefficients in a representation $(V, \rho, \mathbf{d}^V)$.

3.1. Cohomology of 3-Lie algebras

Recall that the complex of 3-Lie algebra $\mathfrak{g}$ with coefficients in representation V is the cochain complex

$$(\mathrm{C}^*_{\text{3-Lie}}(\mathfrak{g}, V) = \oplus_{n=0}^{\infty} \mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g}, V), \partial^*_{\text{3-Lie}}),$$

where for $n \geq 1$, $\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g}, V) = \mathrm{Hom}(\underbrace{\wedge^2\mathfrak{g} \otimes \cdots \otimes \wedge^2\mathfrak{g}}_{n-1} \wedge \mathfrak{g}, V)$ (in particular, $\mathrm{C}^0_{\text{3-Lie}}(\mathfrak{g}, V) = V$) and the coboundary operator $\partial^n_{\text{3-Lie}} : \mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g}, V) \longrightarrow \mathrm{C}^{n+1}_{\text{3-Lie}}(\mathfrak{g}, V)$, $n \geq 1$ is given by

$$\partial^n_{\text{3-Lie}} f(X_1, \ldots, X_n, x_{n+1})$$

$$
\begin{aligned}
&= (-1)^{n+1}\rho(y_n, x_{n+1})f(X_1,\ldots,X_{n-1},x_n)+\\
&\quad (-1)^{n+1}\rho(x_{n+1}, x_n)f(X_1,\ldots,X_{n-1},y_n)+\\
&\quad \sum_{j=1}^{n}(-1)^{j+1}\rho(x_j,y_j)f(X_1,\ldots,\hat{X}_j,\ldots,X_n,x_{n+1})+\\
&\quad \sum_{j=1}^{n}(-1)^{j}f(X_1,\ldots,\hat{X}_j,\ldots,X_n,[x_j,y_j,x_{n+1}])+\\
&\quad \sum_{1\le j<k\le n}(-1)^{j}f(X_1,\ldots,\hat{X}_j,\ldots,X_{k-1},[x_j,y_j,x_k]\wedge y_k+\\
&\quad x_k\wedge[x_j,y_j,y_k],X_{k+1},\ldots,X_n,x_{n+1})
\end{aligned}
$$

for all $f\in \mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},V)$, $X_i=x_i\wedge y_i\in\wedge^2\mathfrak{g}$, $i=1,2,\ldots,n$ and $x_{n+1}\in\mathfrak{g}$. The corresponding cohomology is denoted by $\mathrm{H}^*_{\text{3-Lie}}(\mathfrak{g},V)$. When V is the adjoint representation, we write

$$\mathrm{H}^n_{\text{3-Lie}}(\mathfrak{g})=\mathrm{H}^n_{\text{3-Lie}}(\mathfrak{g},V),\quad n\ge 1.$$

3.2. Cohomology of 3-LieHDer pairs

Now, we introduce a cohomology theory of the 3-LieHDer pairs $(\mathfrak{g},\mathbf{d})$ by the cohomology of 3-Lie algebras. With a slight abuse of notation, we still use d_l, $l=1,\ldots,N$ to denote the endomorphism on $\wedge^2\mathfrak{g}$ given by

$$d_l(X)=d_l(x)\wedge y+x\wedge d_l(y),\quad \forall\, X=x\wedge y\in\wedge^2\mathfrak{g}.$$

Define the set of n-cochains by

$$\mathrm{C}^n_{\text{3-LieHDer}}(\mathfrak{g},V):=\begin{cases}\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},V)\oplus\mathrm{C}^{n-1}_{\text{3-Lie}}(\mathfrak{g},V), & n\ge 2,\\ \mathrm{C}^1_{\text{3-Lie}}(\mathfrak{g},V)=\mathrm{Hom}(\mathfrak{g},V), & n=1.\end{cases}$$

For $n\ge 1$, we define a linear map

$$\delta_l:\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},V)\to\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},V),\ \text{for } l=1,\ldots,N$$

by

$$\delta_l f:=\sum_{i_1+\cdots+i_n=l} f\circ(d_{i_1}\otimes\cdots\otimes d_{i_n})-d^V_l\circ f$$

for any $f\in\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},V)$.

Lemma 3.1 *The map $\partial_{\text{3-Lie}}$ and δ are commutative with each other, i.e., $\partial_{\text{3-Lie}}\circ\delta_l=\delta_l\circ\partial_{\text{3-Lie}}$.*

