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# Entire Solutions of Fermat-Type Partial Differential-Difference Equations in $\mathbb{C}^2$

**Caoqiang TANG, Zhigang HUANG***
*School of Mathematical Sciences, Suzhou University of Science and Technology, Jiangsu* 215009*, P. R. China*

**Abstract** In this paper, we mainly investigate the forms of entire solutions for certain Fermat-type partial differential-difference equations in $\mathbb{C}^2$ by using Nevanlinna's theory of several complex variables.

**Keywords** Fermat-type; entire solution; partial differential-difference equation

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## 1. Introduction and main results

In physics and engineering, partial differential equations are frequently used to explain a variety of phenomena, such as the Laplace and Poisson equations and steady-state phenomena. Classical partial differential equations in physics include the wave equation and the heat equation, for instance. It is well known that the Fermat-type equation is strongly related to both the heat equation and the wave equation. A Lagrangian function in the Lagrangian functional describing the "action" of a system, for example, is the Fermat-type equation $(\frac{\partial f(z_1,z_2)}{\partial z_1})^2+(\frac{\partial f(z_1,z_2)}{\partial z_2})^2=1$ in $\mathbb{C}^2$. It also occurs in particle mechanics. In recent years, there has been a great deal of interest in the fascinating field of research known as the study of solutions of Fermat-type functional equations in $\mathbb{C}^n$. The main purpose of this article is to deal with the forms of solutions for several complex differential-difference equations of Fermat-type.

In 1995, Wiles [1] and Wiles and Taylor [2] proved that there are nontrivial rational solutions to the equation

$$x^n+y^n=1, \tag{1.1}$$

when $n=2$, while there are no nontrivial rational solutions when $n\geq 3$. There have been similar functional investigations to this prestigious result. In 1933, Cartan [3] studied the generalized Fermat-type functional equations

$$f^n(z)+g^m(z)=1, \tag{1.2}$$

and deduced that all entire solutions $f,g$ must be constants if $mn>m+n$. For $m=n=2$, Iyer [4] has proved that the entire solutions of Eq. (1.2) are $f(z)=\sin h(z)$ and $g(z)=\cos h(z)$,

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* Corresponding author
E-mail address: tang334349@163.com (Caoqiang TANG); alexehuang@sina.com (Zhigang HUANG)

where $h(z)$ is an entire function. No other solutions exist for this case. For $m = n \geq 3$, Montel [5] has concluded that Eq. (1.2) has no transcendental entire solutions. Gross [6] then proved that for $m = n \geq 4$, Eq. (1.2) has no transcendental meromorphic solutions. We can see that there is a long history of research on the existence and form of the solutions of Fermat-type functional equations, and this topic still deserves further investigation. For various aspects of solutions of Fermat-type difference, differential or difference-differential equations, we refer the reader to [7–14].

Recently, there has been a lot of interest in the captivating area of research known as the study of solutions of Fermat-type equations in $\mathbb{C}^n$. One goal in this field is to achieve symmetry in the case of a single variable. Before delving into a detailed study of such equations in $\mathbb{C}^n$, let us review some established results for the analogous study of Fermat-type functional equations in several complex variables. Hereinafter, we denote $z + \omega = (z_1 + \omega_1, z_2 + \omega_2)$ for any $z = (z_1, z_2), \omega = (\omega_1, \omega_2) \in \mathbb{C}^2$. In 1995, Khavinson [15] obtained the first result, that is, the entire solution $u$ of the partial differential equation $u_{z_1}^2 + u_{z_2}^2 = 1$ in $\mathbb{C}^2$ must be linear. From the fact that the surface $z_1^m + z_2^m = 1$ in $\mathbb{C}^2$ is a Kobayashi hyperbolic manifold, it can be deduced that there are no non-constant holomorphic curves $(f_1, f_2)$ from the complex plane $\mathbb{C}^2$ to the surface, and thus every entire solution of the partial differential equations

$$(\frac{\partial u}{\partial z_1})^m + (\frac{\partial u}{\partial z_2})^m = 1, \quad m \geq 3 \tag{1.3}$$

must also be linear. In 2018, Xu and Cao [16] investigated the entire solutions of Fermat-type partial differential-difference equations in $\mathbb{C}^2$ by utilizing the difference version of logarithmic derivative lemma of several complex variables.

**Theorem 1.1** *Let $c = (c_1, c_2)$ be a constant in $\mathbb{C}^2$. Then any transcendental entire solution with finite order of the partial differential-difference equation*

$$(\frac{\partial f}{\partial z_1})^2 + f^2(z) = 1 \tag{1.4}$$

*has the form of $f(z_1, z_2) = \sin(Az_1 + B)$, where $A, B \in \mathbb{C}$ are constants satisfying $Ae^{iAc_1} = 1$; in the special case whenever $c_2 \neq 0$, we have $f(z_1, z_2) = \sin(z_1 + B)$.*

**Remark 1.2** From Theorem 1.1, we find that it is surprising that all entire solutions of finite order should be a function of one variable $z_1$. Given that $f(z)$ have two partial derivatives, it is natural to consider the following case, which is proved by Gui et al. [17].

**Theorem 1.3** ([17]) *Let $c = (c_1, c_2) \in \mathbb{C}^2$, and $\alpha, \beta$ be constants in $\mathbb{C}$ that are not equal to zero at the same time. Then any transcendental entire solution with finite order for the Fermat-type partial differential-difference equation*

$$(\alpha\frac{\partial f}{\partial z_1} + \beta\frac{\partial f}{\partial z_2})^2 + f^2(z + c) = 1 \tag{1.5}$$

*has the following form:*

$$f(z_1, z_2) = \frac{A_1 e^{a_1 z_1 + a_2 z_2 + H(z)} + A_2 e^{-a_1 z_1 - a_2 z_2 - H(z)}}{2},$$

*where $H(z)=H(s_1)$ is a polynomial in $s_1$ satisfying $(\alpha c_2-\beta c_1)H'\equiv 0$, $s_1=c_2z_1-c_1z_2$, $A_1A_2=1$, $A_1, A_2$ are constants in $\mathbb{C}$, and $c, \alpha_1, \alpha_2, \alpha, \beta$ satisfy one of the following cases:*

*(i) $\alpha\alpha_1+\beta\alpha_2=i$ and $L(c)=2k\pi i$, where $L(c)=\alpha_1c_1+\alpha_2c_2$, here and below $k$ is an integer;*

*(ii) $\alpha\alpha_1+\beta\alpha_2=-i$ and $L(c)=(2k+1)\pi i$.*

**Example 1.4** In view of Theorem 1.2, if $\alpha=1$, $\beta=0$, and $c_2\neq 0$, then we can conclude that every finite order transcendental entire solution $f$ of Eq. (1.5) has the form

$$f(z)=\frac{e^{L(z)+B}+e^{-(L(z)+B)}}{2},$$

where $a_1=i$ and $L(c)=2k\pi i$, or $a_1=-i$ and $L(c)=(2k+1)\pi i$.

Naturally, Gui [17] proceeded to discuss the existence and form of solutions for the following Fermat-type partial differential-difference equation:

$$(\alpha\frac{\partial f}{\partial z_1}+\beta\frac{\partial f}{\partial z_2})^2+[\gamma_1 f(z+c)-\gamma_2 f(z)]^2=1, \tag{1.6}$$

where $\alpha$, $\beta$, $\gamma_1$, $\gamma_2$ are constants in $\mathbb{C}$.

