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Sharp Bounds for Upper and Bottom Spectrum of Hermitizable Tridiagonal Matrices

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Abstract We display sharp bounds for upper and lower spectrum of a Hermitizable tridiagonal matrix. The representations are brought to light by exploiting the characteristic for eigenpairs (eigenvalue and its corresponding eigenvector) of tridiagonal matrices, isospectral transforms and sharp bounds for speed stability of birth-death processes.

Keywords Hermitizable matrix; isospectral matrices; upper and bottom spectrum

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1. Introduction

Hermitizable matrix is a class of self-conjugate operators, as generalized by Chen [1, Definition 1] from Hermite matrices. Recall that a complex matrix $A=(a_{ij})$ is Hermitizable if there exists a positive sequence $\mu=(\mu_i: i\in E)$ such that

$$\mu_i a_{ij}=\mu_j \bar{a}_{ji},\quad \forall\, i,j\in E,$$

where $\overline{a}$ denotes the conjugate of a, $E=\{k: 0\le k<N+1\}$ with $N\le\infty$. Thus the spectra of A denoted by $\sigma(A)$ in $L^2(\mu)$ is real. For a general reversible transition matrix (or a Hermitizable matrix) A, few results are known on the sharp bounds of its upper and bottom spectra (denoted by $\lambda_+(A)$ and $\lambda_-(A)$, respectively). It is the purpose of this paper to consider the sharp upper and lower bounds for $\lambda_+(A)$ and $\lambda_-(A)$, respectively.

Notice that a symmetric real matrix is a particular Hermitizable matrix. The upper and bottom spectrum of such a matrix have many applications. For instance, they are closely related to the rate of geometric ergodicity of discrete time Markov chains. Precisely, suppose $\pi=(\pi_i, i\in E)$ is the stationary distribution of the finite reversible transition matrix $P=(p_{ij})_{i,j\in E}$ (i.e., $\pi_i p_{ij}=\pi_j p_{ji}$). The spectra of P is denoted by $\sigma(P)$, then $\sigma(P)$ is contained in $[-1,1]$. Moreover, define

$$\lambda_+(P)=\sup\{\lambda\in\sigma(P)\setminus\{1\}\},\quad \lambda_-(P)=-\inf\{\lambda\in\sigma(P)\}.$$

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Diaconis and Stroock [2, Proposition 3] proved that the geometric convergence rate of P is determined by $\rho := \lambda_+(P) \vee \lambda_-(P)$, which only depends on the upper spectrum $\lambda_+(P)$ and bottom one $\lambda_-(P)$. If P is nonnegative definite, then $\rho = \lambda_+(P)$. If P has even period, then -1 is an element of $\sigma(P)$, thus $\rho = 1$ (see [3,4]). If P is irreducible with aperiodic period, [3,5] presented some results using probabilistic method (which is somehow results of $\rho(P^2)$).

To our knowledge, an effective method to obtain the bounds, for the spectra of a discrete time Markov chain, is taking advantage of the well studied sharp bounds for convergence rate of continuous time reversible Markov chain. On one hand, let $Q = P - I$, then Q is a generator of some continuous time Markov chain, the spectral gap of Q equals to $1 - \lambda_+(P)$, which denotes some convergence rate of the chain, its sharp bound is well studied [6], thus we obtain the information of $\sigma_+(P)$. On the other hand, when $\sigma_-(P) \neq \emptyset$, we need to study the part on $\sigma(P) \cap [-1, 0]$, which has an essential difference between continuous time and discrete time Markov chains. To overcome the difficulty, one may study the operator $-Q = I - P$. However, $\sigma(Q) = -\sigma(-Q)$, we only also obtain the information of $\sigma_+(P)$, rather than $\sigma_-(P)$. Generally, it is hard to get the information of $\sigma_-(P)$. New skills are needed to solve the problem.

