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Improved Bound of the Fourth Hankel Determinant for a Class of Analytic Functions with Bounded Turnings Involving Cardioid Domain

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Abstract In the paper, a class of functions with bounded turnings involving cardioid domain, are studied in the region of the unit disc. The bounds of $|a_5|,|a_6|,|a_7|$ and the fourth Hankel determinant are obtained, which are more accurate than those obtained by Srivastava.

Keywords analytic functions; starlike functions; convex functions; Schwarz function; cardioid domain; Hankel determinant

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the unit disc $\Delta=\{z\in\mathcal{C}:|z|<1\}$ and $\mathcal{H}$ denote the subclass of $\mathcal{A}$ consisting of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad z\in\Delta. \tag{1.1}$$

Also, let $\mathcal{S}$ be the subclass of $\mathcal{H}$ consisting of all univalent functions in Δ. For a function $f\in\mathcal{S}$, Bieberbach conjectured in 1916 that $|a_n|\le n$ for all $n\ge 2$. In the year 1985, De Branges [1] proved Bieberbach conjecture. During the period, a lot of coefficients results were established for some subfamilies of $\mathcal{S}$. For example, the class $\mathcal{S}^*$ of starlike functions, $\mathcal{C}$ of convex functions and $\mathcal{R}$ of bounded turning functions:

$$\mathcal{S}^*=\big\{f\in\mathcal{S}:\frac{zf'(z)}{f(z)}\prec\frac{1+z}{1-z}\big\},$$
$$\mathcal{C}=\big\{f\in\mathcal{S}:1+\frac{zf''(z)}{f'(z)}\prec\frac{1+z}{1-z}\big\}, \tag{1.2}$$

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$$\mathcal{R} = \{f \in \mathcal{S} : f'(z) \prec \frac{1+z}{1-z}\},$$

where $\prec$ represents the subordination.

By varying the function right hand side of subordinations in (1.2), various subclasses of $\mathcal{S}$ were introduced by recent researchers [2–6]. From among these subfamilies, the families that are associated with cardioid domain are the following:

$$\mathcal{S}^*_{\text{car}} = \{f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \vartheta(z)\},$$

$$\mathcal{C}_{\text{car}} = \{f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \vartheta(z)\},$$

$$\mathcal{R}_{\text{car}} = \{f \in \mathcal{S} : f'(z) \prec \vartheta(z)\}, \tag{1.3}$$

where $\vartheta(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, which maps the open unit disc Δ onto a region bounded by the cardioid equation given as $(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1)^2 = 0$. The families $\mathcal{S}^*_{\text{car}}$ and $\mathcal{C}_{\text{car}}$ were introduced by Sharma et al. [4] who studied the coefficient estimates. The subfamily R_{car} of analytic functions was established by Shi et al. [5] who obtained the non-sharp bound of the third Hankel determinant. In [6], the sharp bound of the third Hankel determinant for subfamily R_{car} was obtained.

For given parameters $q, n \in \mathcal{N} = \{1, 2, 3...\}$, the Hankel determinant $\mathcal{H}D_{q,n}(f)$ for a function $f \in \mathcal{S}$ of the series form (1.1) was given by Pommerenke [7,8] as

$$\mathcal{HD}_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+k} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

It is not hard to note that the third and the fourth Hankel determinants of f can be given by

$$\mathcal{HD}_{3,1}(f) = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5,$$

$$\mathcal{HD}_{4,1}(f) = a_7\mathcal{HD}_{3,1}(f) - a_6\Lambda_1 + a_5\Lambda_2 - a_4\Lambda_3, \tag{1.4}$$

where Λ_1, Λ_2 and Λ_3 are third-order determinants given by

$$\Lambda_1 = a_3a_6 - a_4a_5 - a_2^2a_6 + a_2a_3a_5 + a_2a_4^2 - a_4a_3^2, \tag{1.5}$$

$$\Lambda_2 = a_4a_6 - a_5^2 - a_2a_3a_6 + a_2a_4a_5 + a_3^2a_5 - a_3a_4^2, \tag{1.6}$$

$$\Lambda_3 = a_2a_4a_6 - a_2a_5^2 - a_3^2a_6 + 2a_3a_4a_5 - a_4^3, \tag{1.7}$$

respectively.

The computation of the third-order determinant of $\mathcal{HD}_{3,1}(f)$ is very difficult. Recently, a few authors achieved the sharp bounds of $|\mathcal{HD}_{3,1}(f)|$ for some subfamilies of univalent functions, see, for example, [9–18]. The estimation of fourth Hankel determinant $\mathcal{HD}_{4,1}(f)$ for the first time for the class of bounded turning functions was obtained by Arif et al. [19]. Recently, the fourth

Hankel determinant $\mathcal{HD}_{4,1}(f)$ for functions with bounded turnings involving cardioid domain was achieved by Srivastava et al. [20], it was proved that

$$|\mathcal{HD}_{4,1}(f)| \leq 2.7555..., \quad f \in \mathcal{R}_{\text{car}}.$$

Motivated by the aforementioned investigations, we restudy the bound of fourth Hankel determinant $\mathcal{HD}_{4,1}(f)$ for the family defined in (1.3), which is more accurate than that obtained by Srivastava et al. [20]. Let $\mathcal{B}_0$ denote the class of Schwarz functions v normalized by

$$v(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad z \in \Delta. \tag{1.8}$$

Lemma 1.1 ([21]) *Let $v \in \mathcal{B}_0$ be given by (1.8). Then*

$$|c_2| \leq 1 - |c_1|^2, \ |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}, \ |c_4| \leq 1 - |c_1|^2 - |c_2|^2,$$
$$|c_5| \leq 1 - |c_1|^2 - |c_2|^2 - \frac{|c_3|^2}{1 + |c_1|}, \quad |c_6| \leq 1 - |c_1|^2 - |c_2|^2 - |c_3|^2.$$

Lemma 1.2 ([22, 23]) *Let $v \in \mathcal{B}_0$ be given by (1.8). Then*

$$|c_2 c_4 - c_3^2| \leq 1 - |c_1|^2, \quad |c_1 c_3 - c_2^2| \leq 1 - |c_1|^2.$$

Lemma 1.3 ([6]) *If $f \in \mathcal{R}_{\text{car}}$, then*

$$|\mathcal{HD}_{3,1}(f)| \leq \frac{1}{9}.$$

This bound is achieved with optimal precision.

Lemma 1.4 ([20]) *If $f \in \mathcal{R}_{\text{car}}$, then $|a_2| \leq \frac{2}{3}$, $|a_3| \leq \frac{4}{9}$, $|a_4| \leq \frac{1}{3}$.*

2. Main results

In the following theorem, we find the bounds of $|a_5|, |a_6|$ and $|a_7|$ for $f \in \mathcal{R}_{\text{car}}$.