**Theorem 1.5** ([17]) *Let $c=(c_1,c_2)\in\mathbb{C}^2$ and $c_1$, $c_2$, $\alpha$, $\beta$, $\gamma_1$, $\gamma_2$ be nonzero constants in $\mathbb{C}$ such that $\alpha c_2-\beta c_1\neq 0$. Let $f$ be a finite order transcendental entire solution of the Fermat type partial differential-difference equation (1.6).*

*(i) If $\alpha\frac{\partial f}{\partial z_1}+\beta\frac{\partial f}{\partial z_2}$ is not a constant, then the solution $f(z)$ has the following form:*

$$f(z)=\frac{e^{a_1z_1+a_2z_2+B}-e^{-(a_1z_1+a_2z_2+B)}}{2(a_1\alpha+a_2\beta)}+e^{\eta(\alpha z_2-\beta z_1)}G(\alpha z_2-\beta z_1), \tag{1.7}$$

*where $G(u)$ is a finite order period function with period $\alpha c_2-\beta c_1$, $a_1, a_2, B, \eta$ are constants in $\mathbb{C}$.*

*Moreover,*

*($i_1$) if $\gamma_1\neq\pm\gamma_2$, then $\eta=\frac{\log\gamma_2-\log\gamma_1}{\alpha c_2-\beta c_1}$, and either $a_1\alpha+a_2\beta=i(\gamma_1-\gamma_2)$, $L(c)=2k\pi i$, or $a_1\alpha+a_2\beta=-i(\gamma_2+\gamma_1)$, $L(c)=(2k+1)\pi i$, where $L(c)=a_1c_1+a_2c_2$;*

*($i_2$) if $\gamma_1=\gamma_2$, then $\eta\equiv 0, i(a_1\alpha+a_2\beta)=2\gamma_1$ and $L(c)=(2k+1)\pi i$;*

*($i_3$) if $\gamma_1=-\gamma_2$, then $\eta=\frac{\log(-1)}{\alpha c_2-\beta c_1}, i(a_1\alpha+a_2\beta)=-2\gamma_1$ and $L(c)=2k\pi i$;*

*(ii) If $\alpha\frac{\partial f}{\partial z_1}+\beta\frac{\partial f}{\partial z_2}$ is a constant, then the solution $f(z)$ has the following form:*

$$f(z)=D_1z_1+D_2(\alpha z_2-\beta z_1)+[e^{\eta(\alpha z_2-\beta z_1)}G(\alpha z_2-\beta z_1)+\tau], \tag{1.8}$$

*where $\tau$, $D_1$, $D_2$ are constants in $\mathbb{C}$.*

*Moreover,*

*($ii_1$) if $\gamma_1=\gamma_2$, then $\eta=0, \tau\in\mathbb{C}$ and*

$${\gamma_1}^2[D_1c_1+D_2(\alpha c_2-\beta c_2)]^2=1-(\alpha D_1)^2; \tag{1.9}$$

*($ii_2$) if $\gamma_1\neq\gamma_2$, then $\eta=\frac{\log\gamma_2-\log\gamma_1}{\alpha c_2-\beta c_1}$, $D_1=D_2=0$, and $\tau$, $\gamma_1$, $\gamma_2$ satisfy*

$$(\gamma_1-\gamma_2)\tau=\pm 1. \tag{1.10}$$

**Remark 1.6** Obviously, we can see that Eq. (1.6) can be reduced to Eq. (1.5) when $\gamma_1 = 1$ and $\gamma_2 = 0$. Therefore, in Theorem 1.3, we consider only the case where $\alpha$, $\beta$, $\gamma_1$, $\gamma_2$ are non-zero constants in $\mathbb{C}$.

Thus, Theorems 1.2 and 1.3 suggest the following question as an open problem.

**Problem 1.7** What will happen when the first-order partial derivatives are replaced by second-order partial derivatives and the number of difference operators changes to three?

To generalize and establish a result that combines Theorems 1.2 and 1.3, in our investigation we consider the following partial differential-difference equation.

**Theorem 1.8** *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *with* $c_1 \neq 0$, $c_2 \neq 0$ *and* $c_1 + c_2 \neq 0$. *Assume that* $\alpha, \beta, \lambda, \gamma, \delta, \varphi \in \mathbb{C}$ *and* $\alpha, \beta, \lambda$ *are not all zero. If* $f$ *is a finite order transcendental entire solution of the Fermat-type partial differential-difference equation*

$$\left(\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2} + \beta\frac{\partial^2 f(z)}{\partial {z_2}^2} + \lambda\frac{\partial^2 f(z)}{\partial z_1 \partial z_2}\right)^2 + (\gamma f(z+2c) + \delta f(z+c) + \varphi f(z))^2 = 1, \tag{1.11}$$

*then one of the following cases holds.*

*(i) If two out of* $\gamma, \delta, \varphi$ *are equal to zero, then the following three subcases arise:*

*($i_1$) if* $\gamma = \varphi = 0$, *then*

$$f(z) = \frac{i}{2\delta}(e^{-L(z)-B+L(c)} - e^{L(z)+B-L(c)}),$$

*where* $L(z) = a_1 z_1 + a_2 z_2$, $a_1, a_2, B \in \mathbb{C}$ *satisfies* $e^{2L(c)} = -1$;

*($i_2$) if* $\delta = \varphi = 0$, *then*

$$f(z) = \frac{i}{2\gamma}(e^{L(z)+B} - e^{-L(z)-B}),$$

*where* $L(z) = a_1 z_1 + a_2 z_2$, $a_1, a_2, B \in \mathbb{C}$ *satisfies* $e^{2L(c)} = -1$;

*($i_3$) if* $\gamma = \delta = 0$, *then*

$$f(z) = \frac{i}{2\varphi}(e^{-L(z)-B} - e^{L(z)+B}),$$

*where* $L(z) = a_1 z_1 + a_2 z_2$, $a_1, a_2, B \in \mathbb{C}$ *satisfies* $e^{2L(c)} = -1$;

*(ii) If only one of* $\gamma, \delta, \varphi$ *is equal to zero, then*

$$f(z_1, z_2) = \frac{e^{L(z)+B} + e^{-(L(z)+B)}}{2(a_1 r + a_2 m)(a_1 q + a_2 n)} + \phi_1\left(z_2 - \frac{m z_1}{r}\right) + \phi_2\left(z_2 - \frac{n z_1}{q}\right),$$

*where* $\phi_1$ *and* $\phi_2$ *are entire functions with finite order in* $\mathbb{C}^2$, $m, n, r, q \in \mathbb{C}$ *satisfying* $rq = \alpha$, $mn = \beta$, $rn + qm = \lambda$ *and* $L(z) = a_1 z_1 + a_2 z_2$, $a_1, a_2, B \in \mathbb{C}$.

*Moreover,*

*($ii_1$) if* $\gamma = 0$, *then*

$$e^{L(c)}(i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2) - \varphi) = e^{-L(c)}(-i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2) - \varphi) = \delta;$$

*($ii_2$) if* $\delta = 0$, *then*

$$e^{L(c)}(i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2) - \varphi) = e^{-L(c)}(-i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2) - \varphi) = \gamma;$$

*($ii_3$) if $\varphi = 0$, then either $\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2 = 0$, we have $e^{2L(c)} = 1$, or*

$$\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2 \neq 0,$$

*we have*

$$\delta e^{L(c)} = -\gamma,\ A_1 e^{2L(z)+2B} - A_2 = 0 \text{ or } A_2 e^{L(c)} = \delta,\ A_1 e^{2L(z)+2B} + \gamma = 0,$$

*where* $A_1 = \gamma e^{2L(c)} + \delta e^{L(c)} + i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2)$, $A_2 = i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2)$;

*(iii) if $\gamma, \delta, \varphi$ are all non-zero, then $f(z)$ must satisfy the following form:*

$$f(z_1, z_2) = \frac{e^{L(z)+B} + e^{-(L(z)+B)}}{2(a_1 r + a_2 m)(a_1 q + a_2 n)} + \phi_1(z_2 - \frac{m z_1}{r}) + \phi_2(z_2 - \frac{n z_1}{q}),$$

*where $\phi_1$ and $\phi_2$ are entire functions with finite order in $\mathbb{C}^2$, $m, n, r, q \in \mathbb{C}$ satisfying $rq = \alpha$, $mn = \beta$, $rn + qm = \lambda$ and $L(z) = a_1 z_1 + a_2 z_2$, $a_1, a_2, B \in \mathbb{C}$ satisfying one of the following two situations:*

$$\delta e^{L(c)} = -\gamma, A_1 e^{2L(z)+2B} + A_2 = 0 \text{ or } A_2 e^{L(c)} = -\delta, A_1 e^{2L(z)+2B} + \gamma = 0,$$

*where* $A_1 = \gamma e^{2L(c)} + \delta e^{L(c)} + \varphi + i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2)$, $A_2 = \varphi - i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2)$.

