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On Finite Solvable Groups G with $m(G) - d(G) = 1$

Hailin LIU[1,*], **Liping ZHONG**[1], **Shoushuang CHEN**[1], **Yulong MA**[2]
1. *School of Science, Jiangxi University of Science and Technology, Jiangxi* 341000*, P. R. China;*
2. *School of Mathematics, Northwest University, Shaanxi* 710127*, P. R. China*

Abstract Let G be a finite group. A generating set X of G is said to be minimal if no proper subset of X generates G. Let $d(G)$ and $m(G)$ denote the smallest size and the largest size of a minimal generating set of G, respectively. In this paper we present a characterization for finite solvable groups G such that $m(G) - d(G) = 1$ and $m(G) \geq m(G/N) + m(N)$ for any non-trivial normal subgroup N of G.

Keywords finite solvable group; minimal generating set; normal subgroup; cyclic group

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1. Introduction

For a finite group G, we use the symbols $|G|$, $\Phi(G)$, $d(G)$ and $m(G)$ to denote the order of G, the Frattini subgroup of G, the smallest size of a minimal generating set of G, and the largest size of a minimal generating set of G, respectively. Let p be a prime factor of $|G|$. We denote by $\mathrm{Syl}_p(G)$ the set of all Sylow p-subgroups of G. Let N and H be subgroups of G. Then $C_H(N)$ denotes the centraliser of N in H. For two groups M and L, we use $M{:}L$ and $M \times L$ to denote a semidirect product and the direct product of M by L. For an integer n and a prime p, we denote by Z_p and Z_p^n the cyclic group of order p and the elementary abelian p-group of order p^n.

The characterization of finite groups regarding minimal generating sets has been studied extensively in the past a few decades [1–6]. The Burnside Basis Theorem shows that the minimal generating sets of a p-group have the same cardinality. A group has property $\mathfrak{B}$ if its minimal generating sets have the same cardinality. Further, a group is said to have the basis property if it and all its subgroups have property $\mathfrak{B}$. In [5], McDougall-Bagnall and Quick attempted to classify all finite groups with the basis property. Unfortunately, a mistake was discovered in their main theorem by Apisa and Klopsch in [1]. And Krempa and Stocka corrected and generalized some characterizations of groups with the basis property in [7]. In [1], Apisa and Klopsch proved that the class of finite $\mathfrak{B}$-groups is closed under taking quotients and that every finite $\mathfrak{B}$-group is solvable, and they obtained a general structure theorem for finite $\mathfrak{B}$-groups.

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* Corresponding author
E-mail address: hailinliuqp@163.com (Hailin LIU); zhongliping2022@126.com (Liping ZHONG); Chen7164@126.com (Shoushuang CHEN); yulongma1@163.com (Yulong MA)

Saxl and Whiston showed that for projective special linear groups $G = \mathrm{PSL}(2, p^r)$, $m(G) - d(G)$ depends on the number of prime divisors of r in [6]. In particular, $m(G) - d(G) = 1$ for all $G = \mathrm{PSL}(2, p)$ with p not congruent to ± 1 modulo 8 or 10. Therefore, generalizing Apisa and Klopsch's result to groups G with $m(G) - d(G) = 1$ is an interesting open problem. In this paper, we consider partially the problem when G is solvable. Our result is as follows.

Theorem 1.1 *Let G be a finite solvable group with $m(G) - d(G) = 1$. Suppose that $m(G) \geq m(G/N) + m(N)$ for any non-trivial normal subgroup N of G. Then one of the following holds:*

(i) $G = P \times Q$, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ for distinct primes p and q;

(ii) $G = P{:}R$, where P is a p-subgroup of G for a prime p, and R is an r-subgroup of G for a prime r;

(iii) $G = P{:}R{:}Q$, where P is a p-subgroup of G for a prime p, R is an r-subgroup of G for a prime r, and Q is a q-subgroup of G for a prime q;

(iv) $G = P{:}R$, where P is a p-subgroup of G for a prime p and R is an $\{r_1, r_2\}$-subgroup of G for distinct prime r_1 and r_2;

(v) $G = H{:}R$, where H is a $\{p, q\}$-subgroup of G for distinct primes p and q, and R is an r-subgroup of G for a prime r; or

(vi) $G = H{:}R$, where H is a $\{p, q\}$-subgroup of G for distinct primes p and q, and R is an $\{r, r_1\}$-subgroup of G for distinct primes r and r_1.