Fortunately, the celebrated Householder transformation makes a Hermitizable matrix A be similar to a birth-death matrix [7,8] for $N < \infty$. Thus, the bounds for upper and bottom spectra of tridiagonal matrix are essential for the general Hermitizable matrix. In what follows, suppose $N < \infty$, we deal with the Hermitiazble tridiagonal matrix T. The characteristic for eigenpairs of T is given in Theorem 3.2. By the skills of h-transform and sharp bounds for stability of birth-death processes, the upper and lower bounds for $\lambda_+(T)$ and $\lambda_-(T)$ are presented in Theorem 3.3. Two numerical examples are investigated to verify the effectiveness of the results.

2. Preliminaries

Let T denote a Hermitizable tridiagonal matrix:

$$T = \begin{pmatrix} -c_0 & b_0 & & & & \\ a_1 & -c_1 & b_1 & & 0 & \\ & a_2 & -c_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & a_{N-1} & -c_{N-1} & b_{N-1} \\ & & & & a_N & -c_N \end{pmatrix}, \tag{2.1}$$

$\{c_k\}_{k=0}^N$ is real, $\{a_k\}_{k=1}^N$ and $\{b_k\}_{k=0}^{N-1}$ are complex but $a_{k+1}b_k > 0$, $0 \le k < N$. T is determined by three sequences $\{a_k\}_{k=1}^N$, $\{-c_k\}_{k=0}^N$ and $\{b_k\}_{k=0}^{N-1}$. For simplicity, we write the tridiagonal matrix (2.1) by $T \sim (a_k, -c_k, b_k)$. Suppose $\sup_k(-c_k+|a_k|+|b_k|) = m$, $\sup_k(c_k+|a_k|+|b_k|) = \widehat{m}$. Let $\sigma(T)$ denote the spectra of T. $\lambda_+(T) = \sup\{\lambda \in \sigma(T) \setminus \{m\}\}$ and $\lambda_-(T) = -\inf\{\lambda \in \sigma(T) \setminus \{-\widehat{m}\}\}$ is the upper and lower spectra of T, respectively, i.e.,

$$\sigma(T) \setminus \{m, \widehat{m}\} \subset [\lambda_+(T), -\lambda_-(T)].$$

Before giving the bounds for $\lambda_-(T),\ \lambda_+(T)$, we need the following basis Lemma 2.1, see [6, Corollary 3.3] for complete version and the bound is exact in some examples there.

Lemma 2.1 *Let $Q \sim (a_k, -c_k, b_k)_{k\in E}$ be a birth-death Q-matrix satisfying*

$$\{b_k\}_{k=0}^{N-1} > 0,\ \{a_k\}_{k=1}^{N} > 0,\ b_0 = c_0,\ c_k = a_k + b_k,\ 1 \le k \le N-1,\ c_N > a_N.$$

Make a convention that $b_N = c_N - a_N$. Then $\lambda_-(-Q)$ has the estimate:

$$(4\delta^Q)^{-1} \le (\delta_1^Q)^{-1} \le \lambda_-(-Q) \le (\delta_1^{'Q})^{-1} \le (\delta^Q)^{-1},$$

here $\delta^\sharp$, $\delta_1^\sharp$ and $\delta_1^{'\sharp}$ are three numbers related to the given matrix $\sharp = Q$:

$$\delta^Q = \sup_{0\le n\le N} \sum_{j=0}^{n} \mu_j \sum_{k=n}^{N} \frac{1}{\mu_k b_k}, \tag{2.2}$$

$$\delta_1^Q = \sup_{0\le i\le N} \Big(\sqrt{\varphi_i}\sum_{k=0}^{i} \mu_k \sqrt{\varphi_k} + \frac{1}{\sqrt{\varphi_i}} \sum_{k=i+1}^{N} \mu_k \varphi^{3/2}\Big), \tag{2.3}$$

$$\delta_1^{\prime Q} = \sup_{0\le \ell\le N} \Big(\varphi_\ell \mu[0,\ell] + \frac{1}{\varphi_\ell} \sum_{k=\ell+1}^{N} \mu_k \varphi_k^2\Big) \in [\delta^Q, 2\delta^Q]. \tag{2.4}$$

Where $\{\mu_k\}_{k=0}^N$, $\mu[0,k]$ and $\{\varphi_k\}_{k=0}^N$ are defined as follows:

$$\mu_0 = 1,\ \mu_k = \mu_{k-1}\frac{b_{k-1}}{a_k},\quad \mu[0,k] := \sum_{\ell=0}^{k} \mu_\ell,\quad \varphi_k = \sum_{\ell=k}^{N} \frac{1}{\mu_\ell b_\ell},\ 0 \le k \le N.$$

3. Main results

Before presenting the bounds for $\lambda_\pm(T)$ of a tridiagonal matrix T, we need to understand the characteristic for eigenpairs of T, which are illustrated visually in Example 3.1 for some special case and listed in Theorem 3.2 for general case.