**Remark 1.9** The following examples illustrate the existence of transcendental entire solutions of three situations of (1.11).

**Example 1.10** Let $a_1 = 1$, $a_2 = -1$ such that $L(z) = z_1 - z_2$, $c_1 = \pi i$, $c_2 = 0$, $\alpha = 1$, $\beta = 1$, $\lambda = 2$, $\delta = 1$, $\gamma = \varphi = 0$ and $B = 0$. Then in view of Theorem 1.8 (i), it can be easily verified that $f(z) = -\frac{1}{2i} e^{L(z)} + \frac{1}{2i} e^{-L(z)}$ is a solution of (1.11).

**Example 1.11** Let $a_1 = \frac{1}{2}$, $a_2 = \frac{\sqrt{3}}{2}$ such that $L(z) = \frac{1}{2} z_1 + \frac{\sqrt{3}}{2} z_2$, $c_1 = 2\pi i$, $c_2 = -\sqrt{3}\pi i$, $\alpha = 1$, $\beta = 1$, $\lambda = 0$, $\delta = i$, $\gamma = \varphi = 0$ and $B = 0$. Then in view of Theorem 1.8 (i), it can be easily verified that $f(z) = -\frac{1}{2} e^{L(z) + \frac{1}{2}\pi i} + \frac{1}{2} e^{-L(z) - \frac{1}{2}\pi i}$ is a solution of (1.11).

**Example 1.12** Let $a_1 = 1$, $a_2 = 0$ such that $L(z) = z_1$, $c_1 = -\pi$, $c_2 = \frac{1}{2}\pi i$, $\alpha = 1$, $\beta = 1$, $\lambda = 0$, $\delta = i$, $\varphi = -i$, $\gamma = 0$ and $B = 0$. Then in view of Theorem 1.8 (ii), it can be easily verified that $f(z_1, z_2) = \frac{2}{15}(e^{iz_1 + z_2} + e^{-iz_1 - z_2})$ is a solution of (1.11).

Before proceeding, we assume that the reader is familiar with the basic notations of Nevanlinna theory for a function $f = f(z_1, ..., z_m)$ on $\mathbb{C}^m$, such as the counting function $N(r, f)$, the proximity function $m(r, f)$, the characteristic function $T(r, f) = m(r, f) + N(r, f)$ and $S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure.

## 2. Preliminary lemmas

In this section, we present some necessary lemmas which will contribute to prove the results of this paper.

**Lemma 2.1** ([18]) *For an entire function $F$ in $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then* *there exist a canonical function* $f_F$ *and a function* $g_F \in \mathbb{C}^n$ *such that* $F(z) = f_F(z)e^{g_F(z)}$*. For the special case* $n = 1$*,* $f_F$ *is the canonical product of Weierstrass.*

**Remark 2.2** Here we denote by $\rho(n_F)$ the order of $n_F$ which is the counting function of zeros of $F$.

**Lemma 2.3** ([19]) *If* $g$ *and* $h$ *are entire functions in the complex plane* $\mathbb{C}$ *and* $g(h)$ *is an entire function of finite order, then there are only two possible cases: either*

*(a) the internal function* $h$ *is a polynomial and the external function* $g$ *is of finite order; or else*

*(b) the internal function* $h$ *is not a polynomial but a function of finite order, and the external function* $g$ *is of zero order.*

**Lemma 2.4** ([20]) *Suppose that* $a_0(z), a_1(z), \ldots, a_n(z)$ $(n \geq 1)$ *are meromorphic function on* $\mathbb{C}^m$ *and* $g_0(z), g_1(z), \ldots, g_n(z)$ *are entire function on* $\mathbb{C}^m$*, such that* $g_j(z) - g_k(z)$ *are not constants for* $0 < j < k < n$*. If*

$$\sum_{j=0}^{n} a_j(z)e^{g_j(z)} \equiv 0$$

*and*

$$T(r, a_j) = o(T(r)), \quad j = 0, 1, \ldots, n,$$

*outside of a possible exceptional set of finite linear measure, where* $T(r) = \min_{0\leq j\leq k\leq n} T(r, e^{g_j - g_k})$*, then* $a_j \equiv 0$ $(j = 1, 2, \ldots, n)$*.*

**Lemma 2.5** ([20]) *Let* $f_j$ $(j = 1, 2, \ldots, n)$ *be meromorphic functions in* $\mathbb{C}^n$ *such that* $f_k$ $(k = 1, 2, \ldots, n-2)$ *is not a constant, and* $\sum_{j=1}^{n} f_j \equiv 1$*, and such that*

$$\sum_{j=1}^{n} \{N(r, \frac{1}{f_j}) + (n-1)\sum_{j=1}^{n} \bar{N}(r, f_j)\} < (\lambda + o(1))T(r, f_k)$$

*for all* $r$ *outside possibly a set with finite logarithmic measure, where* $\lambda < 1$ *is a positive number. Then either* $f_{n-1} \equiv 1$ *or* $f_n \equiv 1$*.*

**Remark 2.6** Here $N_2(r, \frac{1}{f})$ is the counting function of zeros of $f$ in $|z| \leq r$, where the simple zero is counted once, and the multiple zero is counted twice.

## 3. Proof of Theorem 1.8

Now, we will start to prove Theorem 1.8.

**Proof of Theorem 1.8** Firstly, we assume that $f(z)$ is a finite order transcendental entire solution of (1.11).

Case I. If two out of $\gamma, \delta, \varphi$ are equal to zero, then we have three subcases to deal with. Since the other two cases can be proved by similar reasoning, we will only prove the case where

$\gamma=\varphi=0$. So (1.11) can be reduced to

$$(\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2})^2+(\delta f(z+c))^2=1. \tag{3.1}$$

We rewrite (3.1) as

$$[P(f)+i\delta f(z+c)][P(f)-i\delta f(z+c)]=1, \tag{3.2}$$

where

$$P(f)=\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2}.$$

Since $f$ is a transcendental entire solution of finite order, by (3.2) we know that both $P(f)+i\delta f(z+c)$ and $P(f)-i\delta f(z+c)$ have no zeros and no poles. Thus, by applying Lemmas 2.1 and 2.2, we can conclude that there exists a non-constant polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\begin{cases}(\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2})+i\delta f(z+c)=e^{p(z)},\\ (\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2})-i\delta f(z+c)=e^{-p(z)}.\end{cases} \tag{3.3}$$

By (3.3), a simple calculation shows that

$$\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2}=\frac{e^{p(z)}+e^{-p(z)}}{2} \tag{3.4}$$

and

$$\delta f(z+c)=\frac{e^{p(z)}-e^{-p(z)}}{2i}. \tag{3.5}$$

We now need to verify the form of $p(z)$. From Eqs. (3.4) and (3.5), we can deduce that

$$Q_1(z)e^{p(z+c)+p(z)}+Q_2(z)e^{p(z+c)-p(z)}-e^{2p(z+c)}\equiv 1, \tag{3.6}$$

where

$$Q_1(z)=-\frac{i}{\delta}(\alpha\frac{\partial^2 p}{\partial {z_1}^2}+\alpha(\frac{\partial p}{\partial z_1})^2+\beta\frac{\partial^2 p}{\partial {z_2}^2}+\beta(\frac{\partial p}{\partial z_2})^2+\lambda\frac{\partial^2 p}{\partial z_1\partial z_2}+\lambda\frac{\partial p}{\partial z_1}\frac{\partial p}{\partial z_2})$$

and

$$Q_2(z)=-\frac{i}{\delta}(\alpha\frac{\partial^2 p}{\partial {z_1}^2}-\alpha(\frac{\partial p}{\partial z_1})^2+\beta\frac{\partial^2 p}{\partial {z_2}^2}-\beta(\frac{\partial p}{\partial z_2})^2+\lambda\frac{\partial^2 p}{\partial z_1\partial z_2}-\lambda\frac{\partial p}{\partial z_1}\frac{\partial p}{\partial z_2}).$$

Now we claim that $Q_1(z)$ and $Q_2(z)$ cannot be identical to zero at the same time. Otherwise, it follows from (3.6) that $-e^{2p(z+c)}\equiv 1$, which implies that $p(z+c)$ is a constant, a contradiction.