Remark 1.2 The conditions (i) to (vi) are necessary for Theorem 1.1, not necessarily sufficient conditions. Thus the converse to this theorem does not hold in general. However, it is very interesting and important to explore the sufficient conditions of Theorem 1.1.

2. Preliminary results

First, we obtain a characterization of finite cyclic groups G with $m(G) - d(G) = 1$.

Proposition 2.1 *Let G be a finite cyclic group. Then $m(G) - d(G) = 1$ if and only if G is a $\{p, q\}$-group, where p and q are distinct primes.*

Proof Suppose that $G = \langle g \rangle$. Then $d(G) = 1$. Assume that $m(G) - d(G) = 1$. Then $m(G) = 2$ and the primary decomposition of G shows that $m(G)$ is equal to the number of primes dividing $|G|$. It follows that G is a $\{p, q\}$-group for distinct primes p and q.

Conversely, if G is a $\{p, q\}$-group, then $G = \langle g_1 \rangle \times \langle g_2 \rangle$, where g_1 has order a power of p and g_2 has order a power of q. This means that $m(G) = 2$. Therefore, $m(G) - d(G) = 1$. □

Let G be a finite group with $m(G) - d(G) = 1$. Suppose that $m(G) \geq m(G/N) + m(N)$ for any non-trivial normal subgroup N of G. Then one of the following holds:

$$m(G/N) = d(G/N), \quad m(N) = d(N); \tag{2.1}$$

$$m(G/N) - d(G/N) = 1, \quad m(N) = d(N); \tag{2.2}$$

$$m(G/N) = d(G/N), \quad m(N) - d(N) = 1. \tag{2.3}$$

Moreover, (2.2) and (2.3) imply that $m(G) = m(G/N) + m(N)$ and $d(G) = d(G/N) + d(N)$.

The next lemma gives a relation between $d(G)$ and $d(G/N)$ when N is a minimal normal subgroup of G with $m(G) - d(G) = 1$.

Lemma 2.2 *Let G be a finite group such that $m(G) - d(G) = 1$, and let N be a minimal normal subgroup of G. Then*

(i) $d(G) = \begin{cases} d(G/N), & N \leq \Phi(G), \\ d(G/N) + 1, & \text{otherwise}; \end{cases}$

(ii) $m(G) = \begin{cases} d(G/N) + 1, & N \leq \Phi(G), \\ d(G/N) + 2, & \text{otherwise}. \end{cases}$

Proof Note that each generating set of G projects to a generating set of G/N. Thus $d(G) = d(G/N)$ if and only if $N \leq \Phi(G)$. We now suppose that $d(G) > d(G/N)$. Since $d(G) \leq d(G/N) + 1$ by [3], we have $d(G) = d(G/N) + 1$. Hence (i) holds.

As $m(G) - d(G) = 1$, (ii) follows from (i). □

Now we focus on Frattini-free groups G with $m(G) - d(G) = 1$ and $m(G) \geq m(G/N) + m(N)$ for any non-trivial normal subgroup N of G.

Lemma 2.3 *Let G be a noncyclic, Frattini-free finite group such that $m(G) - d(G) = 1$. Suppose that $m(G) \geq m(G/N) + m(N)$ for any non-trivial normal subgroup N of G. If G has unique minimal normal subgroup N, then*

(i) $d(G) = 2$; and

(ii) G/N is cyclic and $|G/N|$ has at most two prime factors.