Proof For any $f\in\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},V)$, $l=1,\ldots,N$, we have

$$
\begin{aligned}
&\partial_{\text{3-Lie}}\circ\delta_l(f)(X_1,\ldots,X_n,x_{n+1})\\
&\quad= (-1)^{n+1}\rho(y_n,x_{n+1})\delta_l(f)(X_1,\ldots,X_{n-1},x_n)+\\
&\qquad (-1)^{n+1}\rho(x_{n+1},x_n)\delta_l(f)(X_1,\ldots,X_{n-1},y_n)+\\
&\qquad \sum_{j=1}^{n}(-1)^{j+1}\rho(x_j,y_j)\delta_l(f)(X_1,\ldots,\hat{X}_j,\ldots,X_n,x_{n+1})+
\end{aligned}
$$

$$\sum_{j=1}^{n}(-1)^{j}\delta_l(f)(X_1,\dots,\hat{X}_j,\dots,X_n,[x_j,y_j,x_{n+1}])+$$
$$\sum_{1\le j<k\le n}(-1)^{j}\delta_l(f)(X_1,\dots,\hat{X}_j,\dots,X_{k-1},[x_j,y_j,x_k]\wedge y_k+$$
$$x_k\wedge[x_j,y_j,y_k],X_{k+1},\dots,X_n,x_{n+1})$$
$$=(-1)^{n+1}\sum_{i_1+\dots+i_n=l}\rho(y_n,x_{n+1})f\circ(d_{i_1}(X_1),\dots,d_{i_{n-1}}(X_{n-1}),d_{i_n}(x_n))-$$
$$(-1)^{n+1}\rho(y_n,x_{n+1})d_l^V\circ f(X_1,\dots,X_{n-1},x_n)+$$
$$(-1)^{n+1}\sum_{i_1+\dots+i_n=l}\rho(x_{n+1},x_n)f\circ(d_{i_1}(X_1),\dots,d_{i_{n-1}}(X_{n-1}),d_{i_n}(y_n))-$$
$$(-1)^{n+1}\rho(x_{n+1},x_n)d_l^V\circ f(X_1,\dots,X_{n-1},y_n)+$$
$$\sum_{j=1}^{n}\sum_{i_1+\dots+i_n=l}(-1)^{j+1}\rho(x_j,y_j)f\circ(d_{i_1}(X_1),\dots,\hat{X}_j,\dots,d_{i_{n-1}}(X_n),d_{i_n}(x_{n+1}))-$$
$$\sum_{j=1}^{n}(-1)^{j+1}\rho(x_j,y_j)d_l^V\circ f(X_1,\dots,\hat{X}_j,\dots,X_n,x_{n+1})+$$
$$\sum_{j=1}^{n}\sum_{i_1+\dots+i_n=l}(-1)^{j}f\circ(d_{i_1}(X_1),\dots,\hat{X}_j,\dots,d_{i_{n-1}}(X_n),d_{i_n}([x_j,y_j,x_{n+1}]))+$$
$$\sum_{j=1}^{n}(-1)^{j}d_l^V\circ f(X_1,\dots,\hat{X}_j,\dots,X_n,x_n,[x_j,y_j,x_{n+1}])+$$
$$\sum_{1\le j<k\le n}\sum_{i_1+\dots+i_n=l}(-1)^{j}f\circ(d_{i_1}(X_1),\dots,\hat{X}_j,\dots,d_{i_{k-1}}(X_{k-1}),d_{i_k}([x_j,y_j,x_k])\wedge y_k+$$
$$x_k\wedge d_{i_k}([x_j,y_j,y_k]),d_{i_{k+1}}(X_{k+1}),\dots,d_{i_n}(X_n),d_{i_{n+1}}(x_{n+1}))-$$
$$\sum_{1\le j<k\le n}(-1)^{j}d_l^V\circ f(X_1,\dots,\hat{X}_j,\dots,X_{k-1},[x_j,y_j,x_k]\wedge y_k+$$
$$x_k\wedge[x_j,y_j,y_k],X_{k+1},\dots,X_n,x_{n+1})$$
$$=\sum_{i_1+\dots+i_n=l}(-1)^{n+1}\rho(y_n,x_{n+1})f\circ(d_{i_1}(X_1),\dots,d_{i_{n-1}}(X_{n-1}),d_{i_n}(x_n))-$$
$$(-1)^{n+1}d_l^V\circ\rho(y_n,x_{n+1})f(X_1,\dots,X_{n-1},x_n)+$$
$$\sum_{i_1+\dots+i_n=l}(-1)^{n+1}\rho(x_{n+1},x_n)f\circ(d_{i_1}(X_1),\dots,d_{i_{n-1}}(X_{n-1}),d_{i_n}(y_n))-$$
$$(-1)^{n+1}d_l^V\circ\rho(x_{n+1},x_n)f(X_1,\dots,X_{n-1},y_n)+$$
$$\sum_{i_1+\dots+i_n=l}\sum_{j=1}^{n}(-1)^{j+1}\rho(x_j,y_j)f\circ(d_{i_1}(X_1),\dots,\hat{X}_j,\dots,d_{i_{n-1}}(X_n),d_{i_n}(x_{n+1}))-$$
$$\sum_{j=1}^{n}(-1)^{j+1}d_l^V\circ\rho(x_j,y_j)f(X_1,\dots,\hat{X}_j,\dots,X_n,x_{n+1})+$$
$$\sum_{i_1+\dots+i_n=l}\sum_{j=1}^{n}(-1)^{j}f\circ(d_{i_1}(X_1),\dots,\hat{X}_j,\dots,d_{i_{n-1}}(X_n),d_{i_n}([x_j,y_j,x_{n+1}]))+$$