Example 3.1 Given a real tridiagonal matrix of form (2.1),

$$T \sim (a_j, -c_j, b_j) \in \mathbb{R}^{N\times N},$$

where $a_j \equiv a,\ b_j \equiv b$, and $c_j \equiv c$ are positive. Define the relevant matrix $T^- \sim (a_j, c_j, b_j)$. The exact eigenpairs of T and T^- are denoted by $(\lambda_k^{\text{exact}}, g_k^{\text{exact}})$ and $(\lambda_k^{-\text{exact}}, g_k^{-\text{exact}})$, respectively. Then $\lambda_k^{-\text{exact}} = -\lambda_{N+1-k}^{\text{exact}}$ and $g_k^{-\text{exact}}(\ell) = (-1)^{\ell-1} g_{N+1-k}^{\text{exact}}(\ell)$.

Proof By [7, Example 20], the exact eigenpairs of T are defined by

$$\lambda_k^{\text{exact}} = 2\sqrt{ab}\cos(\frac{k\pi}{N+1}) - c,$$

$$g_k^{\text{exact}}(\ell) = (\sqrt{\frac{a}{b}})^\ell \sin(\frac{k\ell\pi}{N+1}),\quad \ell = 1,\dots,N.$$

Combining this with

$$\cos(\frac{k\pi}{N+1}) = -\cos(\frac{(N+1-k)\pi}{N+1}),$$

$$\sin(\frac{k\ell\pi}{N+1}) = (-1)^{\ell-1}\sin(\frac{\ell(N+1-k)\pi}{N+1}).$$

The exact eigenpirs of T^- are

$$\lambda_k^{-\text{exact}} = 2\sqrt{ab}\cos(\frac{k\pi}{N+1}) + c = -\lambda_{N+1-k}^{\text{exact}},$$

$$g_k^{-\text{exact}}(\ell) = (\sqrt{\frac{a}{b}})^{\ell}\sin(\frac{k\ell\pi}{N+1}) = (-1)^{\ell-1}g_{N+1-k}^{\text{exact}}(\ell),\ \ell = 1,\dots,N.$$

Thus $(\lambda_k^{\text{exact}}, g_k^{\text{exact}})$ is an eigenpair of T iff $(-\lambda_k^{\text{exact}}, \text{diag}(\nu)g_k^{\text{exact}})$ is an eigenpair of T^-, where $\text{diag}(\nu)$ is a diagonal matrix having diagonal elements $\nu = \{(-1)^k\}_{k=1}^N \in \mathbb{R}^N$. □

The fact in Example 3.1 is still valid for any tridiagonal matrix.

Theorem 3.2 *Given a tridiagonal matrix $T \sim (a_k, -c_k, b_k)_{k\in E}$ of form (2.1), define a tridiagonal matrix $T^- \sim (a_k, c_k, b_k)_{k\in E}$. Then (g, λ) is an eigenpair of T iff $(\text{diag}(u)g, -\lambda)$ is an eigenpair of T^-, where $\text{diag}(u)$ is a diagonal matrix with diagonal elements $(u_k) : u_0 = 1, u_k = -u_{k-1}, 1 \le k < N+1$.*

Proof For $T \sim (a_k, -c_k, b_k)_{k=0}^N$, define $\widetilde{T} \sim (-a_k, -c_k, -b_k)$ and diagonal matrix $\text{diag}(u)$ with elements $u = (u_k)$. Then

$$T = \text{diag}(u^{-1})\widetilde{T}\text{diag}(u), \quad \text{i.e., } T \simeq \widetilde{T}.$$