Theorem 2.1 *If $f \in \mathcal{R}_{\text{car}}$ has the power expansion series given by (1.1), then*

$$|a_5| \leq \frac{128}{405}, \ |a_6| \leq \frac{64}{243}, \ |a_7| \leq \frac{128}{567}.$$

The estimates are sharp.

Proof If $f \in \mathcal{R}_{\text{car}}$ has the form (1.1), then there exists $v \in \mathcal{B}_0$ with $v(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ such that

$$\begin{aligned} f'(z) =& \vartheta((v(z))) = 1 + \frac{4c_1}{3} z + \frac{4c_2 + 2c_1^2}{3} z^2 + \frac{4c_3 + 4c_1 c_2}{3} z^3 + \frac{4c_4 + 4c_1 c_3 + 2c_2^2}{3} z^4 + \\ & \frac{4c_5 + 4c_1 c_4 + 4c_2 c_3}{3} z^5 + \frac{4c_6 + 4c_1 c_5 + 4c_2 c_4 + 2c_3^2}{3} z^6 + \cdots. \end{aligned}$$

From the above relation, it follows that

$$\begin{cases} a_2 = \frac{2c_1}{3}, \\ a_3 = \frac{4c_2+2c_1^2}{9}, \\ a_4 = \frac{c_3+c_1c_2}{3}, \\ a_5 = \frac{4c_4+4c_1c_3+2c_2^2}{15}, \\ a_6 = \frac{2c_5+2c_1c_4+2c_2c_3}{9}, \\ a_7 = \frac{4c_6+4c_1c_5+4c_2c_4+2c_3^2}{21}. \end{cases} \tag{2.1}$$

Applying the triangle inequality and Lemma 1.1 to a_5 in formula (2.1), we obtain

$$\begin{aligned} |a_5| &\le \frac{1}{15}[4|c_4| + 4|c_1||c_3| + 2|c_2|^2] \\ &\le \frac{1}{15}[4(1-|c_1|^2-|c_2|^2) + 4|c_1|(1-|c_1|^2 - \frac{|c_2|^2}{1+|c_1|}) + 2|c_2|^2] \\ &= \frac{1}{15}[4+4|c_1| - 4|c_1|^2 - 4|c_1|^3 - \frac{2+6|c_1|}{1+|c_1|}|c_2|^2] \\ &= \frac{1}{15}[4+4x-4x^2-4x^3 - \frac{2+6x}{1+x}y^2] \\ &= \frac{1}{15}F(x,y), \end{aligned}$$

where $x = |c_1| \in [0,1]$, $y = |c_2| \in [0, 1-x^2]$.

Consider

$$\frac{\partial F}{\partial x} = -\frac{4+12x}{1+x}y \le 0,$$

$F(x,y)$ is a decreasing function of $y \in [0,1-x^2]$. Hence $F(x,y) \le F(x,0) \le F(\frac{1}{3},0) = \frac{128}{27}$.

Thus, we have

$$|a_5| \le \frac{128}{405}.$$

The equality holds for $c_1 = \frac{1}{3}$, $c_3 = c_4 = \frac{8}{9}$ and $c_2 = 0$.

The application of the triangle inequality and Lemma 1.1 to a_6 in formula (2.1), we yield

$$\begin{aligned} |a_6| &\le \frac{1}{9}[2|c_5| + 2|c_1||c_4| + 2|c_2||c_3|] \\ &\le \frac{1}{9}[2(1-|c_1|^2-|c_2|^2 - \frac{|c_3|^2}{1+|c_1|}) + 2|c_1|(1-|c_1|^2-|c_2|^2) + 2|c_2||c_3|] \\ &= \frac{1}{9}[2+2|c_1| - 2|c_1|^2 - 2|c_1|^3 - (2+2|c_1|)|c_2|^2 + 2|c_2||c_3| - \frac{2}{1+|c_1|}|c_3|^2] \\ &= \frac{1}{9}[2+2x-2x^2-2x^3-(2+2x)y^2+2yz - \frac{2}{1+x}z^2] \\ &= \frac{1}{9}F_1(x,y,z), \end{aligned}$$

where $x = |c_1| \in [0,1]$, $y = |c_2| \in [0,1-x^2]$, $y = |c_3| \in [0, 1-x^2-\frac{y^2}{1+x}]$.

Now we need to maximize the function F_1 in the closed cube $\Theta : [0,1] \times [0,1-x^2] \times [0,1-$

$x^2 - \frac{y^2}{1+x}]$. For this, we observe that

$$\frac{\partial F_1}{\partial x} = 2 - 4x - 6x^2 - 2y^2 + \frac{2z^2}{(1+x)^2},$$
$$\frac{\partial F_1}{\partial y} = -(4+4x)y + 2z, \quad \frac{\partial F_1}{\partial z} = 2y - \frac{4z}{1+x}.$$

Setting $\frac{\partial F_1}{\partial x} = 0, \frac{\partial F_1}{\partial y} = 0$ and $\frac{\partial F_1}{\partial z} = 0$, we achieve $\begin{cases} x = \frac{1}{3}, \\ y = 0, \\ z = 0. \end{cases}$ Therefore, there are no critical points in the closed cube Θ.

Interior of all the four faces of cube Θ.

(1) On the face $z = 0$,

$$F_1(x, y, 0) = 2 + 2x - 2x^2 - 2x^3 - (2+2x)y^2 = \gamma_1(x, y).$$

Differentiating γ_1 partially with respect to y, we get

$$\frac{\partial \gamma_1}{\partial y} = -(4+4x)y \leq 0.$$

Thus, $\gamma_1(x, y) \leq \gamma_1(x, 0) \leq \gamma_1(\frac{1}{3}, 0) = \frac{64}{27}$.

(2) On the face $y = 0$,

$$F_1(x, 0, z) = 2 + 2x - 2x^2 - 2x^3 - \frac{2z^2}{1+x} = \gamma_2(x, z).$$

Differentiating γ_2 partially with respect to z, we have

$$\frac{\partial \gamma_2}{\partial z} = -\frac{4z}{1+x} \leq 0.$$

Thus, γ_2 has no critical points in $(0,1) \times (1, 1-x^2)$.

(3) On the face $x = 0$,

$$F_1(0, y, z) = 2 - 2y^2 + 2yz - 2z^2 = \gamma_3(y, z).$$

It is easy to know that the function γ_3 has no critical points in $(0,1) \times (0, 1-x^2)$.

