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Parameterized Littlewood-Paley Operators and Their Commutators on Two-Weight Grand Homogeneous Variable Herz-Morrey Spaces

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Abstract In this paper, the authors prove that the parameterized area integral $\mu^{\rho}_{\Omega,S}$ and the parameterized Littlewood-Paley g^{*}_{δ}-function $\mu^{*,\rho}_{\Omega,\delta}$ are bounded on two-weight grand homogeneous variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)$, where $\theta>0$, $\lambda\in(2,\infty)$, $q(\cdot)\in\mathbb{B}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^{\infty}(\mathbb{R}^n)$, $\omega_1\in A_{p_{\omega_1}}$ for $p_{\omega_1}\in[1,\infty]$ and ω_2 is a weight. Furthermore, the authors prove that the commutators $[b,\mu^{\rho}_{\Omega,S}]$ which is formed by $b\in\mathrm{BMO}(\mathbb{R}^n)$ and the $\mu^{\rho}_{\Omega,S}$, and the $[b,\mu^{*,\rho}_{\Omega,\delta}]$ generated by $b\in\mathrm{BMO}(\mathbb{R}^n)$ and the $\mu^{*,\rho}_{\Omega,\delta}$ are bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)$, respectively.

Keywords Grand homogeneous variable Herz-Morrey space; parameterized area integral; parameterized Littlewood-Paley g^{*}_{δ}-function; commutator; space $\mathrm{BMO}(\mathbb{R}^n)$

MR(2020) Subject Classification 42B20; 42B35; 47B47

1. Introduction

Suppose that the $\mathbb{S}^{n-1}$ is a unit sphere in $\mathbb{R}^n$ $(n\geq 2)$ equipped with the Lebesgue measure $\mathrm{d}\sigma=\mathrm{d}\sigma(x')$. And let Ω be homogeneous function of degree zero and satisfy

$$\int_{\mathbb{S}^{n-1}}\Omega(x')\mathrm{d}\sigma(x')=0, \tag{1.1}$$

where $x'=\frac{x}{|x|}$ for $x\in\mathbb{R}^n\backslash\{0\}$. In 1999, Sakamoto and Yabuta [1] introduced a parameterized area integral $\mu^{\rho}_{\Omega,S}$ and a Littlewood-Paley g^{*}_{δ}-function $\mu^{*,\rho}_{\Omega,\delta}$, i.e, for any $\rho>0$ and $\delta>1$,

$$\mu^{\rho}_{\Omega,S}(f)(x)=\Big(\int_0^{\infty}\int_{|x-y|<t}\Big|\int_{|y-z|\leq t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+2\rho+1}}\Big)^{\frac12} \tag{1.2}$$

and

$$\mu^{*,\rho}_{\Omega,\delta}(f)(x)=\Big(\int_0^{\infty}\int_{\mathbb{R}^{n+1}_{+}}\Big(\frac{t}{t+|x-y|}\Big)^{\delta n}\Big|\int_{|y-z|\leq t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+2\rho+1}}\Big)^{\frac12}. \tag{1.3}$$

And also they showed that the $\mu^{\rho}_{\Omega,S}$ and the $\mu^{*,\rho}_{\Omega,\delta}$ are bounded on Lebesgue spaces $L^p(\mathbb{R}^n)$, where $1<p<\infty$ and $\Omega\in\mathrm{Lip}_{\beta}(\mathbb{S}^{n-1})$ for $\beta\in(0,1]$. Since then, many papers focus on the boundedness of the $\mu^{\rho}_{\Omega,S}$ and the $\mu^{*,\rho}_{\Omega,\delta}$ on various function spaces. For example, in 2019, Li [2] showed the $\mu^{\rho}_{\Omega,S}$ and the $\mu^{*,\rho}_{\Omega,\delta}$ are bounded from the Musielak-Orlicz Hardy spaces $H^{\varphi}(\mathbb{R}^n)$ into

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the Musielak-Orlicz spaces $L^{\varphi}(\mathbb{R}^n)$, and also obtained the endpoint estimates for the $\mu^{\rho}_{\Omega,S}$ and the $\mu^{*,\rho}_{\Omega,\delta}$, where Ω satisfies the $L^{2,\beta}$-Dini condition or Lipschitz condition of order β. In 2020, Liu et al. [3] proved that the $\mu^{\rho}_{\Omega,S}$ and the $\mu^{*,\rho}_{\Omega,\delta}$ are bounded from the homogeneous variable Herz-type Hardy spaces $H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ into the homogeneous variable Herz spaces $\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$. In 2021, Wang and Guo [4] proved that the $\mu^{\rho}_{\Omega,S}$ and the $\mu^{*,\rho}_{\Omega,\delta}$ are bounded on homogeneous Herz-Morrey spaces with variable exponents $M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}(\mathbb{R}^n)$. More researches about the operators $\mu^{\rho}_{\Omega,S}$ and $\mu^{*,\rho}_{\Omega,\delta}$ on various function spaces can be seen in [5–9].