Thus, (g, λ) is an eigenpair of T iff $(\text{diag}(u)g, \lambda)$ is an eigenpair of $\widetilde{T}$. Noticing that $T^- \sim (a_k, c_k, b_k)_{k=0}^N$, $T^- = -\widetilde{T}$, we get (g, λ) is an eigenpair of T iff $(\text{diag}(u)g, -\lambda)$ is an eigenpair of T^-. □

Theorem 3.2 illustrates an important property for eigenpairs of T, especially the variation principle of symbols for eigenvectors of T. Combining with the sharp bounds in Lemma 1.1, it inspires us that we can use T and T^- to estimate $\lambda_+(T)$ and $\lambda_+(T^-)$ (i.e., $\lambda_-(T)$), respectively. For $T \sim (a_k, -c_k, b_k)$, define m and $\widehat{m}$ as before:

$$m = \sup_{k\in E}(-c_k + |a_k| + |b_k|)^+,\ \widehat{m} = \sup_{k\in E}(c_k + |a_k| + |b_k|)^+.$$

Here we make a convention that $a_0 = 0$, $b_N = 0$ in the definition of m and $\widehat{m}$. Set $u_k = a_k b_{k-1}$, $1 \le k \le N$, define two particular birth-death Q-matrices $\widetilde{Q} \sim (\widetilde{a}_k, -\widetilde{c}_k, \widetilde{b}_k)_{k=0}^N$ and $\widehat{Q} \sim (\widehat{a}_k, -\widehat{c}_k, \widehat{b}_k)_{k=0}^N$:

$$\begin{cases} \widetilde{c}_k = c_k + m, \quad k \in E; \\ \widetilde{b}_0 = \widetilde{c}_0 > 0, \quad \widetilde{b}_k = \widetilde{c}_k - u_k/\widetilde{b}_{k-1}, \quad 1 \le k < N; \\ \widetilde{a}_k = \widetilde{c}_k - \widetilde{b}_{k-1}, \quad 1 \le k < N,\ \widetilde{a}_N = u_N/\widetilde{b}_{N-1}. \end{cases} \tag{3.1}$$

$$\begin{cases} \widehat{c}_k = -c_k + \widehat{m}, \quad k \in E; \\ \widehat{b}_0 = \widehat{c}_0 > 0, \quad \widehat{b}_k = \widehat{c}_k - u_k/\widehat{b}_{k-1},\ 1 \le k < N; \\ \widehat{a}_k = \widehat{c}_k - \widehat{b}_{k-1}, \quad 1 \le k < N,\ \widehat{a}_N = u_N/\widehat{b}_{N-1}. \end{cases} \tag{3.2}$$

(3.1) and (3.2) are obtained by h-transform, which can be obtained immediately from Proposition 3.7 below. Then if needed, define another two particular Q-matrices $\widetilde{Q}^{\setminus 0} \sim (\widetilde{a}_k^{\setminus 0}, -\widetilde{c}_k^{\setminus 0}, \widetilde{b}_k^{\setminus 0})_{k=1}^N$,

$\widehat{Q}^{\setminus 0} \sim (\widehat{a}_k^{\setminus 0}, -\widehat{c}_k^{\setminus 0}, \widehat{b}_k^{\setminus 0})_{k=1}^N$ by using $\widetilde{Q}$ and $\widehat{Q}$, respectively:

$$\begin{cases} \widetilde{c}_k^{\setminus 0} = \widetilde{a}_k + \widetilde{b}_{k-1}, \quad k \in E \setminus \{0\}; \\ \widetilde{b}_1^{\setminus 0} = \widetilde{c}_1^{\setminus 0} > 0, \ \ \widetilde{b}_k^{\setminus 0} = \widetilde{c}_k^{\setminus 0} - \widetilde{a}_{k-1}\widetilde{b}_{k-1}/\widetilde{b}_{k-1}^{\setminus 0}, \ 2 \le k < N; \\ \widetilde{a}_k^{\setminus 0} = \widetilde{c}_k^{\setminus 0} - \widetilde{b}_{k-1}^{\setminus 0}, \quad 2 \le k < N, \ \widetilde{a}_N^{\setminus 0} = \widetilde{a}_{N-1}\widetilde{b}_{N-1}/\widetilde{b}_{N-1}^{\setminus 0}. \end{cases} \tag{3.3}$$