$$\sum_{j=1}^{n}(-1)^j d_l^V \circ f(X_1,\ldots,\hat{X}_j,\ldots,X_n,x_n,[x_j,y_j,x_{n+1}])+$$
$$\sum_{i_1+\cdots+i_n=l}\sum_{1\le j<k\le n}(-1)^j f\circ(d_{i_1}(X_1),\ldots,\hat{X}_j,\ldots,d_{i_{k-1}}(X_{k-1}),d_{i_k}([x_j,y_j,x_k])\wedge y_k+$$
$$x_k\wedge d_{i_k}([x_j,y_j,y_k]),d_{i_{k+1}}(X_{k+1}),\ldots,d_{i_n}(X_n),d_{i_{n+1}}(x_{n+1}))-$$
$$\sum_{1\le j<k\le n}(-1)^j d_l^V \circ f(X_1,\ldots,\hat{X}_j,\ldots,X_{k-1},[x_j,y_j,x_k]\wedge y_k+$$
$$x_k\wedge[x_j,y_j,y_k],X_{k+1},\ldots,X_n,x_{n+1})$$
$$=\delta_l\big((-1)^{n+1}\rho(y_n,x_{n+1})f(X_1,\ldots,X_{n-1},x_n)+(-1)^{n+1}\rho(x_{n+1},x_n)f(X_1,\ldots,X_{n-1},y_n)+$$
$$\sum_{j=1}^{n}(-1)^{j+1}\rho(x_j,y_j)f(X_1,\ldots,\hat{X}_j,\ldots,X_n,x_{n+1})+$$
$$\sum_{j=1}^{n}(-1)^j f(X_1,\ldots,\hat{X}_j,\ldots,X_n,[x_j,y_j,x_{n+1}])+$$
$$\sum_{1\le j<k\le n}(-1)^j f(X_1,\ldots,\hat{X}_j,\ldots,X_{k-1},[x_j,y_j,x_k]\wedge y_k+x_k\wedge[x_j,y_j,y_k],X_{k+1},\ldots,X_n,x_{n+1})\big)$$
$$=\delta_l\circ\partial_{\text{3-Lie}}(f)(X_1,\ldots,X_n,x_{n+1}).$$

And the proof is completed. □

Define $\partial_{\text{3-LieHDer}}: C^1_{\text{3-Lie}}(\mathfrak{g},V)\to C^2_{\text{3-Lie}}(\mathfrak{g},V)$ by

$$\partial_{\text{3-LieHDer}}(f)=(\partial_{\text{3-Lie}}(f),(-1)^1\delta f),\quad \forall\, f\in C^1_{\text{3-Lie}}(\mathfrak{g},V).$$

Then for $n\ge 2$, we define $\partial_{\text{3-LieHDer}}: C^n_{\text{3-Lie}}(\mathfrak{g},V)\to C^{n+1}_{\text{3-Lie}}(\mathfrak{g},V)$ by

$$\partial_{\text{3-LieHDer}}(f_n,g_{n-1})=(\partial_{\text{3-Lie}}(f_n),\partial_{\text{3-Lie}}(g_{n-1})+(-1)^n\delta_l f_n)$$

for any $f_n\in C^n_{\text{3-Lie}}(\mathfrak{g},V), g_{n-1}\in C^{n-1}_{\text{3-Lie}}(\mathfrak{g},V)$.

Theorem 3.2 *The pair $(C^*_{\text{3-LieHDer}}(\mathfrak{g},V),\partial_{\text{3-LieHDer}})$ is a cochain complex. So*

$$\partial^2_{\text{3-LieHDer}}=0.$$

Proof For any $v\in \mathrm{C}^1_{\text{3-Lie}}(\mathfrak{g},V)$, $l=1,\ldots,N$, we have

$$\partial^2_{\text{3-LieHDer}}v=\partial_{\text{3-LieHDer}}(\partial_{\text{3-Lie}}v,\delta_l v)=(\partial^2_{\text{3-Lie}}v,\partial_{\text{3-Lie}}\delta_l v-\delta_l\partial_{\text{3-Lie}}v)=0.$$

Given any $f\in \mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},V)$, $g\in \mathrm{C}^{n-1}_{\text{3-Lie}}(\mathfrak{g},V)$ with $n\ge 2$, $l=1,\ldots,N$, we have

$$\begin{aligned}\partial^2_{\text{3-LieHDer}}(f,g)&=\partial_{\text{3-LieHDer}}(\partial_{\text{3-Lie}}f,\partial_{\text{3-Lie}}g+(-1)^n\delta_l f)\\&=(\partial^2_{\text{3-Lie}}f,\partial_{\text{3-Lie}}(\partial_{\text{3-Lie}}g+(-1)^n\delta_l f)+(-1)^{n+1}\delta_l\partial_{\text{3-Lie}}f)\\&=0.\end{aligned}$$