The definition of spaces $\mathrm{BMO}(\mathbb{R}^n)$ is as follows

Definition 1.1 ([10]) *A function $b\in L^1_{\rm loc}(\mathbb{R}^n)$ is said to belong to the space* $\mathrm{BMO}(\mathbb{R}^n)$ *if*

$$\|b\|_*=\sup_B\frac{1}{|B|}\int_B|b(y)-b_B|\mathrm{d}y<\infty,$$

where b_B represents the average of b over ball B, i.e.,

$$b_B=\frac{1}{|B|}\int_B b(t)\mathrm{d}t.$$

The commutators $[b,\mu^{\rho}_{\Omega,S}]$ and the $[b,\mu^{*,\rho}_{\Omega,\delta}]$ are respectively defined by

$$[b,\mu^{\rho}_{\Omega,S}](f)(x)=\Big(\int_0^{\infty}\int_{|x-y|\le t}\Big|\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(y)-b(z)]f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+2\rho+1}}\Big)^{\frac12} \tag{1.4}$$

and

$$\begin{aligned}[b,\mu^{*,\rho}_{\Omega,\delta}](f)(x)=&\Big(\int_0^{\infty}\int_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\delta n}\times\\ &\Big|\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(y)-b(z)]f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+2\rho+1}}\Big)^{\frac12}.\end{aligned} \tag{1.5}$$

Since then, the boundedness of the commutators $[b,\mu^{\rho}_{\Omega,S}]$ and $[b,\mu^{*,\rho}_{\Omega,\delta}]$ are widely studied by many authors. For example, in 2017, Wang and Wu [11] got the boundedness of the $[b^m,\mu^{\rho}_{\Omega,S}]$ and the $[b^m,\mu^{*,\rho}_{\Omega,\delta}]$ on variable Herz spaces $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and on non-homogeneous variable Herz spaces $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. In 2020, Liu et al. in [3] showed that the $[b^m,\mu^{\rho}_{\Omega,S}]$ and the $[b^m,\mu^{*,\rho}_{\Omega,\delta}]$ are bounded from the variable Herz-type Hardy spaces $H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ into the variable Herz spaces $\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$, where Ω satisfies the $L^{2,\beta}$-Dini condition. In 2021, Wang and Guo [4] established their boundedness on homogeneous variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}(\mathbb{R}^n)$ when $\Omega\in L^2(\mathbb{S}^{n-1})$. More progress on the boundedness of the commutators $[b,\mu^{\rho}_{\Omega,S}]$ and $[b,\mu^{*,\rho}_{\Omega,\delta}]$ can be seen [12–16] and the references therein.

Definition 1.2 ([17]) *Let $1\le p<\infty$ and $\theta>0$. Then the grand Lebesgue sequence spaces $l^{p),\theta}$ is defined by*

$$\|X\|_{l^{p),\theta}(\mathbb{X})}=\sup_{\epsilon>0}\Big(\epsilon^{\theta}\sum_{k\in\mathbb{X}}|x_k|^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}=\sup_{\epsilon>0}\epsilon^{\frac{\theta}{p(1+\epsilon)}}\|X\|_{l^{p(1+\epsilon)}(\mathbb{X})}<\infty,$$

where $X=\{x_k\}_{k\in\mathbb{X}}$ and $\mathbb{X}$ represents one of sets $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_0$.

Definition 1.3 ([18]) *Let $0\le\lambda<\infty$, $1<p<\infty$, $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^{\infty}(\mathbb{R}^n)$, $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$,* *and $\theta > 0$, then the grand homogeneous variable Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)$ is defined by*

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n) = \big\{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)} < \infty\big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\epsilon^\theta \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^{p(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\epsilon)}}. \tag{1.6}$$

Remark 1.4 (a) If we take $\lambda = 0$ in (1.6), then the space $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)$ is just the grand homogeneous variable Herz space $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ introduced in [19].