$$\begin{cases} \widehat{c}_k^{\setminus 0} = \widehat{a}_k + \widehat{b}_{k-1}, \quad k \in E \setminus \{0\}; \\ \widehat{b}_1^{\setminus 0} = \widehat{c}_1^{\setminus 0} > 0, \ \ \widehat{b}_k^{\setminus 0} = \widehat{c}_k^{\setminus 0} - \widehat{a}_{k-1}\widehat{b}_{k-1}/\widehat{b}_{k-1}^{\setminus 0}, \ 2 \le k < N; \\ \widehat{a}_k^{\setminus 0} = \widehat{c}_k^{\setminus 0} - \widehat{b}_{k-1}^{\setminus 0}, \quad 2 \le k < N, \ \widehat{a}_N^{\setminus 0} = \widehat{a}_{N-1}\widehat{b}_{N-1}/\widehat{b}_{N-1}^{\setminus 0}. \end{cases} \tag{3.4}$$

Then three numbers defined in (2.2)–(2.4) related to $\widetilde{Q}$, $\widetilde{Q}^{\setminus 0}$, $\widehat{Q}$, and $\widehat{Q}^{\setminus 0}$ are denoted by $(\delta^\sharp,\ \delta_1^\sharp,\ \delta_1'^\sharp)$ when the matrix $\sharp$ is some Q-matrix. The sharp bounds for $\lambda_-(T)$ and $\lambda_+(T)$ in this paper are presented in Theorem 3.3.

Theorem 3.3 *Suppose $T \sim (a_k, -c_k, b_k)_{k\in E}$ is a Hermitizable tridiagonal matrix of form (2.1), define m and $\widehat{m}$ as before. Let $\sigma(T)$ denote all the eigenvalues of T, and set $\lambda_+(T) = \sup\{\lambda \in \sigma(T)\setminus\{m\}\}$, $\lambda_-(T) = -\inf\{\lambda \in \sigma(T)\setminus\{\widehat{m}\}\}$, the estimates for $\lambda_+(T)$ and $\lambda_-(T)$ are presented as follows:*

$$(4\delta^\sharp)^{-1} \le (\delta_1^\sharp)^{-1} \le m - \lambda_+(T) \le (\delta_1'^\sharp)^{-1} \le (\delta^\sharp)^{-1},$$
$$(4\delta^\natural)^{-1} \le (\delta_1^\natural)^{-1} \le \widehat{m} - \lambda_-(T) \le (\delta_1'^\natural)^{-1} \le (\delta^\natural)^{-1},$$

where $\sharp$ and $\natural$ are two Q-matrices corresponding to the following four cases.

(1) If for every $k \in E$, $-c_k + |a_k| + |b_k| \equiv m$, $c_k + |a_k| + |b_k| \equiv \widehat{m}$, then $\sharp = \widetilde{Q}^{\setminus 0}$ and $\natural = \widehat{Q}^{\setminus 0}$.

(2) If for every $k \in E$, $-c_k + |a_k| + |b_k| \equiv m$, but $c_k + |a_k| + |b_k| \not\equiv \widehat{m}$, then $\sharp = \widetilde{Q}^{\setminus 0}$ and $\natural = \widehat{Q}$.

(3) If for every $k \in E$, $c_k + |a_k| + |b_k| \equiv \widehat{m}$, but $c_k + |a_k| + |b_k| \not\equiv m$, then $\sharp = \widetilde{Q}$ and $\natural = \widehat{Q}^{\setminus 0}$.

(4) If there exist $k, \ell \in E$, such that $-c_k + |a_k| + |b_k| \ne m$ and $c_\ell + |a_\ell| + |b_\ell| \ne \widehat{m}$, then we use matrices $\sharp = \widetilde{Q}$ and $\natural = \widehat{Q}$.

Before proving Theorem 3.3, let us compute two examples to illustrate the power of the bounds first. The following example is the continuation of Example 3.1.