Therefore, $(\mathrm{C}^*_{\text{3-LieHDer}}(\mathfrak{g},V),\partial_{\text{3-LieHDer}})$ is a cochain complex. □

Definition 3.3 *The cohomology of the cochain complex $(C^*_{\text{3-LieHDer}}(\mathfrak{g},V),\partial_{\text{3-LieHDer}})$. The corresponding cohomology groups are denoted by $H^*_{\text{3-LieHDer}}(\mathfrak{g},V)$.*

4. Abelian extensions of 3-LieHDer pairs

In this section, we interpret the second cohomology group as the space of all isomorphism classes of abelian extensions of 3-LieHDer pairs.

Definition 4.1 *An abelian extension of the 3-LieHDer pairs* $(\mathfrak{g}, \boldsymbol{d})$ *by the pair* $(V, \boldsymbol{d}^V)$ *is another 3-LieHDer pair* $(\hat{\mathfrak{g}}, \hat{\boldsymbol{d}})$ *with a short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} & \longrightarrow & 0 \\ & & {\scriptstyle V}\big\downarrow & & {\scriptstyle \hat{\boldsymbol{d}}}\big\downarrow & & {\scriptstyle \boldsymbol{d}}\big\downarrow & & \\ 0 & \longrightarrow & V & \xrightarrow{i} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

Diagram 1 Exact sequence diagram

of 3-LieHDer pairs.

Definition 4.2 *Let* $(\hat{\mathfrak{g}}, \hat{\boldsymbol{d}})$ *and* $(\hat{\mathfrak{g}}', \hat{\boldsymbol{d}}')$ *be two abelian extensions of the 3-LieHDer pairs* $(\mathfrak{g}, \boldsymbol{d}_{\mathfrak{g}})$ *by the pair* $(V, \boldsymbol{d}^V)$. *These two abelian extensions are said to be isomorphic if there exists an isomorphism* $\zeta : (\hat{\mathfrak{g}}, \hat{\boldsymbol{d}}) \to (\hat{\mathfrak{g}}, \hat{\boldsymbol{d}}')$ *of 3-LieHDer pairs that make the following diagram commutative:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V, \boldsymbol{d}^V) & \xrightarrow{i} & (\hat{\mathfrak{g}}, \hat{\boldsymbol{d}}) & \xrightarrow{p} & (\mathfrak{g}, \boldsymbol{d}) & \longrightarrow & 0 \\ & & \Big\| & & {\scriptstyle \zeta}\big\downarrow & & \Big\| & & \\ 0 & \longrightarrow & (V, \boldsymbol{d}^V) & \xrightarrow{i} & (\hat{\mathfrak{g}}', \hat{\boldsymbol{d}}') & \xrightarrow{p} & (\mathfrak{g}, \boldsymbol{d}) & \longrightarrow & 0. \end{array}$$

Diagram 2 The isomorphism diagram

A section of an abelian extension $(\hat{\mathfrak{g}}, \hat{\mathbf{d}})$ of $(\mathfrak{g}, \mathbf{d})$ by $(V, \mathbf{d}^V)$ is a linear map $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ such that $p \circ \sigma = \mathrm{id}_{\mathfrak{g}}$.

Now for an abelian extension $(\hat{\mathfrak{g}}, \hat{\mathbf{d}})$ of $(\mathfrak{g}, \mathbf{d})$ by $(V, \mathbf{d}^V)$ with a section $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$, we define a linear map $\rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(V)$ by

$$\rho(x, y)v := \rho(\sigma(x), \sigma(y))v, \quad \forall\, x, y \in \mathfrak{g}, \ v \in V.$$

Proposition 4.3 *With the above notations,* $(V, \boldsymbol{d}^V)$ *is a representation over the 3-LieHDer pair* $(\mathfrak{g}, \boldsymbol{d})$.

Proof It is routine to check that ρ is a representation of the 3-Lie algebra $\mathfrak{g}$ on V. Moreover, $\hat{d}_l(\sigma(x)) - \sigma(d_l(x)) \in V$ means that

$$\rho(\hat{d}_l(\sigma(x)), \hat{d}_l(\sigma(y)))v = \rho(\sigma(d_l(x)), \sigma(d_l(y)))v \text{ for } l = 1, \dots, N.$$

Thus we have

$$\begin{aligned} d_l^V(\rho(x, y)v) &= d_l^V(\rho(\sigma(x), \sigma(y))v) \\ &= \sum_{i+j+k=l} \rho(\hat{d}_i(\sigma(x)), \hat{d}_j(\sigma(y)))d_k^V(v) \end{aligned}$$

$$= \sum_{i+j+k=l} \rho(\sigma(d_i(x)), \sigma(d_j(y)))d_k^V(v)$$
$$= \sum_{i+j+k=l} \rho(d_i(x), d_j(y))d_k^V(v).$$