(b) When $\epsilon = 0$, then $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$; when $\epsilon = 0$ and $\alpha(\cdot) \equiv$ const, then $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

In 2020, Nafis et al. introduced the grand variable Herz spaces $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$, and they proved that the sublinear operators T, Marcinkiewicz integrals operators $\mathcal{M}_\Omega$ and the multilinear Calderón-Zygmund operators $\mathcal{T}$ are bounded on spaces $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ (see [19–21], respectively). In 2022, Sultan et al. [18] introduced the grand variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)$ (see Definition 1.3), and proved that the Riesz potential operators I^γ are bounded from spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q_1(\cdot)}(\mathbb{R}^n)$ into spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q_2(\cdot)}(\mathbb{R}^n)$. Furthermore, further developments of the grand variable spaces can be seen [22–27].

Definition 1.5 ([28]) *Let ω be a weight on $\mathbb{R}^n$ and $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function, for any $p(\cdot)$,*

$$1 \le p_- \le p(x) \le p_+ < \infty,$$

where

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 1; \quad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

We denote $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and if

$$\|f\|_{L^{p(\cdot)}(\omega)} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \big(\frac{|f(x)|}{\lambda}\big)^{p(x)} \omega(x)\mathrm{d}x \le 1\Big\},$$

then $f \in L^{p(\cdot)}(\omega)$. If $\omega = 1$, then we simply write $L^{p(\cdot)}(\omega) = L^{p(\cdot)}$.

Definition 1.6 *Let $0 \le \lambda < \infty$, $1 < p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $\theta > 0$, $\omega_1 \in A_{p_{\omega_1}}$ for $p_{\omega_1} \in [1, \infty]$ and ω_2 be a weight. Then the two-weight grand homogeneous variable Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1, \omega_2)$ is defined by*

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1, \omega_2) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)} = \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\epsilon^\theta \sum_{k=-\infty}^{k_0} \|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} f\chi_k\|^{p(1+\epsilon)}_{L^{q(\cdot)}(\omega_2)}\Big)^{\frac{1}{p(1+\epsilon)}}. \tag{1.7}$$

Definition 1.7 ([8]) *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a log-Hölder continuous function if there exists a constant $C = C_{\log} > 0$ such that,*

$$|g(x) - g(y)| \le \frac{C}{\log(e + 1/|x-y|)}, \quad |x-y| \le \frac{1}{2}, \ x, y \in \mathbb{R}^n.$$

If the following two conditions

$$|g(x) - g_\infty| \le \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n \tag{1.8}$$

and

$$|g(x) - g(0)| \le \frac{C}{\log(e + 1/|x|)}, \quad |x| \le \frac{1}{2} \tag{1.9}$$

hold, then we say $g(\cdot)$ has a log decay at infinity and at the origin, where $g_\infty = \lim_{x\to\infty} g(x)$.

It is now position to state the main theorems of this paper as follows:

Theorem 1.8 *Let $1 < p < \infty$, $\lambda > 2$, $\rho > \frac{n}{2}$, $\Omega \in L^2(\mathbb{S}^{n-1})$ satisfy (2.1), $\omega_1 \in A_{p_{\omega_1}}$ for some $p_{\omega_1} \in [1, \infty)$, ω_2 be a weight, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $q(\cdot) \in \mathbb{B}$ satisfy (1.8) and (1.9). Suppose that $\delta_1, \delta_2 \in (0, 1)$,*

$$-n\delta_1 < \omega^-\alpha^-, \quad \omega^+\alpha^+ < n\delta_2.$$

Then there exists a positive constant C such that, for any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1, \omega_2)$,

$$\|\mu^\rho_{\Omega,S}(f)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)} \le C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}.$$

Theorem 1.9 *Let $1 < p < \infty$, $\delta > 2$, $\lambda > 2$, $\rho > \frac{n}{2}$, $\Omega \in L^2(\mathbb{S}^{n-1})$ satisfy (2.1), $\omega_1 \in A_{p_{\omega_1}}$ for some $p_{\omega_1} \in [1, \infty)$, ω_2 be a weight, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $q(\cdot) \in \mathbb{B}$ satisfy (1.8) and (1.9). Suppose that $\delta_1, \delta_2 \in (0, 1)$,*

$$-n\delta_1 < \omega^-\alpha^-, \quad \omega^+\alpha^+ < n\delta_2.$$

Then there exists a positive constant C such that, for any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1, \omega_2)$,

$$\|\mu^{*,\rho}_{\Omega,\delta}(f)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)} \le C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}.$$