(4) On the face $z = 1 - x^2 - \frac{y^2}{1+x}$,

$$\begin{aligned} F_1(x, y, 1 - x^2 - \frac{y^2}{1+x}) &= 4x - 4x^3 + (2-2x^2)y + \frac{-2x^2 - 8x + 2}{1+x}y^2 - \frac{2y^3}{1+x} - \frac{2y^4}{(1+x)^3} \\ &= \gamma_4(x, y). \end{aligned}$$

Differentiating partially with respect to x, then with respect to y, we achieve

$$\frac{\partial \gamma_4}{\partial x} = 4 - 12x^2 - 4xy + \frac{-2x^2 - 4x - 10}{(1+x)^2}y^2 + \frac{2y^3}{(1+x)^2} + \frac{6y^4}{(1+x)^4}$$

and

$$\frac{\partial \gamma_4}{\partial y} = 2 - 2x^2 + \frac{-4x^2 - 16x + 4}{1+x}y - \frac{6y^2}{1+x} - \frac{8y^3}{(1+x)^3}.$$

A numerical calculation shows that there exists a solution $(0.553702\cdots, 0.114243)$ at which the function γ_4 attains the maximum of $1.666663\cdots$.

On the edges of cube Θ:

(1) On the edge $y=0$ and $z=0$,

$$F_1(x,0,0)=2+2x-2x^2-2x^3=\gamma_5(x)\le\gamma_5(\frac{1}{3})=\frac{64}{27}.$$

(2) On the edge $x=0$ and $y=0$,

$$F_1(0,0,z)=2-2z^2=\gamma_6(z)\le\gamma_6(0)=2.$$

(3) On the edge $x=0$ and $z=0$,

$$F_1(0,y,0)=2-2y^2=\gamma_7(y)\le\gamma_7(0)=2.$$

(4) On the edge $z=0$ and $y=1-x^2$,

$$F_1(x,1-x^2,0)=2x^2+2x^3-2x^4-2x^5=\gamma_8(x)\le\gamma_8(\frac{1+\sqrt{41}}{10})=0.8621\cdots.$$

(5) On the edge $y=0$ and $z=1-x^2$,

$$F_1(x,0,1-x^2)=4x-4x^3=\gamma_9(x)\le\gamma_9(\frac{\sqrt{3}}{3})=\frac{8\sqrt{3}}{9}=1.5396\cdots.$$

(6) On the edge $x=0$ and $z=1-y^2$,

$$F_1(0,y,1-y^2)=2y+2y^2-2y^3-2y^4=\gamma_{10}(y)\le\gamma_{10}(\frac{1+\sqrt{17}}{8})=1.2394\cdots.$$

Thus, we get $|a_6|\le\frac{64}{243}$. The equality holds for $c_1=\frac{1}{3}$, $c_4=c_5=\frac{8}{9}$ and $c_2=c_3=0$.

Applying the triangle inequality and Lemma 1.1 to a_7 in formula (2.1), we achieve

$$\begin{aligned}|a_7|\le&\frac{1}{21}[4|c_6|+4|c_1||c_5|+4|c_2||c_4|+2|c_3|^2]\\ \le&\frac{1}{21}[4(1-|c_1|^2-|c_2|^2-|c_3|^2)+4|c_1|(1-|c_1|^2-|c_2|^2-\frac{|c_3|^2}{1+|c_1|})+\\ &4|c_2|(1-|c_1|^2-|c_2|^2)+2|c_3|^2]\\ =&\frac{1}{21}[-\frac{6|c_1|+2}{1+|c_1|}|c_3|^2+(4-4|c_1|^2)|c_2|-(4+4|c_1|)|c_2|^2-4|c_2|^3+4+\\ &4|c_1|-4|c_1|^2-4|c_1|^3]\\ \le&\frac{1}{21}[(4-4|c_1|^2)|c_2|-(4+4|c_1|)|c_2|^2-4|c_2|^3+4+4|c_1|-4|c_1|^2-4|c_1|^3].\end{aligned}$$

Setting $x=|c_1|\in[0,1]$, $y=|c_2|\in[0,1-x^2]$, we yield

$$|a_7|\le\frac{1}{21}[(4-4x^2)y-(4+4x)y^2-4y^3+4+4x-4x^2-4x^3]=\zeta(x,y).$$

Partial derivative of ζ with respect to x and then with respect to y, we get

$$\frac{\partial\zeta}{\partial x}=-8xy-4y^2+4-8x-12x^2,\quad\frac{\partial\zeta}{\partial y}=-8(1+x)y-12y^2+4-4x^2.$$

A numerical calculation shows that the function ζ has its maximum at $x=y=\frac{-1+\sqrt{7}}{6}$ and therefore,

$$\zeta(x,y)\le\zeta(\frac{-1+\sqrt{7}}{6},\frac{-1+\sqrt{7}}{6})=4.247574\cdots.$$

Consider the end points of $[0,1]\times[0,1-x^2]$. When $x=0$, $\zeta(0,y)=4+4y-4y^2-4y^3\le\zeta(0,\frac{1}{3})=\frac{128}{27}$. When $y=0$, $\zeta(x,0)=4+4x-4x^2-4x^3\le\zeta(\frac{1}{3},0)=\frac{128}{27}$. When $y=1-x^2$, $\zeta(x,1-x^2)=8x^2+4x^3-12x^4-4x^5+4x^6$ which has its maximum value 2.2094 at $x=0.6958$.

Therefore, we obtain

$$|a_7|\le\frac{128}{567}.$$

The equality holds for $c_1=\frac{1}{3}$, $c_4=c_5=c_6=\frac{8}{9}$ and $c_2=c_3=0$. We complete our proof. □

Remark 2.2 Theorem 2.1 provides an improvement estimates obtained by Srivastava et al. [20, Theorem 3.1].

Theorem 2.3 *If $f\in\mathcal{R}_{\rm car}$ has the power expansion series given by (1.1), then*

$$|\mathcal{HD}_{4,1}(f)|\le 0.123268\cdots.$$

Proof Let $f\in\mathcal{R}_{\rm car}$. From (2.1) and (1.5), we achieve

$$\begin{aligned}\Lambda_1=&\frac{1}{1215}[108c_1c_2c_4-18c_1c_3^2+28c_1^2c_2c_3-86c_1c_2^3-12c_1^3c_4+34c_1^3c_2^2+\\&28c_1^4c_3-14c_2^2c_3-20c_1^5c_2-108c_3c_4+120c_2c_5-60c_1^2c_5].\end{aligned}$$

The above equation can be written as

$$\begin{aligned}\Lambda_1=&\frac{1}{1215}\{18c_1(c_2c_4-18c_3^2)+90c_1c_2c_4+28c_1c_2(c_1c_3-c_2^2)-58c_1c_2^3-12c_1^3c_4+\\&34c_1^3c_2^2+28c_1^4c_3-14c_2^2c_3-20c_1^5c_2-108c_3c_4+120c_2c_5-60c_1^2c_5\}.\end{aligned}$$