Hence, $(V, \mathbf{d}^V)$ is a representation over $(\mathfrak{g}, \mathbf{d})$. $\square$

We further define linear maps $\psi : \wedge^3\mathfrak{g} \to V$ and $\chi_l : \mathfrak{g} \to V$ for $l = 1, \dots, N$, respectively by

$$\psi(x, y, z) = [\sigma(x), \sigma(y), \sigma(z)] - \sigma([x, y, z]), \quad \forall\, x, y, z \in \mathfrak{g},$$
$$\chi_l(x) = \hat{d}_l(\sigma(x)) - \sigma(d_l(x)), \quad \forall\, x \in \mathfrak{g}.$$

We set $\hat{\mathfrak{g}}$ to $\mathfrak{g} \oplus V$. Then we define a trilinear multiplication $[\cdot,\cdot,\cdot]_\psi : \wedge^3\hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ and a linear map $\mathbf{d}_l^\chi : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ by

$$[a + u, b + v, c + w]_\psi = [a, b, c] + \rho(a, b)v + \rho(b, c)u + \rho(c, a)v + \psi(a, b, c), \tag{4.1}$$
$$d_l^\chi(a + v) = d_l(a) + \chi_l(x) + d_l^V(v), \quad l = 1, \dots, N \tag{4.2}$$

for any $a, b, c \in \mathfrak{g}$, $u, v, w \in V$.

Proposition 4.4 *The triple $(\mathfrak{g} \oplus V, [\cdot,\cdot,\cdot]_\psi, \boldsymbol{d}_\chi)$ equipped with the above product and the tuple $\boldsymbol{d}^\chi = (d_1^\chi, \dots, d_N^\chi)$ of linear maps forms a 3-LieHDer pair of rank N if and only if $(\psi, \chi_1, \dots, \chi_N)$ is a 2-cocycle of the 3-LieHDer pair $(\mathfrak{g}, \boldsymbol{d})$ with the coefficient in $(V, \boldsymbol{d}^V)$.*

Proof Suppose that $(\mathfrak{g} \oplus V, [\cdot,\cdot,\cdot]_\psi, \mathbf{d}^\chi)$ is a 3-LieHDer pair for any $a, b, c \in \mathfrak{g}$, one can easily observe that $[\cdot,\cdot,\cdot]_\psi$ satisfies

$$\psi(a, [b, c]) + \psi(a, \psi(b, c)) + \psi(b, [c, a]) + \psi(b, \psi(c, a)) + \psi(c, [a, b]) + \psi(c, \psi(a, b)) = 0. \tag{4.3}$$

Since $\mathbf{d}^\chi$ satisfies (2.1) for $l = 1, \dots, N$, we deduce that

$$\chi_l([a, b, c]) + d_l^V(\psi(a, b, c)) = \sum_{i+j+k=l} \psi(d_i(a), d_j(b), d_k(c)). \tag{4.4}$$

Hence, $(\psi, \chi_1, \dots, \chi_N)$ is a 2-cocycle.

Conversely, if $(\psi, \chi_1, \dots, \chi_N)$ satisfies equalities (4.1) and (4.2). Since $\partial_{\text{3-Lie}}(\psi) = 0$, it follows that $[\cdot,\cdot,\cdot]_\psi$ defines a 3-Lie algebra structure on $\hat{\mathfrak{g}}$. On the other hand,

$$\partial_{\text{3-Lie}}(\chi_l) + \delta_l(\psi) = 0$$

implies that $\mathbf{d}^\chi$ is a higher derivation. And the proof is completed. $\square$

Now we give the important theorems in this section.

Theorem 4.5 *Let $(\mathfrak{g}, \boldsymbol{d})$ be a 3-LieHDer pair and $(V, \boldsymbol{d}^V)$ be a trivial 3-LieHDer pair. Then, the isomorphism classes of abelian extensions of $\mathfrak{g}$ by V are classified by the second cohomology group.*

Proof Let $(\hat{\mathfrak{g}}, \hat{\mathbf{d}})$ be an abelian extension of $(\mathfrak{g}, \mathbf{d})$ by $(V, \mathbf{d}^V)$. If $\sigma_1 : \mathfrak{g} \to \hat{\mathfrak{g}}$ is a section of the map p and $(\psi^1, \chi_1^1, \dots, \chi_N^1)$ is a corresponding 2-cocycle. Let $(\psi^2, \chi_1^2, \dots, \chi_N^2)$ be a 2-cocycles and the section σ_1. We define $\omega:\mathfrak{g}\to V$ by

$$\omega(x)=\sigma_1(x)-\sigma_2(x).$$

Then

$$\begin{aligned}\psi^1(x,y,z)&=[\sigma_1(x),\sigma_1(y),\sigma_1(z)]-\sigma_1([x,y,z])\\&=[\sigma_2(x)+\omega(x),\sigma_2(y)+\omega(y),\sigma_2(z)+\omega(z)]-(\sigma_2([x,y,z])+\omega([x,y,z]))\\&=\psi^2(x,y,z)+\partial\omega(x,y,z)\end{aligned}$$

and

$$\begin{aligned}\chi_l^1(x)&=\hat{d}_l(\sigma_1(x))-\sigma_1(d_l(x))\\&=\hat{d}_l(\sigma_2(x)+\omega(x))-(\sigma_2(d_l(x))+\omega(d_l(x)))\\&=(\hat{d}_l(\sigma_2(x))-\sigma_2(d_l(x)))+\hat{d}_l(\omega(x))-\omega(d_l(x))\\&=\chi_l^2(x)+d_l^V(\omega(x))-\omega(d_l(x))\\&=\chi_l^2(x)-\delta\omega(x).\end{aligned}$$