Example 3.4 (Example 3.1 Continued) For $a = 2$, $b = 1$, $c = 3$ and $a = 1$, $b = 2$, $c = 3$ (see [7, Example 20]) tested the algorithms there. For the two cases, we have

$$\lambda_+(T) = 3 - 2\sqrt{2}\cos\frac{\pi}{N+1}, \quad \lambda_-(T) = 3 - 2\sqrt{2}\cos\frac{N\pi}{N+1},$$

N	$-\lambda_+(T)$			$-\lambda_-(T)$		
	Lower	Exact	Upper	Lower	Exact	Upper
50	0.1716	0.1769	0.3333	5.6667	5.8231	5.8284
100	0.1729	0.1769	0.3333	5.6667	5.8271	5.8284
1000	0.1716	0.1716	0.3333	5.6667	5.8284	5.8284

Table 1 Estimates in Theorem 3.3 for $\lambda_+(T)$ and $\lambda_-(T)$

the estimates in Theorem 3.3 for $\lambda_+(T)$ and $\lambda_-(T)$ are presented in Table 1, where $m = 1$ for $T \sim (a_k, -c_k, b_k)$, and $\widehat{m} = 6$ for $T^- \sim (a_k, c_k, b_k)$.

The next example is a modification of [9, Example 7].

Example 3.5 Let $\{\beta_n\}_{n=0}^N$ be a given arbitrarily real sequence,

$$b_n = n^4 e^{i\beta_n},\ a_{n+1} = (n(n+1))^2 e^{-i\beta_n},\ c_n = |a_n| + |b_n|,\quad 1 \le n \le N-1,$$

$$a_1 = 1,\ b_0 = 1,\ c_0 = 1,\ c_N = |a_N|.$$

Here $m = 0$ for $T \sim (a_n, -c_n, b_n)$ and 0 is a trivial eigenvalue of T, thus $\lambda_+(T)$ is an eigenvalue of T adjacent to 0. For $T^- \sim (a_n, c_n, b_n)$, $\widehat{m} = 2(a_{N-1} + b_{N-1})$, $\lambda_-(T)$ is the maximal eigenvalue of T^-. The estimates in Theorem 3.3 for $\lambda_+(T)$ and $\lambda_-(T)$ are presented in Tables 2 and 3.

N	Lower$_1$	Lower$_2$	Exact	Upper$_1$	Upper$_2$
10	0.41236	1.2365	1.3008	1.4549	1.6494
25	0.40585	1.1737	1.2218	1.4020	1.6234

Table 2 Estimates in Theorem 3.3 for $\lambda_+(T)$

N	Lower$_1$	Lower$_2$	Exact	Upper$_1$	Upper$_2$
10	973.73	2582.8	2870.8	3178.1	3894.9
25	65722	177810	202340	234750	262890

Table 3 Estimates in Theorem 3.3 for $\lambda_-(T)$

Here the Lower$_1$ and Lower$_2$ denote $(4\delta^\sharp)^{-1}$ and $(\delta_1^\sharp)^{-1}$, respectively. Upper$_1$ and Upper$_2$ denote $(\delta_1^{'\sharp})^{-1}$ and $(\delta^\sharp)^{-1}$, respectively.

When $N = \infty$, [9, Example 7] proved that both $\mathrm{Spec}(T_{\min})$ and $\mathrm{Spec}(T_{\max})$ are discrete. The estimates in Theorem 3.3 are valid for infinite tridiagonal matrices with discrete spectrum.

Remark 3.6 (1) When $N = \infty$, the bounds for Hermitizable tridiagonal matrix with discrete spectrum can be obtained by some modifications.

(2) Similar results can be obtained for Block tridiagonal matrices.

(3) For Hermitizable matrices, combining with Householder transform, the bounds can be obtained.

To prove Theorem 3.3, we present the h-transform in [8, Theorem 18] and prove it here for reader's convenience. Here the idea of h-transformation is taken from the manuscript "The case having negative spectrum of P" written by Mufa CHEN.