Theorem 1.10 *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, $1 < p < \infty$, $\sigma > 2$, $\lambda > 2$, $\rho > \frac{n}{2}$, $\Omega \in L^2(\mathbb{S}^{n-1})$ satisfy (2.1), $\omega_1 \in A_{p_{\omega_1}}$ for some $p_{\omega_1} \in [1, \infty)$, ω_2 be a weight, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $q(\cdot) \in \mathbb{B}$ satisfy (1.8) and (1.9). Suppose that $\delta_1, \delta_2 \in (0, 1)$,*

$$-n\delta_1 < \omega^-\alpha^-, \quad \omega^+\alpha^+ < n\delta_2.$$

Then there exists a positive constant C such that, for any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1, \omega_2)$,

$$\|[b, \mu^\rho_{\Omega,S}](f)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)} \le C\|b\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}.$$

Theorem 1.11 *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, $\delta > 2$, $1 < p < \infty$, $\sigma > 2$, $\lambda > 2$, $\rho > \frac{n}{2}$, $\Omega \in L^2(\mathbb{S}^{n-1})$ satisfy (2.1), $\omega_1 \in A_{p_{\omega_1}}$ for some $p_{\omega_1} \in [1, \infty)$, ω_2 be a weight, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $q(\cdot) \in \mathbb{B}$ satisfy (1.8) and (1.9). Suppose that $\delta_1, \delta_2 \in (0, 1)$,*

$$-n\delta_1 < \omega^-\alpha^-, \quad \omega^+\alpha^+ < n\delta_2.$$

Then there exists a positive constant C such that, for any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)$,

$$\|[b,\mu^{*,\rho}_{\Omega,\delta}](f)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)} \leq C\|b\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}.$$

Remark 1.12 ([29,30]) For any $x \in \mathbb{R}^n$,

$$\mu^{\rho}_{\Omega,S}(f)(x) \leq C2^{n\lambda}\mu^{*,\rho}_{\Omega,\delta}(f)(x)$$

and

$$[b,\mu^{\rho}_{\Omega,S}](f)(x) \leq C2^{n\lambda}[b,\mu^{*,\rho}_{\Omega,\delta}](f)(x),$$

it is easy to see that Theorems 1.8 and 1.10 hold. Thus, in this paper, we only state the proofs of Theorems 1.9 and 1.11.

Finally, confirm the symbols and notions of this article. C represents a constant being independent of the main parameters, but may vary from row to row. $p(\cdot)$ represents the conjugate exponents defined by $1/p(\cdot) + 1/p'(\cdot) = 1$. The expression $f \approx g$ means $C_1 f \leq g \leq C_2 f$. We denote $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $D_k = B_k \backslash B_{k-1}$. We also need note that if $p(\cdot) \in \mathcal{P}$, then the Hardy-Littlewood maximal operator M is bounded on spaces $L^{p}(\cdot)$, namely, $p(\cdot) \in \mathbb{B}$.

2. Preliminaries

To prove the main theorems, in this section, we need to recall some necessary lemmas.

Lemma 2.1 ([9]) *Suppose $\rho > \frac{n}{2}$, $\lambda > 2$ and $\Omega \in L^2(\mathbb{S}^{n-1})$ satisfies*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^{\sigma} < \infty \tag{2.1}$$

for some $\sigma > 1$. If $1 < p < \infty$ and $\omega \in A_p$, then the $\mu^{\rho}_{\Omega,S}$ and $\mu^{*,\rho}_{\Omega,\delta}$ are bounded on weighted Lebesgue spaces $L^p(\omega)$.