Applying the triangle inequality, we obtain

$$\begin{aligned}|\Lambda_1|\le&\frac{1}{1215}[18|c_1||c_2c_4-18c_3^2|+90|c_1||c_2||c_4|+28|c_1||c_2||c_1c_3-c_2^2|+58|c_1||c_2|^3+12|c_1|^3|c_4|+\\&34|c_1|^3|c_2|^2+28|c_1|^4|c_3|+14|c_2|^2|c_3|+20|c_1|^5|c_2|+108|c_3||c_4|+120|c_2||c_5|+60|c_1|^2|c_5|].\end{aligned}$$

Now, using Lemmas 1.1 and 1.2, we yield

$$\begin{aligned}|\Lambda_1|\le&\frac{1}{1215}[18|c_1|+60|c_1|^2-6|c_1|^3-60|c_1|^4-12|c_1|^5+(120+46|c_1|-120|c_1|^2-46|c_1|^3+\\&20|c_1|^5)|c_2|+(-60|c_1|^2+22|c_1|^3)|c_2|^2+(-120+40|c_1|)|c_2|^3+(108-108|c_1|^2+\\&28|c_1|^4-94|c_2|^2)|c_3|-\frac{60|c_1|^2+120|c_2|}{1+|c_1|}|c_3|^2].\end{aligned}$$

Putting $x=|c_1|$, $y=|c_2|$ and $z=|c_3|$, we obtain $|\Lambda_1|\le\frac{1}{1215}\Upsilon(x,y,z)$, where

$$\begin{aligned}\Upsilon(x,y,z)=&18x+60x^2-6x^3-60x^4-12x^5+(120+46x-120x^2-46x^3+20x^5)y+\\&(-60x^2+22x^3)y^2+(-120+40x)y^3+\\&(108-108x^2+28x^4-94y^2)z-\frac{60x^2+120y}{1+x}z^2.\end{aligned}$$

Now, we have to search the maximum value of Υ in the closed cube Θ: $[0,1]\times[0,1-x^2]\times[0,1-x^2-\frac{y^2}{1+x}]$. For this, we have

$$\frac{\partial\Upsilon}{\partial x}=18+120x-18x^2-240x^3-60x^4+(46-240x-138x^2+100x^4)y+$$

$$(-120x+66x^2)y^2+40y^3+(-216x+112x^3)z-\frac{120x+60x^2-120y}{(1+x)^2}z^2,$$

$$\frac{\partial\Upsilon}{\partial y}=120+46x-120x^2-46x^3+20x^5+(-120x^2+44x^3)y+ (-360+120x)y^2-188yz-\frac{120}{1+x}z^2$$

and

$$\frac{\partial\Upsilon}{\partial z}=108-108x^2+28x^4-94y^2-\frac{120x^2+240y}{1+x}z.$$

Setting $\frac{\partial\Upsilon}{\partial x}=0$, $\frac{\partial\Upsilon}{\partial y}=0$ and $\frac{\partial\Upsilon}{\partial z}=0$, we get

$$\begin{cases} x_1=2.393956\cdots, \\ y_1=2.202810\cdots, \\ z_1=-0.132319\cdots, \end{cases} \quad \begin{cases} x_2=2.412800\cdots, \\ y_2=-3.442046\cdots, \\ z_2=18.348070\cdots, \end{cases} \quad \begin{cases} x_3=-1.766450\cdots, \\ y_3=-1.285548\cdots, \\ z_3=1.299184\cdots. \end{cases}$$

Thus, there is no critical point in the closed cube Θ.

Interior of all the four faces of cube Θ.

(1) On the face $z=0$,

$$\begin{aligned}\Upsilon(x,y,0)=&18x+60x^2-6x^3-60x^4-12x^5+(120+46x-120x^2-46x^3+20x^5)y+\\ &(-60x^2+22x^3)y^2+(-120+40x)y^3=\chi_1(x,y)\\ \le&\chi_1(0.403245...,0.584647...)=59.263493\cdots.\end{aligned}$$

(2) On the face $y=0$,

$$\begin{aligned}\Upsilon(x,0,z)=&18x+60x^2-6x^3-60x^4-12x^5+(108-108x^2+28x^4)z-\frac{60x^2}{1+x}z^2\\ =&\chi_2(x,z).\end{aligned}$$

A numerical calculation shows that there is no critical point in $(0,1)\times(0,1-x^2)$.

(3) On the face $x=0$,

$$\Upsilon(0,y,z)=120y-120y^3+(108-94y^2)z-120yz^2=\chi_3(y,z).$$

A numerical calculation shows that there's no critical point in $(0,1)\times(0,1-y^2)$.

(4) On the face $z=1-x^2-\frac{y^2}{1+x}$,

$$\begin{aligned}\Upsilon(x,y,1-x^2-\frac{y^2}{1+x})=&-\frac{120}{(1+x)^3}y^5+\frac{34x^2+188x+94}{(1+x)^3}y^4+\frac{40x^2-320x+120}{1+x}y^3+\\ &\frac{-6x^4-64x^3+262x^2-94x-202}{1+x}y^2+\frac{20x^6+20x^5-166x^4-166x^3+166x^2+166x}{1+x}y+\\ &\frac{-28x^7-100x^6+64x^5+190x^4-162x^3-198x^2+126x+108}{1+x}\\ =&\chi_4(x,y).\end{aligned}$$

Partial derivative of $\chi_4(x,y)$ with respect to x and then with respect to y, we yield

$$\frac{\partial\chi_4}{\partial x}=\frac{360}{(1+x)^4}y^5+\frac{-34x^2-308x-94}{(1+x)^4}y^4+\frac{40x^2+80x-440}{(1+x)^2}y^3+$$

$$\frac{-18x^4-152x^3+70x^2+524x+108}{(1+x)^2}y^2+$$
$$\frac{100x^6+200x^5-398x^4-996x^3-332x^2+332x+166}{(1+x)^2}y+$$
$$\frac{-168x^7-696x^6-344x^5+896x^4+436x^3-684x^2-396x+18}{(1+x)^2}$$

and

$$\frac{\partial\chi_4}{\partial y}=-\frac{600}{(1+x)^3}y^4+\frac{136x^2+752x+376}{(1+x)^3}y^3+\frac{120x^2-960x+360}{1+x}y^2+$$
$$\frac{-12x^4-128x^3+524x^2-188x-404}{1+x}y+\frac{20x^6+20x^5-166x^4-166x^3+166x^2+166x}{1+x}.$$

Setting $\frac{\partial\chi_4}{\partial x}=0$ and $\frac{\partial\chi_4}{\partial y}=0$, we find a solution of $(0.051358..., 0.022059...)$. Therefore, we have

$$\chi_4(x,y)\le\chi_4(0.051358...,0.022059...)=108.453905....$$

On the edges of Θ.