Proposition 3.7 *Suppose $T \sim (a_k, -c_k, b_k)_{k\in E}$ is a Hermitizable tridiagonal matrix of form (2.1). Set*

$$\widetilde{Q} = \mathrm{diag}(h)^{-1}(T - mI)\,\mathrm{diag}(h), \tag{3.5}$$

where $m = \sup_{k\in E}(-c_k + |a_k| + |b_k|)^+$, $h = (h_k)_{k\in E}$ with $h_0 = 1$ is the solution to

$$((T - mI)h)(k) = 0,\quad k \in E \setminus \{N\}. \tag{3.6}$$

Then the explicit representation of $\widetilde{Q}$ is the birth-death Q-matrix $\widetilde{Q} \sim (\widetilde{a}_k, -\widetilde{c}_k, \widetilde{b}_k)_{k=0}^N$ given in (3.1) satisfying $\widetilde{c}_N \geq \widetilde{a}_N$.

Proof First, we prove $h_k \neq 0$ $k \in E$. By (3.6), we have $h_1 = \frac{c_0+m}{b_0}$, combining with $m = \sup_{k\in E}(-c_k + |a_k| + |b_k|)^+$, we have $|h_1| \geq |h_0| = 1 > 0$ and by induction

$$|h_{k+1}| = |\frac{c_k+m}{b_k}h_k - \frac{a_k}{b_k}h_{k-1}| \geq |\frac{c_k+m}{b_k}h_k| - |\frac{a_k}{b_k}h_{k-1}|$$
$$\geq \frac{|a_k|+|b_k|}{|b_k|}|h_k| - \frac{|a_k|}{|b_k|}|h_{k-1}| \geq \frac{|a_k|+|b_k|}{|b_k|}|h_k| - \frac{|a_k|}{|b_k|}|h_k| = |h_k| > 0.$$

Next, we prove the explicit representation of $\widetilde{Q}$ in (3.5) is the one in (3.1) and

$$\widetilde{a}_k > 0\ (1 \leq k \leq N),\ \widetilde{b}_k > 0\ (0 \leq k \leq N-1),\ \widetilde{c}_N \geq \widetilde{a}_N.$$

Indeed, by (3.5), we have

$$\widetilde{c}_k = c_k + m,\ \widetilde{a}_k = \frac{h_{k-1}}{h_k}a_k,\ \widetilde{b}_k = \frac{h_{k+1}}{h_k}b_k. \tag{3.7}$$

Recall $h_{k+1} = \frac{c_k+m}{b_k}h_k - \frac{a_k}{b_k}h_{k-1}$, $1 \leq k \leq N-1$, thus

$$a_k\frac{h_{k-1}}{h_k} = \widetilde{c}_k - b_k\frac{h_{k+1}}{h_k}\ \text{(i.e., } \widetilde{a}_k = \widetilde{c}_k - \widetilde{b}_k),\quad 1 \leq k \leq N-1.$$

By $h_1 = \frac{c_0+m}{b_0}$, we have $\widetilde{b}_0 = \frac{h_1}{h_0}b_0 = c_0 + m = \widetilde{c}_0 > 0$. Notice that $u_k := \widetilde{a}_k\widetilde{b}_{k-1} = a_kb_{k-1} > 0$, $k \geq 1$ from (3.7), thus $\widetilde{a}_1 = \frac{a_1b_0}{\widetilde{b}_0} > 0$. Recall that $|h_{k+1}| \geq |h_k|$, thus $|\widetilde{a}_k| = |\frac{h_{k-1}}{h_k}a_k| \leq |a_k|$. Furthermore,

$$\widetilde{b}_1 = \widetilde{c}_1 - |\widetilde{a}_1| \geq |a_1| + |b_1| - |a_1| = |b_1| > 0.$$

By induction

$$\widetilde{a}_k = \frac{a_kb_{k-1}}{\widetilde{b}_{k-1}} > 0,\ \widetilde{b}_k = \widetilde{c}_k - |\widetilde{a}_k| > 0,\quad 1 \leq k \leq N-1.$$

Besides, $0 < \widetilde{a}_N = \frac{a_Nb_{N-1}}{\widetilde{b}_{N-1}} \leq |a_N| \leq m + c_N = \widetilde{c}_N$. $\square$

The shift m in the proof of h-transform guarantees the $\widetilde{Q}$ obtained in Proposition 3.7 is an irreducible birth-death Q-matrix, which is given by Chen [1]. If $-c_k + |a_k| + |b_k| \equiv m$, then $\widetilde{a}_k = |a_k|$ $\widetilde{b}_k = |b_k|$, $\widetilde{a}_N = \widetilde{c}_N$. Thus 0 is a trivial eigenvalue of $\widetilde{Q}$ and m is an eigenvalue of T.