Lemma 2.2 ([14]) *Suppose $\rho > \frac{n}{2}$, $\lambda > 2$ and $\Omega \in L^2(\mathbb{S}^{n-1})$ satisfies (2.1) for some $\sigma > 2$. If $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\omega \in A_p$, both of $[b,\mu^{\rho}_{\Omega,S}]$ and $[b,\mu^{*,\rho}_{\Omega,\delta}]$ are bounded on the weighted Lebesgue spaces $L^p(\omega)$.*

Lemma 2.3 ([28]) *Let $k,l \in \mathbb{Z}$, $\omega \in A_q$ with $q \in [1,\infty)$, $\delta \in (0,1)$, $\omega_- = \begin{cases} \delta, & \alpha^- \geq 0 \\ q, & \alpha^- \leq 0 \end{cases}$ and $\omega_+ = \begin{cases} q, & \alpha^+ \geq 0 \\ \delta, & \alpha^+ \leq 0 \end{cases}$. If $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n)$ is a* log-Hölder *continuous function satisfying (1.8) and (1.9), then for any $x \in D_k$ and $y \in D_l$ such that*

$$[\omega(B_k)]^{\alpha(x)} \leq C[\omega(B_l)]^{\alpha(y)} \times \begin{cases} 2^{(k-l)n\omega^+\alpha^+}, & 0 < 2^l \leq 2^{(k-1)}; \\ 1, & 2^{(k-1)} < 2^l \leq 2^{(k+1)}; \\ 2^{(k-l)n\omega^-\alpha^-}, & 2^l > 2^{(k+1)}, \end{cases}$$

where the implicit is independent of x, y, k and l.

Lemma 2.4 ([31]) *If $w \in A_{p(\cdot)}$, $p(\cdot) \in \mathbb{B}$, thus there exist positive constants $\delta_1, \delta_2 \in (0,1)$ such* *that*

$$\frac{\|\chi_k\|_{L^{p(\cdot)}(\omega)}}{\|\chi_l\|_{L^{p(\cdot)}(\omega)}}\le C(\frac{|D_k|}{|D_l|})^{\delta_1},\quad \frac{\|\chi_k\|_{(L^{p(\cdot)}(\omega))'}}{\|\chi_l\|_{(L^{p(\cdot)}(\omega))'}}\le C(\frac{|D_k|}{|D_l|})^{\delta_2}$$

for all $k,l\in\mathbb{Z}$ with $k\ge l$.

Lemma 2.5 ([32]) *Let $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. A positive measurable function $\omega\in A_{q(\cdot)}$, if there exists a positive constant C for all balls B such that*

$$\frac{1}{|B|}\|\omega\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\omega^{-1}\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\le C.$$

Lemma 2.6 ([33,34]) *Let $b\in \mathrm{BMO}(\mathbb{R}^n)$, $k>l$ and $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, we have*

$$\sup_{\mathrm{ball}B}\frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\omega)}}\|(b-b_B)\chi_B\|_{L^{q(\cdot)}(\omega)}\approx\|b\|_*$$

and

$$\|(b-b_{B_l})\chi_{B_k}\|_{L^{q(\cdot)}(\omega)}\le C(k-l)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\omega)}.$$

3. Proofs of Theorems 1.9 and 1.11

The proofs of main results of this section are stated as follows:

Proof of Theorem 1.9 Let $f\in M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)$, and decompose

$$f(x)=\sum_{l=-\infty}^{\infty}f(x)\chi_l(x)=\sum_{l=-\infty}^{\infty}f_l(x).$$

By the Minkowski's inequality and (1.7), we obtain

$$\begin{aligned}
&\|\mu^{*,\rho}_{\Omega,\delta}(f)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}\\
&\le\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\epsilon^\theta\sum_{k=-\infty}^{k_0}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}\mu^{*,\rho}_{\Omega,\delta}(f)\chi_k\|^{p(1+\epsilon)}_{L^{q(\cdot)}(\omega_2)}\Big)^{\frac{1}{p(1+\epsilon)}}\\
&\le\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\epsilon^\theta\sum_{k=-\infty}^{k_0}\Big(\sum_{l=-\infty}^{k-3}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}\mu^{*,\rho}_{\Omega,\delta}(f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}+\\
&\quad\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\epsilon^\theta\sum_{k=-\infty}^{k_0}\Big(\sum_{l=k-2}^{k+2}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}\mu^{*,\rho}_{\Omega,\delta}(f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}+\\
&\quad\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\epsilon^\theta\sum_{k=-\infty}^{k_0}\Big(\sum_{l=k+3}^{\infty}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}\mu^{*,\rho}_{\Omega,\delta}(f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}\\
&=\mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3.
\end{aligned}$$