That is,

$$(\psi^1,\chi_1^1,\ldots,\chi_N^1)=(\psi^2,\chi_1^2,\ldots,\chi_N^2)+\partial_{\text{3-LieHDer}}(\omega).$$

Thus $(\psi^1,\chi_1^1,\ldots,\chi_N^1)$ and $(\psi^2,\chi_1^2,\ldots,\chi_N^2)$ are in the same cohomological class in $H^2_{\text{3-LieHDer}}(\mathfrak{g},V)$.

Let $(\hat{\mathfrak{g}},\hat{\mathbf{d}})$ and $(\hat{\mathfrak{g}}',\hat{\mathbf{d}}')$ be two isomorphic abelian extensions, σ_1 be a section of $(\hat{\mathfrak{g}},\hat{\mathbf{d}})$. As $\pi_2\circ\xi=\pi_1$, we have

$$\pi_2\circ(\xi\circ\sigma_1)=\pi_1\circ\sigma_1=\mathrm{id}_{\mathfrak{g}}.$$

Therefore, $\zeta\circ\sigma_1$ is a section of $(\hat{\mathfrak{g}}',\hat{\mathbf{d}}')$. Denote $\sigma_2:=\xi\circ\sigma_1$. Since $\xi:(\hat{\mathfrak{g}},\hat{\mathbf{d}})\to(\hat{\mathfrak{g}}',\hat{\mathbf{d}}')$ is a homomorphism of 3-LieHDer pairs such that $\xi|_V=\mathrm{id}_V$ for $l=1,\ldots,N$, we have

$$\begin{aligned}\psi^2(x,y,z)&=[\sigma_2(x),\sigma_2(y),\sigma_2(z)]-\sigma_2([x,y,z])\\&=[\xi(\sigma_1(x)),\xi(\sigma_1(y)),\xi(\sigma_1(z))]-\xi(\sigma_1([x,y,z]))\\&=\xi([\sigma_1(x),\sigma_1(y),\sigma_1(z)]-\sigma_1([x,y,z]))=\xi(\psi^1(x,y,z))\\&=\psi^1(x,y,z)\end{aligned}$$

and

$$\begin{aligned}\chi_l^2(x)&=\hat{d}_l'(\sigma_2(x))-\sigma_2(\hat{d}_l(x))=\hat{d}_l'(\zeta(\sigma_1(x)))-\zeta(\sigma_1(\hat{d}_l(x)))\\&=\xi(\hat{d}_l(\sigma_1(x))-\sigma_1(\hat{d}_l(x)))=\xi(\chi_l^1(x))\\&=\chi_l^1(x).\end{aligned}$$

Consequently, all isomorphic abelian extensions give rise to the same element in $\mathrm{H}^2_{\text{3-LieHDer}}(\mathfrak{g},V)$.

Conversely, let $(\psi^2,\chi_1^2,\ldots,\chi_N^2)$ be another 2-cocycle cohomologous to $(\psi^1,\chi_1^1,\ldots,\chi_N^1)$. In other words, they represent the same cohomological class in $\mathrm{H}^2_{\text{3-LieHDer}}(\mathfrak{g},V)$, then there exists a linear map $\omega:\mathfrak{g}\to V$ such that

$$(\psi^1,\chi_1^1,\ldots,\chi_N^1)=(\psi^2,\chi_1^2,\ldots,\chi_N^2)+\partial_{\text{3-LieHDer}}(\omega).$$

Define

$$\xi : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V \text{ by } \xi(x, v) := x + \omega(x) + v.$$

Then ξ is an isomorphism of these two abelian extensions. □

5. Formal deformations of 3-LieHDer pairs

Let $(\mathfrak{g}, \mathbf{d})$ be a 3-LieHDer pair. Denote by $\mu_{\mathfrak{g}}$ the multiplication of $\mathfrak{g}$.