(1) For $z=0$ and $x=0$,

$$\Upsilon(0,y,0)=120y-120y^3=\chi_5(y)\le\chi_5(\frac{\sqrt{3}}{3})=\frac{80\sqrt{3}}{3}=46.1880\cdots.$$

(2) For $y=0$ and $z=0$,

$$\Upsilon(x,0,0)=18x+60x^2-6x^3-60x^4-12x^5=\chi_6(x)\le\chi_6(0.6866...)=23.5367\cdots.$$

(3) For $x=0$ and $y=0$,

$$\Upsilon(0,0,z)=108z\le 108.$$

(4) For $z=0$ and $y=1-x^2$,

$$\Upsilon(x,1-x^2,0)=-38x^7+60x^6+130x^5-180x^4-196x^3+120x^2+104x$$
$$=\chi_7(x)\le\chi_7(0.565627...)=52.1116\cdots.$$

(5) For $y=0$ and $z=1-x^2$,

$$\Upsilon(x,0,1-x^2)=108+18x-216x^2+54x^3+136x^4-72x^5-28x^6$$
$$=\chi_8(x)\le\chi_8(0.042435\cdots)=108.3794\cdots.$$

(6) For $x=0$ and $z=1-y^2$,

$$\Upsilon(0,0,1-x^2)=108-202y^2+120y^3+94y^4-120y^5=\chi_9(y)\le\chi_9(0)=108.$$

Thus, we achieve

$$|\Lambda_1|\le\frac{108.453905...}{1215}=0.08926247\cdots. \tag{2.2}$$

From (2.1) and (1.6), we achieve

$$\Lambda_2=\frac{1}{6075}[150c_2c_3^2-112c_2^2c_4-482c_1c_2^2c_3+52c_2^4+80c_1^5c_3-150c_1^4c_2^2+$$
$$730c_1^2c_2c_4-222c_1^2c_3^2+450c_3c_5-54c_1c_3c_4+50c_1c_2c_5-432c_4^2-200c_1^3c_5-$$

$$120c_1^4c_4+180c_1^3c_2c_3+40c_1^2c_2^3+40c_1^4c_2]. \tag{2.3}$$

The Eq. (2.3) can be written as

$$\begin{aligned}\Lambda_2 =&\frac{1}{6075}[150c_2(c_3^2-c_2c_4)+38c_2^2c_4-482c_2^2(c_1c_3-c_2^2)-430c_2^4+80c_1^4(c_1c_3-c_2^2)-\\&70c_1^4c_2^2+222c_1^2(c_2c_4-c_3^2)+508c_1^2c_2c_4+450c_3c_5-54c_1c_3c_4+50c_1c_2c_5-432c_4^2-\\&200c_1^3c_5-120c_1^4c_4+180c_1^3c_2c_3+40c_1^2c_2^3+40c_1^4c_2].\end{aligned}$$

By applying the triangle inequality, we yield

$$\begin{aligned}|\Lambda_2| \le&\frac{1}{6075}[150|c_2||c_3^2-c_2c_4|+38|c_2|^2|c_4|+482|c_2|^2|c_1c_3-c_2^2|+430|c_2|^4+\\&80|c_1|^4|c_1c_3-c_2^2|+70|c_1|^4|c_2|^2+222|c_1|^2|c_2c_4-c_3^2|+508|c_1|^2|c_2||c_4|+\\&450|c_3||c_5|+54|c_1||c_3||c_4|+50|c_1||c_2||c_5|+432|c_4|^2+200|c_1|^3|c_5|+\\&120|c_1|^4|c_4|+180|c_1|^3|c_2||c_3|+40|c_1|^2|c_2|^3+40|c_1|^4|c_2|].\end{aligned}$$

By using Lemmas 1.1 and 1.2, we get

$$\begin{aligned}|\Lambda_2| \le&\frac{1}{6075}\{432-642|c_1|^2+200|c_1|^3+410|c_1|^4-200|c_1|^5-200|c_1|^6+(150+50|c_1|+\\&358|c_1|^2-50|c_1|^3-468|c_1|^4)|c_2|+(-344+344|c_1|^2-200|c_1|^3-50|c_1|^4)|c_2|^2+\\&(-50|c_1|-468|c_1|^2)|c_2|^3+824|c_2|^4+[450+54|c_1|-450|c_1|^2-54|c_1|^3+\\&180|c_1|^3|c_2|-(450+54|c_1|)|c_2|^2]|c_3|-\frac{200|c_1|^3+50|c_1||c_2|}{1+|c_1|}|c_3|^2-\frac{450}{1+|c_1|}|c_3|^3\}.\end{aligned}$$

By setting $x=|c_1|$, $y=|c_2|$ and $z=|c_3|$, we yield

$$|\Lambda_2| \le \frac{1}{6075}\Psi(x,y,z),$$

where

$$\begin{aligned}\Psi(x,y,z) =&432-642x^2+200x^3+410x^4-200x^5-200x^6+(150+50x+\\&358x^2-50x^3-468x^4)y+(-344+344x^2-200x^3-50x^4)y^2+\\&(-50x-468x^2)y^3+824y^4+[450+54x-450x^2-54x^3+\\&180x^3y-(450+54x)y^2]z-\frac{200x^3+50xy}{1+x}z^2-\frac{450}{1+x}z^3.\end{aligned}$$

By differentiating partially with respect to x, y and z, respectively, we have

$$\begin{aligned}\frac{\partial\Psi}{\partial x} =&-1284x+600x^2+1640x^3-1000x^4-1200x^5+(50+716x-150x^2-1872x^3)y+\\&(688x-600x^2-200x^3)y^2+(-50-936x)y^3+\\&(54-900x-162x^2+540x^2y-54y^2)z-\frac{600x^2+400x^3+50y}{(1+x)^2}z^2+\frac{450}{(1+x)^2}z^3,\\ \frac{\partial\Psi}{\partial y} =&150+50x+358x^2-50x^3-468x^4+(-688+688x^2-400x^3-100x^4)y+\\&(-150x-1404x^2)y^2+3296y^3+[180x^3-(900+108x)y]z-\frac{50x}{1+x}z^2\end{aligned}$$

and

$$\frac{\partial \Psi}{\partial z} = 450 + 54x - 450x^2 - 54x^3 + 180x^3y - (450 + 54x)y^2 - \frac{400x^3 + 100xy}{1 + x}z - \frac{1350}{1 + x}z^2.$$

A numerical calculation shows that there is no critical point in $(0,1)\times(0,1-y^2)\times(0,1-x^2-\frac{y^2}{1+x})$.

Interior of all the four faces of cube Θ.

(1) On the face $z = 0$,

$$\begin{aligned}\Psi(x, y, 0) =&432 - 642x^2 + 200x^3 + 410x^4 - 200x^5 - 200x^6+\\ &(150 + 50x + 358x^2 - 50x^3 - 468x^4)y+\\ &(-344 + 344x^2 - 200x^3 - 50x^4)y^2 + (-50x - 468x^2)y^3 + 824y^4\\ =&\varsigma_1(x, y).\end{aligned}$$

A numerical calculation shows that there is no critical point in $(0, 1) \times (0, 1 - y^2)$.