Proof By [4, Theorem 1.1], $d(G) = \max\{2, d(G/N)\}$, as G is a noncyclic finite group with a unique minimal normal subgroup N. And by Lemma 2.2, $d(G) = d(G/N) + 1$, as G is Frattini-free. Now we can conclude that $d(G) = 2$ and $d(G/N) = 1$. It follows that G/N is cyclic. By (2.1)–(2.3), we have $m(G/N) = d(G/N)$ or $m(G/N) - d(G/N) = 1$. If $m(G/N) = d(G/N)$, then $|G/N| = p^n$ by [1, Proposition 2.1], where p is a prime and n is a positive integer. If $m(G/N) - d(G/N) = 1$, then $|G/N| = p^n q^m$ by Proposition 2.1, where p and q are distinct primes, and m and n are positive integers. □

The socle $\mathsf{Soc}(G)$ of G is the product of all minimal normal subgroups of G. Then $\mathsf{Soc}(G)$ is a direct product of simple groups.

Lemma 2.4 *Let G be a Frattini-free finite solvable group. Suppose that $m(G) - d(G) = 1$. Then*

(i) $G = N{:}H$, where $N = \mathsf{Soc}(G) = Z_p^m \times Z_q$ or Z_p^m for some positive integer m, and distinct primes p and q; and

(ii) $C_H(N) = 1$.

Proof Since G is solvable, $N = \mathsf{Soc}(G)$ is a direct product of abelian simple groups. By [8, Proposition 5.2.13], we have $G = N{:}H$ for some subgroup H of G.

We claim that $C_H(N)$ is a normal subgroup of G. Since N is normal in G, $C_H(N) =$

$H \cap C_G(N)$ is normal in H, while by definition it is centralized by N. Hence $C_H(N)$ is normal in $NH = G$. If $C_H(N) \neq 1$, then $C_H(N) \cap N \neq 1$, a contradiction. Thus $C_H(N) = 1$.

Next we show that $N = Z_p^m \times Z_q$ or Z_p^m for some positive integer m and distinct primes p and q. Assume that $N \cong Z_p^{m_1} \times Z_q^{m_2} \times Z_r^{m_3} \times N'$, where $Z_p^{m_1} \in \mathrm{Syl}_p(N)$, $Z_q^{m_2} \in \mathrm{Syl}_q(N)$ and $Z_r^{m_3} \in \mathrm{Syl}_r(N)$, for distinct primes p, q and r. Then again by [8, Proposition 5.2.13], we have $G = (Z_p^{m_1} \times Z_q^{m_2} \times Z_r^{m_3}) : M$ for a suitable subgroup M of G. We choose a minimal generating set $\{m_1, \ldots, m_f\}$ of M and extend this to a minimal generating set

$$\{x_1, \ldots, x_c, y_1, \ldots, y_d, z_1, \ldots, z_e, m_1, \ldots, m_f\} \tag{2.4}$$

of G, where $\{x_1, \ldots, x_c\}$, $\{y_1, \ldots, y_d\}$ and $\{z_1, \ldots, z_e\}$ are minimal generating sets of $Z_p^{m_1}$, $Z_q^{m_2}$ and $Z_r^{m_3}$, respectively. But we can also obtain another generating set

$$\{x_1y_1z_1, x_2, \ldots, x_c, y_2, \ldots, y_d, z_2, \ldots, z_e, m_1, \ldots, m_f\} \tag{2.5}$$

of G. Note that (2.4) and (2.5) imply that $m(G) - d(G) > 1$, but this contradicts our assumption.

Assume that $N = Z_p^{m_1} \times Z_q^{m_2}$ for positive integers m_1 and m_2 and distinct primes p and q. Note that $G = N{:}H$. We can choose a minimal generating set $\{h_1, \ldots, h_f\}$ of H and extend this to a minimal generating set

$$\{x_1, \ldots, x_c, y_1, \ldots, y_d, h_1, \ldots, h_f\} \tag{2.6}$$

of G, where $\{x_1, \ldots, x_c\}$ and $\{y_1, \ldots, y_d\}$ are minimal generating sets of $Z_p^{m_1}$ and $Z_q^{m_2}$, respectively. If $m_1 \geq 2$ and $m_2 \geq 2$, then

$$\{x_1y_1, x_2y_2, x_3, \ldots, x_c, y_3, \ldots, y_d, h_1, \ldots, h_f\} \tag{2.7}$$

is another generating set of G. This in conjunction with (2.6) implies that $m(G) - d(G) > 1$, a contradiction.