Suppose that $Q_1(z)\equiv 0$ and $Q_2(z)\not\equiv 0$. Then (3.6) leads to

$$Q_2(z)e^{p(z+c)-p(z)}-e^{2p(z+c)}\equiv 1. \tag{3.7}$$

It follows from (3.7) that

$$N(r,\frac{1}{e^{2p(z+c)}+1})=N(r,\frac{1}{Q_2(z)e^{p(z+c)-p(z)}})=S(r,e^{2p(z+c)}).$$

Also note that

$$N(r,e^{2p(z+c)})=S(r,e^{2p(z+c)}).$$

Therefore, applying the Nevanlinna Second Main Theorem we obtain

$$T(r,e^{2p(z+c)})\le\bar{N}(r,e^{2p(z+c)})+\bar{N}(r,\frac{1}{e^{2p(z+c)}})+$$

$$\bar{N}(r, \frac{1}{e^{2p(z+c)}+1}) + S(r, e^{2p(z+c)}) = S(r, e^{2p(z+c)}),$$

which implies that $p(z)$ is a constant, a contradiction.

Similarly, if $Q_1(z) \not\equiv 0$ and $Q_2(z) \equiv 0$, we can also get a contradiction. So $Q_1(z) \not\equiv 0$ and $Q_2(z) \not\equiv 0$.

Therefore, by Lemma 2.1 we get from (3.6) that

$$Q_2(z)e^{p(z+c)-p(z)} \equiv 1. \tag{3.8}$$

By (3.6) and (3.8), it is easy to obtain

$$Q_1(z)e^{p(z)-p(z+c)} \equiv 1. \tag{3.9}$$

Since $p$ is a non-constant polynomial and $Q_2(z)$ is a non-zero polynomial, it follows that the L.H.S. of (3.8) is transcendental, whereas the R.H.S. is a constant. Therefore, the only possibility is that $p(z+c) - p(z)$ is a constant.

Suppose that $p(z+c)-p(z) \equiv \xi_1$, where $\xi_1$ is a constant. So we have that $p(z) = L(z)+H(s)+B$ where $L(z) = a_1z_1 + a_2z_2, a_1, a_2, B \in \mathbb{C}, a_1 \neq 0$ and $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$. Next, we will prove that $H(s) \equiv 0$. If $\deg_s H = n$ ($n \geq 1$ is an integer), then (3.8) implies

$$A_1\frac{\partial^2 H}{\partial s^2} + A_2(\frac{\partial H}{\partial s})^2 + A_3\frac{\partial H}{\partial s} - A_4 = \xi_0,$$

that is,

$$A_1\frac{\partial^2 H}{\partial s^2} + A_3\frac{\partial H}{\partial s} - A_4 = \xi_0 - A_2(\frac{\partial H}{\partial s})^2,$$

where $A_1 = \alpha {c_2}^2+\beta {c_1}^2+\lambda c_1c_2$, $A_2 = \lambda c_1c_2-\alpha {c_2}^2-\beta {c_1}^2$, $A_3 = \lambda a_1c_1-\lambda a_2c_2+2\beta a_2c_1-2\alpha a_1c_2$, $A_4 = \alpha {a_2}^2 + \beta {a_1}^2 + \lambda a_1a_2, \xi_0 \in \mathbb{C}$. By comparing the degree of $s$ in both sides of the above equation we have $2(n-1) = n-1$, that is, $n = 1$. Thus the form of $L(z) + H(s) + B$ is still the linear form of $a_1z_1+a_2z_2+B$ which means that $H(s) \equiv 0$. Hence, it follows that $p(z) = L(z)+B$, where $L(z) = a_1z_1 + a_2z_2$. Substituting these into (3.8) and (3.9), we have

$$e^{L(c)}(i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2)) = (i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2))e^{(a_1c_1+a_2c_2)} = \delta$$

and

$$e^{-L(c)}(-i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2)) = (-i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2))e^{-(a_1c_1+a_2c_2)} = \delta,$$

which imply that

$$e^{L(c)} = \frac{\delta}{i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2)} \tag{3.10}$$

and

$$e^{-L(c)} = \frac{\delta}{-i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2)}, \tag{3.11}$$

that is, $e^{2L(c)} = -1$.

Therefore, by (3.5) we have

$$f(z) = \frac{e^{p(z-c)} - e^{-p(z-c)}}{2\delta i} = \frac{e^{L(z)+B-L(c)} - e^{-L(z)-B+L(c)}}{2\delta i}$$

$$= \frac{i}{2\delta}(e^{-L(z)-B+L(c)} - e^{L(z)+B-L(c)}).$$

This completes the proof of Case I.

Case II. If only one of $\gamma, \delta, \varphi$ is equal to zero, then we need to treat three subcases.

Subcase 1. If $\gamma = 0$, then we rewrite (1.11) as

$$(\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2} + \beta\frac{\partial^2 f(z)}{\partial {z_2}^2} + \lambda\frac{\partial^2 f(z)}{\partial z_1 \partial z_2})^2 + (\delta f(z+c) + \varphi f(z))^2 = 1. \tag{3.12}$$

It follows that

$$[P(f) + i(\delta f(z+c) + \varphi f(z))][P(f) - i(\delta f(z+c) + \varphi f(z))] = 1, \tag{3.13}$$

where

$$P(f) = \alpha\frac{\partial^2 f(z)}{\partial {z_1}^2} + \beta\frac{\partial^2 f(z)}{\partial {z_2}^2} + \lambda\frac{\partial^2 f(z)}{\partial z_1 \partial z_2}.$$

Since $f$ is entire, we obtain that both

$$P(f) + i(\delta f(z+c) + \varphi f(z)) \text{ and } P(f) - i(\delta f(z+c) + \varphi f(z))$$

have no zeros and no poles. Thus, by Lemmas 2.1 and 2.2, there exists a non-constant polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\begin{cases} (\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2} + \beta\frac{\partial^2 f(z)}{\partial {z_2}^2} + \lambda\frac{\partial^2 f(z)}{\partial z_1 \partial z_2}) + i(\delta f(z+c) + \varphi f(z)) = e^{p(z)}, \\ (\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2} + \beta\frac{\partial^2 f(z)}{\partial {z_2}^2} + \lambda\frac{\partial^2 f(z)}{\partial z_1 \partial z_2}) - i(\delta f(z+c) + \varphi f(z)) = e^{-p(z)}. \end{cases} \tag{3.14}$$

By (3.14), a simple calculation shows that

$$\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2} + \beta\frac{\partial^2 f(z)}{\partial {z_2}^2} + \lambda\frac{\partial^2 f(z)}{\partial z_1 \partial z_2} = \frac{e^{p(z)} + e^{-p(z)}}{2} \tag{3.15}$$

and

$$\delta f(z+c) + \varphi f(z) = \frac{e^{p(z)} - e^{-p(z)}}{2i}. \tag{3.16}$$

We now need to verify the form of $p(z)$. From (3.15) and (3.16), we can obtain that

$$Q_3(z)e^{p(z+c)+p(z)} + Q_4(z)e^{p(z+c)-p(z)} - e^{2p(z+c)} \equiv 1, \tag{3.17}$$

where

$$Q_3(z) = \frac{1}{\delta}(-i(\alpha\frac{\partial^2 p}{\partial {z_1}^2} + \alpha(\frac{\partial p}{\partial z_1})^2 + \beta\frac{\partial^2 p}{\partial {z_2}^2} + \beta(\frac{\partial p}{\partial z_2})^2 + \lambda\frac{\partial^2 p}{\partial z_1 \partial z_2} + \lambda\frac{\partial p}{\partial z_1}\frac{\partial p}{\partial z_2}) - \varphi)$$

and

$$Q_4(z) = \frac{1}{\delta}(-i(\alpha\frac{\partial^2 p}{\partial {z_1}^2} - \alpha(\frac{\partial p}{\partial z_1})^2 + \beta\frac{\partial^2 p}{\partial {z_2}^2} - \beta(\frac{\partial p}{\partial z_2})^2 + \lambda\frac{\partial^2 p}{\partial z_1 \partial z_2} - \lambda\frac{\partial p}{\partial z_1}\frac{\partial p}{\partial z_2}) - \varphi).$$

Now, we claim that $Q_3(z)$ and $Q_4(z)$ cannot be both zero at the same time. Otherwise, it follows from (3.17) that $-e^{2p(z+c)} \equiv 1$, which implies that $p(z+c)$ is a constant, a contradiction.