Proof of Theorem 3.3 Step 1. For Case (4), apply Proposition 3.7 to the tridiagonal matrices $T - mI \sim (a_k, -c_k - m, b_k)_{k\in E}$ and $T^- - \widehat{m}I \sim (a_k, c_k - \widehat{m}, b_k)_{k\in E}$, respectively. Then the explicit representation of $\widetilde{Q}$ and $\widehat{Q}$ are listed in (3.1) and (3.2) with $\widetilde{a}_N < \widetilde{c}_N$, $\widehat{a}_N < \widehat{c}_N$. Thus $T - mI$ (resp., $T^- - \widehat{m}I$) has the same spectrum as the birth-death Q-matrix $\widetilde{Q}$ (resp., $\widehat{Q}$). Combining this with Lemma 2.1 and Theorem 3.2, we complete the proof.

Step 2. To prove Cases (1)–(3). It suffices to prove that when a birth-death Q-matrix $\widetilde{Q} \sim (\widetilde{a}_k, -\widetilde{c}_k, \widetilde{b}_k)_{k\in E}$ satisfies that

$$\widetilde{b}_0 = \widetilde{c}_0,\ \widetilde{a}_k + \widetilde{b}_k = \widetilde{c}_k\ (1 \leq k \leq N-1),\ \widetilde{c}_N = \widetilde{a}_N,$$

then $\widetilde{Q}$ has the same nontrivial eigenvalues as the birth-death Q-matrix $\widetilde{Q}^{\setminus 0} \sim (\widetilde{a}_k^{\setminus 0}, -\widetilde{c}_k^{\setminus 0}, \widetilde{b}_k^{\setminus 0})_{k=1}^N$ defined in (3.3). Indeed, take

$$\mathcal{M}=\begin{pmatrix} \widetilde{\mu}_0 & \widetilde{\mu}_1 & \widetilde{\mu}_2 & \cdots & \widetilde{\mu}_N \\ & \widetilde{\mu}_1 & \widetilde{\mu}_2 & \cdots & \widetilde{\mu}_N \\ & & \widetilde{\mu}_2 & \cdots & \widetilde{\mu}_N \\ & & & \ddots & \vdots \\ 0 & & & & \widetilde{\mu}_N \end{pmatrix},$$

where $\widetilde{\mu}_0 = 1$, $\widetilde{\mu}_n = \widetilde{\mu}_{n-1}\frac{\widetilde{b}_{n-1}}{\widetilde{a}_n}$, $1 \le n \le N$. Then $\widetilde{Q}' = \mathcal{M}\widetilde{Q}\mathcal{M}^{-1}$:

$$\widetilde{Q}'=\begin{pmatrix} 0 & 0 & 0 & & & \\ \widetilde{b}_0 & -(\widetilde{a}_1+\widetilde{b}_0) & \widetilde{a}_1 & & & \\ & \widetilde{b}_1 & -(\widetilde{a}_2+\widetilde{b}_1) & \widetilde{a}_2 & & \\ & \ddots & \ddots & \ddots & & \\ & & \widetilde{b}_{N-2} & -(\widetilde{a}_{N-1}+\widetilde{b}_{N-2}) & \widetilde{a}_{N-1} \\ & & & \widetilde{b}_{N-1} & -(\widetilde{a}_N+\widetilde{b}_{N-1}) \end{pmatrix}.$$

By deleting the first line and column of $\widetilde{Q}'$, we get the Q-matrix $\widetilde{Q}_1' \sim (\widetilde{b}_k, -(\widetilde{a}_k+\widetilde{b}_{k-1}), \widetilde{a}_k)_{k=1}^N$. Now, the spectra of Q ignoring the trivial eigenvalue 0 coincides with the spectra of $\widetilde{Q}_1'$. Then applying h transform to the matrix $\widetilde{Q}_1'$, we obtain the results. □

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Conflict of Interest The authors declare no conflict of interest.

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