For E_1, we set $k\ge l+3$, $x\in\mathrm{D}_k$ and $z\in\mathrm{D}_l$. Then, write

$$\begin{aligned}
&|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}\mu^{*,\rho}_{\Omega,\delta}(f_l)(x)|\\
&\le\int_{D_l}|\omega_1(B_k)^{\frac{\alpha(z)}{n}}f_l(z)|\Big(\int_0^\infty\int_{|y-z|\le t}\Big(\frac{t}{t+|x-y|}\Big)^{\delta n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}}\Big)^{\frac12}\mathrm{d}z
\end{aligned}$$

$$\leq \int_{D_l} |\omega_1(B_k)^{\frac{\alpha(z)}{n}} f_l(z)| \Big(\int_0^{|x|} \int_{|y-z|\leq t} \Big(\frac{t}{t+|x-y|}\Big)^{\delta n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}} \Big)^{\frac{1}{2}} \mathrm{d}z +$$
$$\int_{D_l} |\omega_1(B_k)^{\frac{\alpha(z)}{n}} f_l(z)| \Big(\int_{|x|}^{\infty} \int_{|y-z|\leq t} \Big(\frac{t}{t+|x-y|}\Big)^{\delta n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}} \Big)^{\frac{1}{2}} \mathrm{d}z$$
$$\leq \int_{D_l} |\omega_1(B_k)^{\frac{\alpha(z)}{n}} f_l(z)| \Big(\int_0^{|x|} \int_{|y-z|\leq t} \frac{t^{\delta n}}{|x|^{\delta n}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}} \Big)^{\frac{1}{2}} \mathrm{d}z +$$
$$\int_{D_l} |\omega_1(B_k)^{\frac{\alpha(z)}{n}} f_l(z)| \Big(\int_{|x|}^{\infty} \int_{|y-z|\leq t} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}} \Big)^{\frac{1}{2}} \mathrm{d}z$$
$$\leq C 2^{-kn} \|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} f_l\|_{L^1(\mathbb{R}^n)}.$$

By the generalized Hölder's inequality, we have

$$\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} f_l\|_{L^1(\mathbb{R}^n)} \leq \|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} f_l\|_{L^{q(\cdot)}(\omega_2)} \|\chi_l\|_{L^{q'(\cdot)}(\omega^{-q'(\cdot)/q(\cdot)})}, \tag{3.1}$$

from this, Lemmas 2.3–2.5, it then follows that

$$\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} \mu^{*,\rho}_{\Omega,\delta}(f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}$$
$$\leq C 2^{-kn} \|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} f_l\|_{L^1(\mathbb{R}^n)} \|\chi_k\|_{L^{q(\cdot)}(\omega_2)}$$
$$\leq C \|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} f_l\|_{L^{q(\cdot)}(\omega_2)} \|\chi_l\|_{L^{q'(\cdot)}(\omega^{-q'(\cdot)/q(\cdot)})} \|\chi_k\|^{-1}_{L^{q'(\cdot)}(\omega^{-q'(\cdot)/q(\cdot)})}$$
$$\leq C 2^{(k-l)(\omega^+\alpha^+ - n\delta_2)} \|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}} f_l\|_{L^{q(\cdot)}(\omega_2)}.$$

We can find $p(1+\epsilon) \geq 1$, and have

$$\mathrm{E}_1 \leq C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Big(\epsilon^{\theta} \sum_{k=-\infty}^{k_0} \Big(\sum_{l=-\infty}^{k-3} 2^{(k-l)(\omega^+\alpha^+ - n\delta_2)} \|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}} f_l\|_{L^{q(\cdot)}(\omega_2)} \Big)^{p(1+\epsilon)} \Big)^{\frac{1}{p(1+\epsilon)}}$$
$$\leq C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Big(\epsilon^{\theta} \sum_{k=-\infty}^{k_0} \Big(\sum_{l=-\infty}^{k-3} \|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}} f_l\|^{p(1+\epsilon)}_{L^{q(\cdot)}(\omega_2)} \Big) \times$$
$$\Big(\sum_{l=-\infty}^{k-3} 2^{(k-l)(\omega^+\alpha^+ - n\delta_2)p'(1+\epsilon)} \Big)^{\frac{p(1+\epsilon)}{p'(1+\epsilon)}} \Big)^{\frac{1}{p(1+\epsilon)}}$$
$$\leq C \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}.$$