Suppose that $Q_3(z) \equiv 0$ and $Q_4(z) \not\equiv 0$. Then (3.17) leads to

$$Q_4(z)e^{p(z+c)-p(z)} - e^{2p(z+c)} \equiv 1. \tag{3.18}$$

It follows from (3.18) that

$$N(r, \frac{1}{e^{2p(z+c)}+1}) = N(r, \frac{1}{Q_4(z)e^{p(z+c)-p(z)}}) = S(r, e^{2p(z+c)}).$$

Also note that

$$N(r, e^{2p(z+c)}) = S(r, e^{2p(z+c)}).$$

Therefore, by the Nevanlinna second main theorem of several complex variables, we obtain

$$T(r, e^{2p(z+c)}) \leq \bar{N}(r, e^{2p(z+c)}) + \bar{N}(r, \frac{1}{e^{2p(z+c)}}) + \bar{N}(r, \frac{1}{e^{2p(z+c)}+1}) + S(r, e^{2p(z+c)}) = S(r, e^{2p(z+c)}),$$

which implies that $p(z+c)$ is a constant, a contradiction.

Similarly, if $Q_3(z) \not\equiv 0$ and $Q_4(z) \equiv 0$, we can also get a contradiction.

So, $Q_3(z) \not\equiv 0$ and $Q_4(z) \not\equiv 0$.

Therefore, by Lemma 2.1, we obtain from (3.17)

$$Q_4(z)e^{p(z+c)-p(z)} \equiv 1. \tag{3.19}$$

By (3.17) and (3.19), it is easy to obtain

$$Q_3(z)e^{p(z)-p(z+c)} \equiv 1. \tag{3.20}$$

Since $p$ is a non-constant polynomial and $Q_4(z)$ is a non-zero polynomial, it follows that the L.H.S. of (3.19) is transcendental, while the R.H.S. is constant, so the only possibility is that $p(z+c)-p(z)$ is a constant. Therefore, $e^{p(z+c)-p(z)}$ is a constant.

Suppose that $p(z+c)-p(z) = \xi_2$, where $\xi_2$ is a constant. So we have $p(z) = L(z)+H(s)+B$, where $L(z) = a_1z_1 + a_2z_2, a_1 \neq 0, a_1, a_2, B \in \mathbb{C}$, and $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$.

Similar to the discussion in Case I, we have $H(s) \equiv 0$. So it follows that $p(z) = L(z) + B = a_1z_1 + a_2z_2 + B$. Substituting these into (3.19) and (3.20), we have

$$e^{L(c)}(i(\alpha a_1{}^2 + \beta a_2{}^2 + \lambda a_1a_2) - \varphi) = \delta$$

and

$$e^{-L(c)}(-i(\alpha a_1{}^2 + \beta a_2{}^2 + \lambda a_1a_2) - \varphi) = \delta,$$

which imply

$$e^{L(c)} = \frac{\delta}{i(\alpha a_1{}^2 + \beta a_2{}^2 + \lambda a_1a_2) - \varphi} \tag{3.21}$$

and

$$e^{-L(c)} = -\frac{\delta}{i(\alpha a_1{}^2 + \beta a_2{}^2 + \lambda a_1a_2) + \varphi}. \tag{3.22}$$

In view of the results in [21], (3.15) can be written as

$$(rD + mD')(qD + nD')f(z) = \frac{e^{p(z)} + e^{-p(z)}}{2}, \tag{3.23}$$

where $D \equiv \frac{\partial}{\partial z_1}$, $D' \equiv \frac{\partial}{\partial z_2}$, $\alpha, \beta \in \mathbb{C}$ such that $rq = \alpha$, $mn = \beta$, $rn + qm = \lambda$.

Let $(qD+nD')f(z)=u(z)$. Then (3.23) yields that

$$r\frac{\partial u}{\partial z_1}+m\frac{\partial u}{\partial z_2}=\frac{e^{p(z)}+e^{-p(z)}}{2}. \tag{3.24}$$

The characteristic equations of (3.24) are

$$\frac{\mathrm{d}z_1}{\mathrm{d}t}=r,\ \frac{\mathrm{d}z_2}{\mathrm{d}t}=m,\ \frac{\mathrm{d}u}{\mathrm{d}t}=\frac{e^{p(z)}+e^{-p(z)}}{2}.$$

Using the initial conditions: $z_1=0$, $z_2=s$, and $u=u(0,s):=\phi_0(s)$ with a parameter $s$. Then we obtain the following parametric representation for the solutions of the characteristic equations: $z_1=rt$, $z_2=mt+s$,

$$u(t,s)=\int_0^t\frac{e^{(a_1r+a_2m)t+a_2s+B}+e^{-[(a_1r+a_2m)t+a_2s+B]}}{2}\mathrm{d}t+\phi_0(s), \tag{3.25}$$

where $\phi_0$ is an entire function with finite order in $\mathbb{C}^2$.

Since we have assumed that $(qD+nD')f(z)=u(z)$, it follows from (3.24) that

$$q\frac{\partial f(z)}{\partial z_1}+n\frac{\partial f(z)}{\partial z_2}=\int_0^t\frac{e^{(a_1r+a_2m)t+a_2s+B}+e^{-[(a_1r+a_2m)t+a_2s+B]}}{2}\mathrm{d}t+\phi_0(s). \tag{3.26}$$

So we obtain from (3.26) that

$$f(t,s)=\int_0^t\int_0^t\frac{e^{(a_1r+a_2m)t+a_2s+B}+e^{-[(a_1r+a_2m)t+a_2s+B]}}{2}\mathrm{d}t\mathrm{d}t+\int_0^t\phi_0(s)\mathrm{d}t. \tag{3.27}$$

Therefore, it yields that

$$f(z_1,z_2)=\frac{e^{L(z)+B}+e^{-(L(z)+B)}}{2(a_1r+a_2m)(a_1q+a_2n)}+\phi_1(z_2-\frac{mz_1}{r})+\phi_2(z_2-\frac{nz_1}{q}),$$

where $\phi_1$ and $\phi_2$ are entire functions with finite order in $\mathbb{C}^2$.

Subcase 2. If $\delta=0$, we can prove Subcase 2 by similar discussions as in Subcase 1.