Definition 5.1 *A 1-parameter formal deformation of a 3-LieHDer pair $(\mathfrak{g}, \mathbf{d})$ consists of $N+1$ formal power series $(\mu_t; d_{1,t}, \ldots, d_{N,t})$ of the form.*

$$\mu_t = \sum_{i=0}^{\infty} \mu_i t^i,$$
$$d_{1,t} = \sum_{i=0}^{\infty} t^i d_{1,i} \quad d_{1,0} = d_1,$$
$$\vdots$$
$$d_{N,t} = \sum_{i=0}^{\infty} t^i d_{N,i} \quad d_{N,0} = d_N,$$

such that $(\mathfrak{g}[[t]], \mu_t)$ is a 3-Lie algebra over $\mathbb{K}[[t]]$ and the tuple $d_t = (d_{1,t}, \ldots, d_{N,t})$ is a higher derivation of rank N.

It follows that $(\mu_t; d_{1,t}, \ldots, d_{N,t})$ is a formal one-parameter deformation of the 3-LieHDer pair $(\mathfrak{g}, \mathbf{d})$ if and only if the followings are hold:

$$\mu_t(a, b, \mu_t(c, d, e)) = \mu_t(\mu_t(a, b, c), d, e) + \mu_t(c, \mu_t(a, b, d), e) + \mu_t(c, d, \mu_t(a, b, e)),$$
$$d_{l,t}(\mu_t(a, b, c)) = \sum_{i+j+k=l} \mu_t(d_{i,t}(a), d_{j,t}(b), d_{k,t}(c))$$

for $l = 1, \ldots, N$. The above identities are equivalent to

$$\sum_{i+j=n} \mu_i(a, b, \mu_j(c, d, e)) = \sum_{i+j=n} \mu_i(\mu_j(a, b, c), d, e) + \mu_i(c, \mu_j(a, b, d), e) + \mu_i(c, d, \mu_j(a, b, e)),$$
$$\sum_{i+j=n} d_{l,i}\mu_j(a, b, c) = \sum_{i+j+k=l} \sum_{p+q+r+s=n} (\mu_p(d_{i,q}(a), d_{j,r}(b), d_{k,s}(c)))$$

for $n \geq 0$. The equations are hold for $n = 0$ as $d_{l,0} = d_l$. However, for $n = 1$, we get

$$\begin{aligned} &\mu_1(a, b, [c, d, e]) + [a, b, \mu_1(c, d, e)] \\ &= \mu_1([a, b, c], d, e) + [\mu_1(a, b, c), d, e] + [c, \mu_1(a, b, d), e] + \\ &\quad \mu_1(c, [a, b, d], e) + [c, d, \mu_1(a, b, e)] + \mu_1(c, d, [a, b, e]) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} d_l\mu_1(a, b, c) + d_{l,1}[a, b, c] = &\sum_{i+j+k=l} \{(\mu_1(d_i(a), d_j(b), d_k(c))) + \\ &[d_{i,1}(a), d_j(b), d_k(c)] + [d_i(a), d_{j,1}(b), d_k(c)] + [d_i(a), d_j(b), d_{k,1}(c)]\} \end{aligned} \tag{5.2}$$

for $l=1,\ldots,N$. For $n=1$, (5.1) is equivalent to $\partial_{\text{3-Lie}}\mu_1=0$, and (5.2) is equivalent to

$$\partial_{\text{3-Lie}}d_{l,1}+\delta_l\mu_1=0.$$

Therefore, we have

$$\partial_{\text{3-LieHDer}}(\mu_1;d_{1,1},\ldots,d_{N,1})=0.$$

As a consequence of the above discussions, we obtain the following.

Proposition 5.2 *Let $(\mathfrak{g},\mathbf{d})$ be a 3-LieHDer pair. Suppose $(\mu_t;d_{1,t},\ldots,d_{N,t})$ is a formal one parameter deformation of $(\mathfrak{g},\mathbf{d})$. Then the infinitesimal is a 2-cocycle in the cohomology complex of $(\mathfrak{g},\mathbf{d})$ with coefficients in itself. Moreover, the corresponding cohomology class depends only on the equivalence class of the deformation.*

The 2-cocycle is called the infinitesimal of the formal one-parameter deformation $(\mu_t;d_{1,t},\ldots,d_{N,t})$.

Definition 5.3 *Let $(\mu_t;d_{1,t},\ldots,d_{N,t})$ and $(\mu'_t;d'_{1,t},\ldots,d'_{N,t})$ be two deformations of a 3-LieHDer pair $(\mathfrak{g},\mathbf{d})$. They are said to be equivalent if there exists a formal isomorphism*

$$\Phi_t=\sum_{i\geq 0}\Phi_i t^i:\mathfrak{g}[[t]]\to\mathfrak{g}[[t]],$$

where $\Phi_i:\mathfrak{g}\to\mathfrak{g}$ are linear maps with $\Phi_0=\mathrm{id}_{\mathfrak{g}}$, such that

$$\Phi_t\circ\mu_t=\mu'_t\circ(\Phi_t\times\Phi_t\times\Phi_t),$$
$$\Phi_t\circ d_{l,t}=d'_{l,t}\circ\Phi_t,\ \text{for } l=1,\ldots,N.$$