Thus $N = Z_p \times Z_q^{m_2}$, $Z_p^{m_1} \times Z_q$, $Z_p^{m_1}$ or $Z_q^{m_2}$, and this completes the proof. □

By the results above, we can obtain a characterization of finite solvable and Frattini-free group G with $m(G) - d(G) = 1$ and $m(G) \geq m(G/N) + m(N)$ for any non-trivial normal subgroup N of G.

Proposition 2.5 *Let G be a finite solvable group such that $m(G) - d(G) = 1$. Suppose that G is Frattini-free, and $m(G) \geq m(G/N) + m(N)$ for any non-trivial normal subgroup N of G. Then one of the following holds:*

(i) $G = Z_p^m \times Z_q$, where $p \neq q$ are primes, and m is a positive integer;

(ii) $G = Z_p^m{:}R$ or $Z_p^m{:}(R{:}Z_q)$, where p, q are primes and m is a positive integer, and R is an r-group with $r \neq q$;

(iii) $G = Z_p^m{:}Z_{r_1^n r_2}$, where p, $r_1 \neq r_2$ are primes, and n and m are positive integers; or

(iv) $G = (Z_p^m \times Z_q){:}R$ or $(Z_p^m \times Z_q){:}(R{:}Z_{r_1})$, where $p \neq q$, r_1 are primes and m is a positive integer, and R is an r-group with $r \neq r_1$.

Proof If G is abelian, then $G = Z_p^m \times Z_q$ by Lemma 2.4, where m is a positive integer, and p

and q are distinct primes. Now we suppose that G is nonabelian. Then $G = N{:}H$ by Lemma 2.4, where $N = \mathsf{Soc}(G) = Z_p^m$ or $Z_p^m \times Z_q$, and $H \neq 1$ acts faithful on N under conjugation.

Case 1. $N = Z_p^m$.

By (2.1), we have $m(G/N) = d(G/N)$. It follows that $H = R$ or $H = R{:}Q$ by [1, Theorem 1.6], where R is an r-group and Q is a cyclic group of order q for distinct primes r and q.

By (2.2), we have $m(G/N) - d(G/N) = 1$ as $m(N) = d(N)$. If N is the unique minimal normal subgroup of G, then $H = Z_{r_1^n r_2}$ by Lemma 2.3, where $r_1 \neq r_2$ are primes, and n is a positive integer.

Assume that $N = N_1 \times N_2 \times \cdots \times N_d$, where $d \geq 2$ and N_i is a minimal normal subgroup of G for each $i \in \{1, \ldots, d\}$. Then $\overline{G} = G/(N_2 \times \cdots \times N_d) = \overline{N_1} : \overline{H}$, where $\overline{H} \cong H$. We conclude that $\overline{N_1}$ is the unique minimal normal subgroup of $\overline{G}$. Indeed, let $\overline{A} = A/(N_2 \times \cdots \times N_d)$ be a minimal normal subgroup of $\overline{G}$. Then $A \trianglelefteq G$ and $1 \neq A \cap N_1 \trianglelefteq G$. It follows that $N_1 \trianglelefteq A$. Thus $\overline{A} = \overline{N_1}$. Replace N by $\overline{N_1}$, and G by $\overline{G}$ in Lemma 2.3, we see that $H = Z_{r_1^n r_2}$.

Case 2. $N = Z_p^m \times Z_q$.

By (2.3), we obtain that $m(H) = d(H)$ since $m(N) - d(N) = 1$. From [1, Theorem 1.6], we deduce that $H = R$ or $H = R{:}R_1$, where R is an r-group and R_1 is a cyclic group of order r_1 for distinct primes r and r_1. □

3. Proof of Theorem 1.1

Using Proposition 2.5, we determine the structure of finite solvable groups G with $m(G) - d(G) = 1$ and $m(G) \geq m(G/N) + m(N)$ for any non-trivial normal subgroup N of G.