(2) On the face $y = 0$,

$$\begin{aligned}\Psi(x, 0, z) =&432 - 642x^2 + 200x^3 + 410x^4 - 200x^5 - 200x^6+\\ &(450 + 54x - 450x^2 - 54x^3)z - \frac{200x^3}{1 + x}z^2 - \frac{450}{1 + x}z^3\\ =&\varsigma_2(x, z).\end{aligned}$$

A numerical calculation shows that the function ς has no critical points in $(0, 1) \times (0, 1 - x^2)$.

(3) On the face $x = 0$,

$$\Psi(0, y, z) = 432 + 150y - 344y^2 + 824y^4 + (450 - 450y^2)z - 450z^3 = \varsigma_3(y, z).$$

Differentiating partially with respect to y and z, we get

$$\frac{\partial \varsigma_3}{\partial y} = 150 - 688y + 3296y^3 - 900yz, \quad \frac{\partial \varsigma_3}{\partial z} = 450 - 450y^2 - 1350z^2.$$

By setting $\frac{\partial \varsigma_3}{\partial y} = 0$ and $\frac{\partial \varsigma_3}{\partial z} = 0$, we find two solutions of $(0.118807\cdots, 0.690084\cdots)$ and $(0.541616\cdots, 0.617582\cdots)$. Thus, we obtian

$$\begin{aligned}\varsigma_3(y, z) &\leq \max\{\varsigma_3(0.130804\cdots, 0.572390\cdots), \varsigma_3(0.504638\cdots, 0.498444\cdots)\}\\ &= \max\{614.75504\cdots, 584.983613\cdots\} = 614.75504\cdots.\end{aligned}$$

(4) On the face $z = 1 - x^2 - \frac{y^2}{1+x}$,

$$\begin{aligned}&\Upsilon(x, y, 1 - x^2 - \frac{y^2}{1 + x}) = \frac{450}{(1 + x)^4}y^6 - \frac{50x}{(1 + x)^3}y^5 + \frac{678x^3 + 4380x^2 + 3426x - 76}{(1 + x)^3}y^4+\\ &\frac{-648x^3 - 618x^2 + 50x}{1 + x}y^3 + \frac{-50x^5 - 596x^4 + 2452x^3 - 160x^2 - 2252x + 106}{1 + x}y^2+\\ &\frac{-180x^6 - 698x^5 - 338x^4 + 588x^3}{1 + x}y + \frac{408x^2 + 150x + 150}{1 + x}y+\\ &\frac{-400x^7 + 104x^6 + 1114x^5 - 398x^4 - 1650x^3 - 138x^2 + 936x + 432}{1 + x}\\ &= \varsigma_4(x, y).\end{aligned}$$

Differentiating partially with respect to x and y, we yield

$$\begin{aligned}\frac{\partial\varsigma_4}{\partial x}=&-\frac{1800}{(1+x)^5}y^6+\frac{100x-50}{(1+x)^4}y^5+\frac{-2346x^2+1908x+3654}{(1+x)^4}y^4+\\&\frac{-1296x^3-2562x^2-1236x+50}{(1+x)^2}y^3+\frac{-200x^5-2038x^4+2520x^3+7196x^2-320x-2358}{(1+x)^2}y^2+\\&\frac{-900x^6-3872x^5-4504x^4-176x^3}{(1+x)^2}y+\frac{2172x^2+816x}{(1+x)^2}y+\\&\frac{-2400x^7-2280x^6+5080x^5+4376x^4-4892x^3-5088x^2-276x+504}{(1+x)^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\varsigma_4}{\partial y}=&\frac{2700}{(1+x)^4}y^5-\frac{250x}{(1+x)^3}y^4+\frac{2712x^3+17520x^2+13704x-304}{(1+x)^3}y^3+\\&\frac{-1944x^3-1854x^2+150x}{1+x}y^2+\frac{-100x^5-1192x^4+4904x^3-320x^2-4504x+212}{1+x}y+\\&\frac{-180x^6-698x^5-338x^4+588x^3+408x^2+150x+150}{1+x}.\end{aligned}$$

Setting $\frac{\partial\varsigma_4}{\partial x}=0$, $\frac{\partial\varsigma_4}{\partial y}=0$, we obtain

$$\begin{cases}x_1=0.0613621\cdots,\\y_1=-0.545017\cdots.\end{cases}$$

Therefore, the function $\varsigma_4(x,y)$ has no critical points in $(0,1)\times(0,1-y^2)$.

On the edges of Θ.

(1) For $z=0$ and $x=0$,

$$\Upsilon(0,y,0)=432+150y-344y^2+824y^4=\varsigma_5(y)\le\varsigma_5(1)=1062.$$

(2) For $y=0$ and $z=0$,

$$\Upsilon(x,0,0)=432-642x^2+200x^3+410x^4-200x^5-200x^6=\varsigma_6(x)\le\varsigma_6(0)=432.$$

(3) For $x=0$ and $y=0$,

$$\Upsilon(0,0,z)=432+450z-450z^3=\varsigma_7(z)\le\varsigma_7(\frac{\sqrt{3}}{3})=605.1983\cdots.$$

(4) For $z=0$ and $y=1-x^2$,

$$\begin{aligned}\Upsilon(x,1-x^2,0)&=1242x^8-150x^7-3988x^6+100x^5+4850x^4+50x^3-3166x^2+1062\\&=\varsigma_8(x)\le\varsigma_8(0)=1062.\end{aligned}$$

(5) For $y=0$ and $z=1-x^2$,

$$\begin{aligned}\Upsilon(x,0,1-x^2)&=432+504x-642x^2-1008x^3+610x^4+504x^5-400x^6\\&=\varsigma_9(x)\le\varsigma_9(0.267707\cdots)=505.2537\cdots.\end{aligned}$$

(6) For $x=0$ and $z=1-y^2$,

$$\Upsilon(0,0,1-y^2)=432+150y+106y^2-76y^4+450y^5=\varsigma_{10}(y)\le\varsigma_{10}(1)=1062.$$

Thus, we achieve

$$|\Lambda_2| \leq \frac{1062}{6075} = \frac{354}{2025}. \tag{2.4}$$

From (2.1) and (1.7), we achieve

$$\begin{aligned}\Lambda_3 =&\frac{1}{18225}[315c_1c_2c_3^2 - 224c_1c_2^2c_4 - 80c_2^3c_3 + 820c_1^3c_2c_4 - 144c_1^3c_3^2 + \\ &1440c_2c_3c_4 - 675c_3^3 + 520c_1^4c_2c_3 - 315c_1^3c_2^3 + 900c_1c_3c_5 - 800c_2^2c_5 - 989c_1^2c_2^2c_3 + \\ &504c_1c_2^4 - 108c_1^2c_3c_4 + 100c_1^2c_2c_5 - 864c_1c_4^{-}200c_1^4c_5 - 200c_1^5c_4].\end{aligned} \tag{2.5}$$