They are equivalent to the following equations: for $n\geq 0$,

$$\sum_{i+j=n}\Phi_i\circ\mu_j=\sum_{p+q+r+s=n}\mu'_p\circ(\Phi_q\times\Phi_r\times\Phi_s),$$
$$\sum_{i+j=n}\Phi_i\circ d_{l,j}=\sum_{p+g=n}d'_{l,p}\circ\Phi_q\ \text{for } l=1,\ldots,N.$$

The above identities hold for $n=0$. For $n=1$, we have

$$\mu_1+\Phi_1\circ\mu=\mu'_1+\mu\circ(\Phi_1\times\mathrm{id}\times\mathrm{id})+\mu\circ(\mathrm{id}\times\Phi_1\times\mathrm{id})+\mu\circ(\mathrm{id}\times\mathrm{id}\times\Phi_1),$$
$$d_l\circ\Phi_1+d'_{l,1}=d_{l,1}+\Phi_1\circ d_l\ \text{for } l=1,\ldots,N.$$

Thus, we have

$$(\mu_1;d_{1,1},\ldots,d_{N,1})-(\mu'_1;d'_{1,1},\ldots,d'_{N,1})=\partial_{\text{3-LieHDer}}(\Phi_1).$$

Proposition 5.4 *The corresponding cohomology class depends only on the equivalence class of the deformation.*

Definition 5.5 *If the deformation $(\mu_t;d_{1,t},\ldots,d_{N,t})$ is equivalent to the undeformed deformation $(\mu'_t=\mu;d'_{1,t}=d_1,\ldots,d'_{N,t}=d_N)$, then it is called trivial.*

Theorem 5.6 *If $H^2_{\text{3-LieHDer}}(\mathfrak{g},\mathfrak{g})=0$, then every formal deformation of the 3-LieHDer pair $(\mathfrak{g},\mathbf{d})$ is trivial.*

Proof Suppose $(\mu_t; d_{1,t}, \ldots, d_{N,t})$ is a deformation of the 3-LieHDer pair $(\mathfrak{g}, \mathbf{d})$, we obtain that the linear term $(\mu_1; d_{1,1}, \ldots, d_{N,1})$ is a 2-cocycle according to Proposiiton 5.2. By

$$H^2_{\text{3-LieHDer}}(\mathfrak{g}, \mathfrak{g}) = 0,$$

there exists a 1-cochain $\Phi_1 \in C^1_{\text{3-Lie}}(\mathfrak{g}, \mathfrak{g})$ such that

$$(\mu_1; d_{1,1}, \ldots, d_{N,1}) = \partial_{\text{3-LieHDer}}(\phi_1). \tag{5.3}$$

We set $\Phi_t = \mathrm{id}_{\mathfrak{g}} + \Phi_1 t$ and define

$$\mu'_t = \Phi_t^{-1} \circ \mu_t \circ \Phi_t \times \Phi_t \times \Phi_t, \quad d'_{l,t} = \Phi_t^{-1} \circ d_{l,t} \circ \Phi_t, \; l = 1, \ldots, N.$$

By definition, $(\mu'_t; d'_{1,t}, \ldots, d'_{N,t})$ is equivalent to $(\mu_t; d_{1,t}, \ldots, d_{N,t})$. Moreover, it follows from (5.3), we have

$$\mu'_t = \mu + \mu'_2 t^2 + \cdots, \quad d'_{l,t} = d_k + d'_{l,2} t^2 + \cdots.$$

In other words, the linear terms vanish. By repeating this argument, we conclude the result. □

6. Conclusions

In this paper, we study 3-Lie algebras equipped with higher derivations.

Like higher derivations are generalization of the usual derivative, Rota-Baxter operators on 3-Lie algebras are generalization of the integral operator. Let $\mathfrak{g}$ be a 3-Lie algebra. A linear map $R : \mathfrak{g} \to \mathfrak{g}$ is called a Rota-Baxter operator (of weight 0) on $\mathfrak{g}$ if R satisfies

$$[R(x), R(y), R(z)] = R([R(x), R(y), z] + [x, R(y), R(z)] + [R(x), y, R(z)]), \quad \forall\, x, y, z \in \mathfrak{g}.$$

See [12, 13] for more details about Rota-Baxter operators on 3-Lie algebras. By generalizing higher integral operators, one may define the notion of higher Rota-Baxter operator of rank N. More precisely, a higher Rota-Baxter operator of rank N consists of a tuple $R = (R_1, \ldots, R_N)$ of linear maps satisfying the following identities

$$[R_l(x), R_l(y), R_l(z)] = R_l\Big(\sum_{i+j+k=l} [R_i(x), R_j(y), R_k(z)]\Big) \text{ for } l = 1, \ldots, N.$$

With the convention that $R_0 = \mathrm{id}_{\mathfrak{g}}$. If R is a Rota-Baxter operator on $\mathfrak{g}$, then it is easy to see that

$$R = (R, \frac{R^2}{2!}, \ldots, \frac{R^N}{N!})$$

is a higher Rota-Baxter operator of rank N. In future, we come with more structural properties of higher Rota-Baxter operators.

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