Subcase 3. If $\varphi=0$, then (1.11) reduces to

$$(\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2})^2+(\delta f(z+c)+\gamma f(z+2c))^2=1. \tag{3.28}$$

We rewrite (3.28) as

$$[P(f)+i(\delta f(z+c)+\varphi f(z))][P(f)-i(\delta f(z+c)+\gamma f(z+2c))]=1, \tag{3.29}$$

where

$$P(f)=\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2}.$$

Since $f$ is entire, we have that

$$P(f)+i(\delta f(z+c)+\gamma f(z+2c))\text{ and }P(f)-i(\delta f(z+c)+\gamma f(z+2c))$$

have no zeros and poles. So by Lemmas 2.1 and 2.2, there exists a non-constant polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\begin{cases}(\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2})+i(\delta f(z+c)+\gamma f(z+2c))=e^{p(z)},\\(\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2})-i(\delta f(z+c)+\gamma f(z+2c))=e^{-p(z)}.\end{cases} \tag{3.30}$$

By (3.30), a simple calculation shows that

$$\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2}+\beta\frac{\partial^2 f(z)}{\partial {z_2}^2}+\lambda\frac{\partial^2 f(z)}{\partial z_1\partial z_2}=\frac{e^{p(z)}+e^{-p(z)}}{2}, \tag{3.31}$$

and

$$\delta f(z+c)+\gamma f(z+2c)=\frac{e^{p(z)}-e^{-p(z)}}{2i}. \tag{3.32}$$

We now need to verify the form of $p(z)$. It follows from (3.31) and (3.32) that

$$\frac{\gamma}{\delta}e^{p(z+c)+p(z+2c)}+\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}+e^{2p(z+c)}-\frac{1}{\delta}Q_5e^{p(z+c)+p(z)}-\frac{1}{\delta}Q_6e^{p(z+c)-p(z)}\equiv -1, \tag{3.33}$$

where

$$Q_5(z)=-i(\alpha\frac{\partial^2 p}{\partial {z_1}^2}+\alpha(\frac{\partial p}{\partial z_1})^2+\beta\frac{\partial^2 p}{\partial {z_2}^2}+\beta(\frac{\partial p}{\partial z_2})^2+\lambda\frac{\partial^2 p}{\partial z_1\partial z_2}+\lambda\frac{\partial p}{\partial z_1}\frac{\partial p}{\partial z_2})$$

and

$$Q_6(z)=-i(\alpha\frac{\partial^2 p}{\partial {z_1}^2}-\alpha(\frac{\partial p}{\partial z_1})^2+\beta\frac{\partial^2 p}{\partial {z_2}^2}-\beta(\frac{\partial p}{\partial z_2})^2+\lambda\frac{\partial^2 p}{\partial z_1\partial z_2}-\lambda\frac{\partial p}{\partial z_1}\frac{\partial p}{\partial z_2}).$$

Suppose that $Q_5(z)\equiv 0$ and $Q_6(z)\equiv 0$, it follows from (3.33) that

$$-\frac{\gamma}{\delta}e^{p(z+c)+p(z+2c)}-\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}-e^{2p(z+c)}\equiv 1. \tag{3.34}$$

Therefore, by Lemma 2.1, we obtain from (3.34)

$$-\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}\equiv 1. \tag{3.35}$$

From (3.34) and (3.35), we have

$$-\frac{\gamma}{\delta}e^{p(z+2c)-p(z+c)}\equiv 1. \tag{3.36}$$

Similar to the discussion in Case I, we conclude that $p(z)=a_1z_1+a_2z_2+B$. So from (3.34)–(3.36), we have $-\gamma e^{-L(c)}=\delta, -\gamma e^{L(c)}=\delta$ and $\alpha {a_1}^2+\beta {a_2}^2+\lambda a_1a_2=0$. Suppose that $Q_5(z)\not\equiv 0$ and $Q_6(z)\equiv 0$, it follows from (3.33) that

$$\frac{\gamma}{\delta}e^{p(z+c)+p(z+2c)}+\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}+e^{2p(z+c)}-\frac{1}{\delta}Q_5e^{p(z+c)+p(z)}\equiv -1. \tag{3.37}$$

Therefore, by Lemma 2.1 and (3.37), we have

$$-\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}\equiv 1, \tag{3.38}$$

which implies that $p(z+c)-p(z+2c)$ is a constant. Consequently, we deduce that $p(z+c)-p(z)$ is a constant.

Similar to Case I, we have $p(z)=L(z)+H(s)+B=a_1z_1+a_2z_2+B$. So

$$Q_5(z)=-i(\alpha {a_1}^2+\beta {a_2}^2+\lambda a_1a_2)$$

and

$$Q_6(z)=i(\alpha {a_1}^2+\beta {a_2}^2+\lambda a_1a_2),$$

which contradicts $Q_5(z)\not\equiv 0$ and $Q_6(z)\equiv 0$.

Similarly, $Q_5(z)\equiv 0$ and $Q_6(z)\not\equiv 0$ can also lead to a contradiction.

Now, suppose that $Q_5(z) \not\equiv 0$ and $Q_6(z) \not\equiv 0$. So by Lemma 2.1 and (3.33), we obtain that

$$-\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)} \equiv 1 \tag{3.39}$$

or

$$\frac{1}{\delta}Q_6 e^{p(z+c)-p(z)} \equiv 1. \tag{3.40}$$

Firstly, we assume that $-\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)} \equiv 1$.

Since $p$ is a non-constant polynomial, and $\frac{\gamma}{\delta}$ is a constant, it follows that the L.H.S. of (3.39) is transcendental, whereas the R.H.S. is constant. Therefore, the only possibility is that $p(z+c)-p(z+2c)$ is a constant, that is $p(z+c)-p(z)$ is constant. Hence, $e^{p(z+c)-p(z+2c)}$ is a constant.

Assume that $p(z+c)-p(z) = \xi_3$, $\xi_3$ is a constant. Hence, we have that $p(z) = L(z)+H(s)+B$, where $L(z) = a_1z_1 + a_2z_2, a_1 \neq 0, a_1, a_2, B \in \mathbb{C}$, and $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$.

Similar to Case I, we have $H(s) = 0$. So it follows that

$$p(z) = L(z) + H(s) + B = a_1z_1 + a_2z_2 + B.$$

Substituting these into (3.39), we have $e^{L(c)} = -\frac{\gamma}{\delta}$ and

$$(\gamma e^{2L(c)} + \delta e^{L(c)} + i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2))e^{2L(z)+2B} - i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2) = 0.$$

Similarly, if $\frac{1}{\delta}Q_6e^{p(z+c)-p(z)} \equiv 1$, then we have

$$e^{L(c)} = \frac{\delta}{i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2)}$$

and

$$(\gamma e^{2L(c)} + \delta e^{L(c)} + i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1a_2))e^{2L(z)+2B} + \gamma = 0.$$

By the discussion in [21] and (3.31), we have

$$(rD + mD')(qD + nD')f(z) = \frac{e^{p(z)} + e^{-p(z)}}{2}, \tag{3.41}$$

where $D \equiv \frac{\partial}{\partial z_1}$, $D' \equiv \frac{\partial}{\partial z_2}$, $\alpha, \beta \in \mathbb{C}$ such that $rq = \alpha$, $mn = \beta$, $rn + qm = \lambda$.

Let $(qD + nD')f(z) = u(z)$. Then (3.41) yields that

$$r\frac{\partial u}{\partial z_1} + m\frac{\partial u}{\partial z_2} = \frac{e^{p(z)} + e^{-p(z)}}{2}. \tag{3.42}$$

The characteristic equations of (3.42) are

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = r,\ \frac{\mathrm{d}z_2}{\mathrm{d}t} = m,\ \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{e^{p(z)} + e^{-p(z)}}{2}.$$

Using the initial conditions: $z_1 = 0$, $z_2 = s$, and $u = u(0, s) := \phi_0(s)$ with a parameter $s$. So we get the following parametric representation for the solutions of the characteristic equations: $z_1 = rt$, $z_2 = mt + s$,

$$u(t,s) = \int_0^t \frac{e^{(a_1r+a_2m)t+a_2s+B} + e^{-[(a_1r+a_2m)t+a_2s+B]}}{2}\mathrm{d}t + \phi_0(s), \tag{3.43}$$

where $\phi_0$ is an entire function with finite order in $\mathbb{C}^2$.