The Eq. (2.5) can be written as

$$\begin{aligned}\Lambda_3 =&\frac{1}{18225}[235c_1c_2(c_3^2 - c_2c_4) + 80c_2c_3(c_1c_3 - c_2^2) + 11c_1c_2^2c_4 + 144c_1^3(c_2c_4 - c_3^2) + \\ &676c_1^3c_2c_3 + 675c_3(c_2c_4 - c_3^2) + 765c_2c_3c_4 + 520c_1^3c_2(c_1c_3 - c_2^2) + 205c_1^3c_2^3 + \\ &900c_5(c_1c_3 - c_2^2) + 100c_2^2c_5 + 989c_1c_2^2(c_2^2 - c_1c_3) - 485c_1c_2^4 - 108c_1^2c_3c_4 + \\ &100c_1^2c_2c_5 - 864c_1c_4^2 - 200c_1^4c_5 - 200c_1^5c_4].\end{aligned} \tag{2.6}$$

By using the triangle inequality, Lemmas 1.1 and 1.2 in (2.6), we achieve

$$\begin{aligned}\Lambda_3 \leq&\frac{1}{18225}[235|c_1||c_2||c_3^2 - c_2c_4| + 80|c_2||c_3||c_1c_3 - c_2^2| + 11|c_1||c_2|^2|c_4| + \\ &144|c_1|^3|c_2c_4 - c_3^2| + 676|c_1|^3|c_2||c_3| + 675|c_3||c_2c_4 - c_3^2| + 765|c_2||c_3||c_4| + \\ &520|c_1|^3||c_2||c_1c_3 - c_2^2| + 205|c_1|^3|c_2|^3 + 900|c_5||c_1c_3 - c_2^2| + 100|c_2|^2|c_5| + \\ &989|c_1||c_2|^2|c_2^2 - c_1c_3| + 485|c_1||c_2|^4 + 108|c_1|^2|c_3||c_4| + 100|c_1|^2|c_2||c_5| + \\ &864|c_1||c_4|^2 + 200|c_1|^4|c_5| + 200|c_1|^5|c_4|] \\ \leq&\frac{1}{18225}[900 + 864|c_1| - 1800|c_1|^2 - 1584|c_1|^3 + 1100|c_1|^4 + 920|c_1|^5 - 200|c_1|^6 - 200|c_1|^7 + \\ &(235|c_1| + 100|c_1|^2 + 961|c_1|^3 - 100|c_1|^4 - 1196|c_1|^5)|c_2| + (-800 - 728|c_1| + 800|c_1|^2 + \\ &728|c_1|^3 - 200|c_1|^4 - 200|c_1|^5)|c_2|^2 + (-100|c_1|^2 - 471|c_1|^3)|c_2|^3 + (-100 + 1338|c_1|)|c_2|^4 + \\ &[675 - 567|c_1|^2 - 108|c_1|^4 + (845 - 845|c_1|^2)y - 108|c_1|^2|c_2|^2 - 765|c_2|^3]|c_3| + \\ &\frac{-900 + 900|c_1|^2 - 200|c_1|^4 - 100|c_1|^2|c_2| - 100|c_2|^2}{1 + |c_1|}|c_3|^2].\end{aligned}$$

Setting $x = |c_1|$, $y = |c_2|$ and $z = |c_3|$, we obtain

$$|\Lambda_3| \leq \frac{1}{18225}\Gamma(x, y, z),$$

where

$$\begin{aligned}\Gamma(x, y, z) =&900 + 864x - 1800x^2 - 1584x^3 + 1100x^4 + 920x^5 - 200x^6 - 200x^7 + \\ &(235x + 100x^2 + 961x^3 - 100x^4 - 1196x^5)y + (-800 - 728x + 800x^2 + \\ &728x^3 - 200x^4 - 200x^5)y^2 + (-100x^2 - 471x^3)y^3 + (-100 + 1338x)y^4 + \\ &[675 - 567x^2 - 108x^4 + (845 - 845x^2)y - 108x^2y^2 - 765y^3]z + \\ &\frac{-900 + 900x^2 - 200x^4 - 100x^2y - 100y^2}{1 + x}z^2.\end{aligned}$$

Differentiating partially with respect to x, y and z, respectively, we achieve

$$\begin{aligned}\frac{\partial\Gamma}{\partial x} =&864-3600x-4752x^2+4400x^3+4600x^4-1200x^5-1400x^6+\\&(235+200x+2883x^2-400x^3-5980x^4)y+(-728+1600x+2184x^2-800x^3-\\&1000x^4)y^2+(-200x-1413x^2)y^3+1338y^4+(-1134x-432x^3-1690xy-216xy^2)z+\\&\frac{900+1800x+900x^2-800x^3-600x^4-200xy-100x^2y+100y^2}{(1+x)^2}z^2,\\ \frac{\partial\Gamma}{\partial y} =&235x+100x^2+961x^3-100x^4-1196x^5+(-1600-1456x+1600x^2+\\&1456x^3-400x^4-400x^5)y+(-300x^2-1413x^3)y^2+(-400+5352x)y^3+\\&(845-845x^2-2165x^2y-2295y^2)z-\frac{100x^2+200y}{1+x}z^2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\Gamma}{\partial z} =&675-567x^2-108x^4+(845-845x^2)y-108x^2y^2-765y^3+\\&\frac{-1800+1800x^2-400x^4-200x^2y-200y^2}{1+x}z.\end{aligned}$$

Setting $\frac{\partial\Gamma}{\partial x}=0$, $\frac{\partial\Gamma}{\partial y}=0$, $\frac{\partial\Gamma}{\partial y}=0$ and by numerical calculation, there is no critical points in $(0,1)\times(0,1-x^2)\times(1-x^2-\frac{y^2}{1+x})$.

Interior of all the four faces of cube Θ.