Proof of Theorem 1.1 Let $M = G/\Phi(G)$. Then M is Frattini-Free. Since $m(G) - d(G) = 1$ and $m(G) \geq m(G/N) + m(N)$ for any non-trivial normal subgroup N of G, we have $m(M) = d(M)$ or $m(M) - d(M) = 1$ by (2.1)–(2.3).

If $m(M) = d(M)$, then by [1, Theorem 1.4], we have $M = Z_p^m$ or $M = Z_p^m{:}Z_q$ for some primes $p \neq q$ and a positive integer m.

For $M = Z_p^m$, since M is the image of any Sylow p-subgroup of G module $\Phi(G)$ and $\Phi(G)$ consists of the non-generators of G, we have $|G| = p^n$ with $n > 1$. But by [1, Theorem 1.6], we see that $m(G) = d(G)$, a contradiction.

For $M = Z_p^m{:}Z_q$, by [5, Theorem 3.4], we conclude that $G = P{:}Q$, where P is the unique Sylow p-subgroup and Q is a cyclic Sylow q-subgroup. Again by [1, Theorem 1.6], we obtain that $m(G) = d(G)$, a contradiction.

If $m(M) - d(M) = 1$, then by Proposition 2.5, $M \cong Z_p^m \times Z_q$, $Z_p^m{:}R$, $Z_p^m{:}(R{:}Z_q)$, $Z_p^m{:}Z_{r_1^n r_2}$, $(Z_p^m \times Z_q){:}R$ or $(Z_p^m \times Z_q){:}(R : Z_{r_1})$.

For $M \cong Z_p^m \times Z_q$, we have $G = P \times Q$, where P is a Sylow p-subgroup of G, and Q is a Sylow q-subgroup of G. Indeed, suppose that $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, then $P\Phi(G)/\Phi(G) \in \mathrm{Syl}_p(G/\Phi(G))$ and $Q\Phi(G)/\Phi(G) \in \mathrm{Syl}_q(G/\Phi(G))$. Since $P\Phi(G)/\Phi(G) \trianglelefteq H$ and $Q\Phi(G)/\Phi(G) \trianglelefteq H$, we see that $P\Phi(G) \trianglelefteq G$ and $Q\Phi(G) \trianglelefteq G$. It follows that $P \trianglelefteq G$ and $Q \trianglelefteq G$. Note that

$(P\Phi(G)/\Phi(G))(Q\Phi(G)/\Phi(G)) = G/\Phi(G) = H$, we conclude that $G = PQ\Phi(G) = PQ$, and so $G = P\times Q$.

By the same techniques as above, for $M \cong Z_p^m{:}R$, we have $G = P{:}R$, where P is a p-subgroup of G and R is an r-subgroup of G; for $M \cong Z_p^m{:}(R{:}Z_q)$, we have $G = P{:}R{:}Q$, where P is a p-subgroup of G, R is an r-subgroup of G and Q is a q-subgroup of G; for $M \cong Z_p^m{:}Z_{r_1^n r_2}$, we have $G = P{:}R$, where P is a p-subgroup of G and R is an $\{r_1, r_2\}$-subgroup of G; for $M \cong (Z_p^m\times Z_q){:}R$, we have $G = H{:}R$, where H is a $\{p, q\}$-subgroup of G and R is an r-subgroup of G; for $M \cong (Z_p^m\times Z_q){:}(R : Z_{r_1})$, we have $G = H{:}R$, where H is a $\{p, q\}$-subgroup of G and R is an $\{r, r_1\}$-subgroup of G. Thus we have completed the proof of Theorem 1.1. □

We conclude with the following interesting and important problems.

Problem 3.1 Explore the sufficient conditions for Theorem 1.1, and characterize all finite solvable groups G with $m(G) - d(G) = 1$.

Problem 3.2 Classify all finite unsolvable groups G with $m(G) - d(G) = 1$. In particular, find all finite nonabelian simple groups G with $m(G) - d(G) = 1$.

We will work on these Problems in a subsequent paper in this series of papers.

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