Since $(qD + nD')f(z) = u(z)$, it follows from (3.43) that

$$q\frac{\partial f(z)}{\partial z_1} + n\frac{\partial f(z)}{\partial z_2} = \int_0^t \frac{e^{(a_1r+a_2m)t+a_2s+B} + e^{-[(a_1r+a_2m)t+a_2s+B]}}{2}\mathrm{d}t + \phi_0(s). \tag{3.44}$$

Similarly, from (3.44) we get

$$f(t,s) = \int_0^t \int_0^t \frac{e^{(a_1r+a_2m)t+a_2s+B} + e^{-[(a_1r+a_2m)t+a_2s+B]}}{2}\mathrm{d}t\mathrm{d}t + \int_0^t \phi_0(s)\mathrm{d}t. \tag{3.45}$$

Thus, it yields

$$f(z_1,z_2) = \frac{e^{L(z)+B} + e^{-(L(z)+B)}}{2(a_1r+a_2m)(a_1q+a_2n)} + \phi_1(z_2 - \frac{mz_1}{r}) + \phi_2(z_2 - \frac{nz_1}{q}),$$

where $\phi_1$ and $\phi_2$ are entire functions with finite order in $\mathbb{C}^2$. This completes the proof of Case II.

Case III. If $\gamma, \delta, \varphi$ are all non-zero, then we rewrite (1.11) as

$$[P(f) + i(\gamma f(z+2c) + \delta f(z+c) + \varphi f(z))][P(f) - i(\gamma f(z+2c) + \delta f(z+c) + \varphi f(z))] = 1, \tag{3.46}$$

where

$$P(f) = \alpha\frac{\partial^2 f(z)}{\partial {z_1}^2} + \beta\frac{\partial^2 f(z)}{\partial {z_2}^2} + \lambda\frac{\partial^2 f(z)}{\partial z_1 \partial z_2}.$$

Since $f$ is entire, it follows from (3.46) that $P(f) + i(\gamma f(z+2c) + \delta f(z+c) + \varphi f(z))$ and $P(f) - i(\gamma f(z+2c) + \delta f(z+c) + \varphi f(z))$ have no zeros and poles. Thus, by Lemmas 2.1 and 2.2, there exists a non-constant polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\begin{cases} P(f) + i(\gamma f(z+2c) + \delta f(z+c) + \varphi f(z)) = e^{p(z)}, \\ P(f) - i(\gamma f(z+2c) + \delta f(z+c) + \varphi f(z)) = e^{-p(z)}. \end{cases} \tag{3.47}$$

From (3.47), a simple calculation shows that

$$\alpha\frac{\partial^2 f(z)}{\partial {z_1}^2} + \beta\frac{\partial^2 f(z)}{\partial {z_2}^2} + \lambda\frac{\partial^2 f(z)}{\partial z_1 \partial z_2} = \frac{e^{p(z)} + e^{-p(z)}}{2} \tag{3.48}$$

and

$$\gamma f(z+2c) + \delta f(z+c) + \varphi f(z) = \frac{e^{p(z)} - e^{-p(z)}}{2i}. \tag{3.49}$$

We now need to verify the form of $p(z)$. From (3.48) and (3.49), we have

$$\frac{\gamma}{\delta}e^{p(z+c)+p(z+2c)} + \frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)} + e^{2p(z+c)} + \frac{1}{\delta}(\varphi - Q_7)e^{p(z+c)+p(z)} + \frac{1}{\delta}(\varphi - Q_8)e^{p(z+c)-p(z)} \equiv -1, \tag{3.50}$$

where

$$Q_7(z) = -i(\alpha\frac{\partial^2 p}{\partial {z_1}^2} + \alpha(\frac{\partial p}{\partial z_1})^2 + \beta\frac{\partial^2 p}{\partial {z_2}^2} + \beta(\frac{\partial p}{\partial z_2})^2 + \lambda\frac{\partial^2 p}{\partial z_1 \partial z_2} + \lambda\frac{\partial p}{\partial z_1}\frac{\partial p}{\partial z_2})$$

and

$$Q_8(z) = -i(\alpha\frac{\partial^2 p}{\partial {z_1}^2} - \alpha(\frac{\partial p}{\partial z_1})^2 + \beta\frac{\partial^2 p}{\partial {z_2}^2} - \beta(\frac{\partial p}{\partial z_2})^2 + \lambda\frac{\partial^2 p}{\partial z_1 \partial z_2} - \lambda\frac{\partial p}{\partial z_1}\frac{\partial p}{\partial z_2}).$$

Suppose that $\varphi - Q_7(z) \equiv 0$ and $\varphi - Q_8(z) \equiv 0$, it follows from (3.50) that

$$-\frac{\gamma}{\delta}e^{p(z+c)+p(z+2c)} - \frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)} - e^{2p(z+c)} \equiv 1. \tag{3.51}$$

Therefore, by Lemma 2.1 and (3.51) we have

$$-\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}\equiv 1. \tag{3.52}$$

By (3.51) and (3.52), it is easy to get

$$-\frac{\gamma}{\delta}e^{p(z+2c)-p(z+c)}\equiv 1. \tag{3.53}$$

Similar to Case I, we conclude that $p(z)=a_1z_1+a_2z_2+B$. Thus, by (3.52) and (3.53), we have $-\gamma e^{-L(c)}=\delta$, $-\gamma e^{L(c)}=\delta$. Moreover, $\varphi-Q_7(z)=\varphi+i(\alpha a_1{}^2+\beta a_2{}^2+\lambda a_1a_2)=0$ and $\varphi-Q_8(z)=\varphi-i(\alpha a_1{}^2+\beta a_2{}^2+\lambda a_1a_2)=0$, which imply that $\varphi=0$, a contradiction.

Suppose that $\varphi-Q_7(z)\equiv 0$ and $\varphi-Q_8(z)\not\equiv 0$. Then (3.50) leads to

$$\frac{\gamma}{\delta}e^{p(z+c)+p(z+2c)}+\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}+e^{2p(z+c)}+\frac{1}{\delta}(\varphi-Q_8)e^{p(z+c)-p(z)}\equiv -1. \tag{3.54}$$

It follows from (3.54) that

$$\begin{aligned} N(r,\frac{1}{-e^{2p(z+c)}-1})&=N(r,\frac{1}{\frac{\gamma}{\delta}e^{p(z+c)+p(z+2c)}+\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}+\frac{1}{\delta}(\varphi-Q_8)e^{p(z+c)-p(z)}})\\ &=S(r,e^{2p(z+c)}).\end{aligned}$$

Also note that

$$N(r,e^{2p(z+c)})=N(r,\frac{1}{e^{2p(z+c)}})=S(r,e^{2p(z+c)}).$$

Therefore, applying second main theorem Nevanlinna for several complex variables, we obtain

$$\begin{aligned} T(r,e^{2p(z+c)})\le&\bar{N}(r,e^{2p(z+c)})+\bar{N}(r,\frac{1}{e^{2p(z+c)}})+\\ &\bar{N}(r,\frac{1}{-e^{2p(z+c)}-1})+S(r,e^{2p(z+c)})=S(r,e^{2p(z+c)}),\end{aligned}$$

which implies that $p(z+c)$ is a constant, a contradiction.

Similarly, if $\varphi-Q_7(z)\not\equiv 0$ and $\varphi-Q_8(z)\equiv 0$, we can also get a contradiction.

Now, suppose that $\varphi-Q_7(z)\not\equiv 0$ and $\varphi-Q_8(z)\not\equiv 0$.

By Lemma 2.1 and (3.50), we have

$$-\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}\equiv 1 \tag{3.55}$$

or

$$-\frac{1}{\delta}(\varphi-Q_8)e^{p(z+c)-p(z)}\equiv 1. \tag{3.56}$$

Firstly, we assume that $-\frac{\gamma}{\delta}e^{p(z+c)-p(z+2c)}\equiv 1$.

Since $p$ is a non-constant polynomial, and $\frac{\gamma}{\delta}$ is a constant, it follows that the L.H.S. of (3.55) is transcendental, whereas the R.H.S. is constant. Therefore, the only possibility is that $p(z+c)-p(z+2c)$ is a constant. Then we can deduce that $p(z+c)-p(z)$ is constant. Hence, $e^{p(z+c)-p(z+2c)}$ is a constant.