(1) On the face $z=0$,

$$\begin{aligned}\Gamma(x,y,0) =&900+864x-1800x^2-1584x^3+1100x^4+920x^5-200x^6-200x^7+\\&(235x+100x^2+961x^3-100x^4-1196x^5)y+(-800-728x+800x^2+\\&728x^3-200x^4-200x^5)y^2+(-100x^2-471x^3)y^3+(-100+1338x)y^4\\=&\xi_1(x,y)\le\xi_1(0.201640\cdots,0.032348\cdots)=991.104373\cdots.\end{aligned}$$

(2) On the face $y=0$,

$$\begin{aligned}\Psi(x,0,z) =&900+864x-1800x^2-1584x^3+1100x^4+920x^5-200x^6-200x^7+\\&[675-567x^2-108x^4]z+\frac{-900+900x^2-200x^4}{1+x}z^2\\=&\xi_2(x,z)\le\xi_2(0.214485\cdots,0.458265\cdots)=1138.213005\cdots.\end{aligned}$$

(3) On the face $x=0$,

$$\Psi(0,y,z)=900-800y^2-100y^4+(675+845y-765y^3)z+(-900-100y^2)z^2.$$

By differentiating partially with respect to y and z, we achieve

$$\frac{\partial\xi_3}{\partial y}=-200yz^2+(845-2295y^2)z-1600y-400y^3$$

and

$$\frac{\partial\xi_3}{\partial z}=(-1800-200y^2)z+675+845y-765y^3.$$

Setting $\frac{\partial \xi_3}{\partial y}=0$, $\frac{\partial \xi_3}{\partial z}=0$, we obtian

$$\begin{cases} y_1 = 0.209326\cdots, \\ z_1 = 0.467094\cdots, \end{cases} \quad \begin{cases} y_2 = 1.762705\cdots, \\ z_2 = -.836442\cdots, \end{cases} \quad \begin{cases} y_3 = -1.520013\cdots, \\ z_3 = 0.918262\cdots. \end{cases}$$

Therefore, we have

$$\xi_3(y,z) \le \xi_3(0.209326\cdots, 0.467094\cdots) = 1062.069626\cdots.$$

(4) On the face $z = 1 - x^2 - \frac{y^2}{1+x}$,

$$\begin{aligned}
\Upsilon(x,y,1-x^2-\frac{y^2}{1+x}) = &-\frac{100}{(1+x)^5}y^6 - \frac{765x^4+3060x^3+4490x^2+3060x+765}{(1+x)^5}y^5 \\
&\frac{108x^6+1570x^5+5100x^4+7660x^3+6160x^2+1538x-800}{(1+x)^5}y^4+ \\
&\frac{-471x^4-771x^3+945x^2-765x-1610}{1+x}y^3+ \\
&\frac{-200x^6-692x^5+1044x^4+3220x^3-1069x^2-3328x+225}{1+x}y^2+ \\
&\frac{-1296x^6-451x^5+1906x^4-629x^3-1455x^2+1080x+845}{1+x}y- \\
&400x^7+108x^6+2020x^5+459x^4-21384x^3-1242x^2+1764x+675 \\
=&\xi_4(x,y).
\end{aligned}$$

Differentiating partially with respect to x and y, we yield

$$\begin{aligned}
\frac{\partial \xi_4}{\partial x} =& \frac{500}{(1+x)^6}y^6 - \frac{765x^4+3060x^3+4290x^2+2660x+165}{(1+x)^6}y^5 - 2800x^6+ \\
&648x^5+10100x^4+1836x^3-64152x^2-2484x+1764+ \\
&\frac{108x^6+648x^5+2750x^4+5080x^3+4500x^2+6168x+5538}{(1+x)^6}y^4+ \\
&\frac{-1413x^4-3426x^3-1368x^2+1890x+845}{(1+x)^2}y^3 + \frac{-100x^6-3968x^5-328x^4+10616x^3}{(1+x)^2}y^2+ \\
&\frac{8591x^2-2678x-3553}{(1+x)^2}y^2 + \frac{-6480x^6-9580x^5+3463x^4+6366x^3}{(1+x)^2}y
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \xi_4}{\partial y} =& -\frac{600}{(1+x)^5}y^5 + \frac{3825x^4+15300x^3+22450x^2+15300x+3825}{(1+x)^5}y^4+ \\
&\frac{432x^6+6280x^5+20400x^4}{(1+x)^5}y^3 + \frac{30640x^3+24640x^2+6152x-3200}{(1+x)^5}y^3+ \\
&\frac{-1413x^4-2313x^3+2835x^2-2295x-4830}{1+x}y^2+ \\
&\frac{-400x^6-1384x^5+2088x^4+6440x^3-2138x^2-6656x+450}{1+x}y+ \\
&\frac{-1296x^6-451x^5+1906x^4-629x^3-1455x^2+1080x+845}{1+x}.
\end{aligned}$$

A numerical calculation shows that there is no critical point in $(0,1)\times(0,1-x^2)$.

On the edges of Θ.

(1) For $z=0$ and $x=0$,

$$\Gamma(0,y,0)=900-800y^2-100y^4=\xi_5(y)\le\xi_5(0)=900.$$

(2) For $y=0$ and $z=0$,

$$\begin{aligned}\Gamma(x,0,0)&=900+864x-1800x^2-1584x^3+1100x^4+920x^5-200x^6-200x^7\\&=\xi_6(x)\le\xi_6(0.199174)=990.1687\cdots.\end{aligned}$$

(3) For $x=0$ and $y=0$,

$$\Gamma(0,0,z)=900+675z-900z^2=\xi_7(z)\le\xi_7(0.375\cdots)=1026.5625\cdots.$$

(4) For $z=0$ and $y=1-x^2$,

$$\begin{aligned}\Gamma(x,1-x^2,0)=&1609x^9-200x^8-4641x^7+1200x^6+\\&5820x^5-2000x^4-4497x^3+1000x^2+1709x\\=&\xi_8(x)\le\xi_8(0.486255\cdots)=584.6785\cdots.\end{aligned}$$

(5) For $y=0$ and $z=1-x^2$,

$$\begin{aligned}\Gamma(x,0,1-x^2)&=675+1764x-1242x^2-3384x^3+459x^4+2020x^5+108x^6-400x^7\\&=\xi_9(x)\le\xi_9(0.333266\cdots)=1013.6\cdots.\end{aligned}$$

(6) For $x=0$ and $z=1-y^2$,

$$\begin{aligned}\Upsilon(0,y,1-y^2)&=675+845y+225y^2-1610y^3-800y^4+765y^5-100y^6\\&=\xi_{10}(y)\le\xi_{10}(0.434933\cdots)=935.2290\cdots.\end{aligned}$$

Thus, we achieve

$$|\Lambda_3\le\frac{1062.069626\cdots}{18225}=0.05827543\cdots. \tag{2.7}$$

From (1.4), (2.2), (2.4), (2.7), Lemmas 1.3, 1.4 and Theorem 2.1, we get

$$\begin{aligned}|\mathcal{HD}_{4,1}(f)|&\le|a_7||\mathcal{HD}_{3,1}(f)|+|a_6||\Lambda_1|+|a_5||\Lambda_2|+|a_4||\Lambda_3|\\&=\frac{128}{567}\times\frac{1}{9}+\frac{64}{243}\times0.08926247+\frac{128}{405}\times\frac{354}{2025}+\frac{1}{3}\times0.05827543\\&=0.123268\cdots.\end{aligned}$$

The proof of Theorem 2.3 is completed. □

Remark 2.4 The estimates in Theorem 2.3 provide improvement over the estimates derived by Srivastava et al. [20, Theorem 4.1].

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Conflict of Interest The authors declare no conflict of interest.

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