Suppose that $p(z+c)-p(z)=\xi_4$, where $\xi_4$ is a constant. So we have $p(z)=L(z)+H(s)+B$, where $L(z)=a_1z_1+a_2z_2$, $a_1\neq 0$, $a_1,a_2,B\in\mathbb{C}$, and $H(s)$ is a polynomial in $s=c_2z_1-c_1z_2$.

Similar to Case I, we have $H(s) = 0$. Hence, it follows that $p(z) = L(z)+B = a_1z_1+a_2z_2+B$. Then by (3.55), we have $e^{L(c)} = -\frac{\gamma}{\delta}$ and

$$(\gamma e^{2L(c)} + \delta e^{L(c)} + \varphi + i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2))e^{2L(z)+2B} + \varphi - i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2) = 0.$$

Similarly, if $-\frac{1}{\delta}(\varphi - Q_8)e^{p(z+c)-p(z)} = 1$, we have

$$e^{L(c)} = -\frac{\delta}{\varphi - i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2)},$$
$$(\gamma e^{2L(c)} + \delta e^{L(c)} + \varphi + i(\alpha {a_1}^2 + \beta {a_2}^2 + \lambda a_1 a_2))e^{2L(z)+2B} + \gamma = 0.$$

By the results in [21] and (3.48), we have

$$(rD + mD')(qD + nD')f(z) = \frac{e^{p(z)} + e^{-p(z)}}{2}, \tag{3.57}$$

where $D \equiv \frac{\partial}{\partial z_1}$, $D' \equiv \frac{\partial}{\partial z_2}$, $\alpha, \beta \in \mathbb{C}$ such that $rq = \alpha$, $mn = \beta$, $rn + qm = \lambda$.

Let $(qD + nD')f(z) = u(z)$. Then (3.57) yields that

$$r\frac{\partial u}{\partial z_1} + m\frac{\partial u}{\partial z_2} = \frac{e^{p(z)} + e^{-p(z)}}{2}. \tag{3.58}$$

The characteristic equations of (3.58) are

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = r, \ \frac{\mathrm{d}z_2}{\mathrm{d}t} = m, \ \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{e^{p(z)} + e^{-p(z)}}{2}.$$

Using the initial conditions: $z_1 = 0$, $z_2 = s$, and $u = u(0, s) := \phi_0(s)$ with a parameter $s$. Thus, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1 = rt$, $z_2 = mt + s$,

$$u(t, s) = \int_0^t \frac{e^{(a_1r+a_2m)t+a_2s+B} + e^{-[(a_1r+a_2m)t+a_2s+B]}}{2}\mathrm{d}t + \phi_0(s), \tag{3.59}$$

where $\phi_0$ is an entire function with finite order in $\mathbb{C}^2$.

Since $(qD + nD')f(z) = u(z)$, it follows from (3.59) that

$$q\frac{\partial f(z)}{\partial z_1} + n\frac{\partial f(z)}{\partial z_2} = \int_0^t \frac{e^{(a_1r+a_2m)t+a_2s+B} + e^{-[(a_1r+a_2m)t+a_2s+B]}}{2}\mathrm{d}t + \phi_0(s). \tag{3.60}$$

Similarly, we have

$$f(t, s) = \int_0^t \int_0^t \frac{e^{(a_1r+a_2m)t+a_2s+B} + e^{-[(a_1r+a_2m)t+a_2s+B]}}{2}\mathrm{d}t\mathrm{d}t + \int_0^t \phi_0(s)\mathrm{d}t. \tag{3.61}$$

Thus, it yields that

$$f(z_1, z_2) = \frac{e^{L(z)+B} + e^{-(L(z)+B)}}{2(a_1r + a_2m)(a_1q + a_2n)} + \phi_1(z_2 - \frac{mz_1}{r}) + \phi_2(z_2 - \frac{nz_1}{q}),$$

where $\phi_1$ and $\phi_2$ are entire functions with finite order in $\mathbb{C}^2$. This completes the proof of Case III. □

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## References


[1] A. WILES. *Modular elliptic curves and Fermats last theorem*. Ann. of Math. (2), 1995, **141**(3): 443–551.

[2] R. TAYLOR, A. WILES. *Ring-theoretic properties of certain Hecke algebra*. Ann. of Math. (2), 1995, **141**(3): 553–572.

[3] H. CARTAN. *Sur les zeros des combinaisons linéaires de p fonctions holomorphes données*. Mathematica (Cluj), 1933, **7**: 5–13.

[4] G. IYER. *On certain functional equations*. J. Indian. Math. Soc., 1939, **3**: 312–315.

[5] P. MONTEL. *Lecons sur les familles de nomales fonctions analytiques et leurs applications*. Gauthier BViuars Paris, 1927, 135–136.

[6] F. GROSS. *On the equation $f^n + g^n = 1$*. Bull. Amer. Math. Soc., 1966, **72**: 86–88.

[7] G. HALDAR. *Solutions of Fermat-type partial differential-difference equations in $\mathbb{C}^2$*. Mediterr. J. Math., 2023, **20**(1): Paper No. 50, 21 pp.

[8] G. HALDAR, M. B. AHAMED. *Entire solutions of several quadratic binomial and trinomial partial differential-difference equations in $\mathbb{C}^2$*. Anal. Math. Phys., 2022, **12**(5): Paper No. 113, 38 pp.

[9] Peichu HU, Qiong WANG. *On meromorphic solutions of functional equations of Fermat type*. Bull. Malays. Math. Sci. Soc., 2019, **42**(5): 2497–2515.

[10] Jingjing LI, Zhigang HUANG. *On entire solutions of two certain types of non-Linear differential-difference equations*. Wuhan Univ. J. Nat. Sci., 2022, **27**(3): 195–200.

[11] Junyang ZHANG, Zhigang HUANG. *Existence of meromorphic solutions of Fermat-type functional equations*. J. Math. Res. Appl., 2024, **44**(5): 659–668.

[12] Qiong WANG, Wei Chen, Peichu HU. *On entire solutions of two certain Fermat-type differential-difference equations*. Bull. Malays. Math. Sci. Soc., 2020, **43**(4): 2951–2965.

[13] Hongyan XU, G. HALDAR. *Solutions of complex nonlinear functional equations including second order partial differential and difference in $\mathbb{C}^2$*. Electron. J. Differential Equations, 2023, Paper No. 43, 18 pp.

[14] Hongyan XU, G. HALDAR. *Entire solutions to Fermat-type difference and partial differential-difference equations in $\mathbb{C}^n$*. Electron. J. Differential Equations, 2024, Paper No. 26, 21 pp.

[15] D. KHAVINSON. *A note on entire solutions of the eikonal equation*. Amer. Math. Monthly, 1995, **102**(2): 159–161.

[16] Ling XU, Tingbin CAO. *Solutions of complex Fermat-type partial difference and differential-difference equations*. Mediterr. J. Math., 2018, **15**(6): Paper No. 227, 14 pp.

[17] Xianmin GUI, Hongyan XU, Wenju TANG, et al. *Entire solutions for several complex partial differential-difference equations of Fermat type in $\mathbb{C}^2$*. Open Math., 2021, **19**(1): 1416–1434.

[18] W. STOLL. *Holomorphic Functions of Finite Order in Several Complex Variables*. American Mathematical Society, Providence, RI, 1974.

[19] L. I. RONKIN. *Introduction to the Theory of Entire Functions of Several Variables*. Nauka, Moscow, 1971. (in Russian)

[20] Peichu HU, Ping LI, Chungchun YANG. *Unicity of Meromorphic Mappings*. Kluwer Academic Publishers, Dordrecht, 2003.

[21] Hongyan XU, Keyu ZHANG, Xiumin ZHENG. *Entire and meromorphic solutions for several Fermat type partial differential difference equations in $\mathbb{C}^2$*. Rocky Mountain J. Math., 2022, **52**(6): 2169–2187.