For E_2, applying the $(L^{p(\cdot)}(\omega), L^{p(\cdot)}(\omega))$-boundedness of $\mu^{*,\rho}_{\Omega,\delta}$ and Lemma 2.3, we obtain

$$\sum_{l=k-2}^{k+2} \|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} \mu^{*,\rho}_{\Omega,\delta}(f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}$$
$$\leq \sum_{l=k-2}^{k-1} \|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} \mu^{*,\rho}_{\Omega,\delta}(f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)} + \sum_{l=k}^{k+1} \|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} \mu^{*,\rho}_{\Omega,\delta}(f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)} +$$
$$\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}} \mu^{*,\rho}_{\Omega,\delta}(f_{k+2})\chi_k\|_{L^{q(\cdot)}(\omega_2)}$$
$$\leq C \sum_{l=k-2}^{k-1} 2^{(k-l)\omega_1^+\alpha^+} \|\mu^{*,\rho}_{\Omega,\delta}(\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}} f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)} + C \sum_{l=k}^{k+1} \|\mu^{*,\rho}_{\Omega,\delta}(\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}} f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)} +$$
$$C 2^{-2\omega_1^-\alpha^-} \|\mu^{*,\rho}_{\Omega,\delta}(\omega_1(B_{k+2})^{\frac{\alpha(\cdot)}{n}} f_{k+2})\chi_k\|_{L^{q(\cdot)}(\omega_2)}$$

$$\leq C\sum_{l=k-2}^{k-1}2^{(k-l)\omega_1^+\alpha^+}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}+C\sum_{l=k}^{k+2}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)},$$

by this, we get

$$\begin{aligned}\mathrm{E}_2\leq & C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\epsilon^\theta\sum_{k=-\infty}^{k_0}\Big(\sum_{l=k-2}^{k-1}2^{(k-l)\omega_1^+\alpha^+}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}+\\ & C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\epsilon^\theta\sum_{k=-\infty}^{k_0}\Big(\sum_{l=k}^{k+2}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}\\ \leq & C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}.\end{aligned}$$

For E_3, set $l\geq k+3$, $x\in \mathrm{D}_k$ and $z\in \mathrm{D}_l$. Then, write

$$\begin{aligned}&|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}\mu^{*,\rho}_{\Omega,\delta}(f_l)(x)|\\ &\leq\int_{D_l}|\omega_1(B_k)^{\frac{\alpha(z)}{n}}f_l(z)|\Big(\int_0^\infty\int_{|y-z|\leq t}\Big(\frac{t}{t+|x-y|}\Big)^{\delta n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}}\Big)^{\frac12}\mathrm{d}z\\ &\leq\int_{D_l}|\omega_1(B_k)^{\frac{\alpha(z)}{n}}f_l(z)|\Big(\int_0^{|z|}\int_{|y-z|\leq t}\Big(\frac{t}{t+|x-y|}\Big)^{\delta n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}}\Big)^{\frac12}\mathrm{d}z+\\ &\quad\int_{D_l}|\omega_1(B_k)^{\frac{\alpha(z)}{n}}f_l(z)|\Big(\int_{|z|}^\infty\int_{|y-z|\leq t}\Big(\frac{t}{t+|x-y|}\Big)^{\delta n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}}\Big)^{\frac12}\mathrm{d}z\\ &\leq\int_{D_l}|\omega_1(B_k)^{\frac{\alpha(z)}{n}}f_l(z)|\Big(\int_0^{|z|}\int_{|y-z|\leq t}\frac{t^{\delta n}}{|z|^{\delta n}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}}\Big)^{\frac12}\mathrm{d}z+\\ &\quad\int_{D_l}|\omega_1(B_k)^{\frac{\alpha(z)}{n}}f_l(z)|\Big(\int_{|z|}^\infty\int_{|y-z|\leq t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1+2\rho}}\Big)^{\frac12}\mathrm{d}z\\ &\leq C2^{-ln}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

by (3.1), Lemmas 2.3–2.5, it then follows that

$$\begin{aligned}&\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}\mu^{*,\rho}_{\Omega,\delta}(f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\ &\quad\leq C2^{-ln}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\ &\quad\leq C\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}\|\chi_l\|^{-1}_{L^{q(\cdot)}(\omega_2)}\|\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\ &\quad\leq C2^{(k-l)(\omega^-\alpha^-+n\delta_1)}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}.\end{aligned}$$

With an argument similar to that used in the estimate for E_1, it is easy to get

$$\mathrm{E}_3\leq C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)},$$

which, combining the estimates for E_1 and E_2, yields our desired result. Hence, the proof of Theorem 1.9 is completed. □

Proof of Theorem 1.11 Let $f\in M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\mathbb{R}^n)$. Write

$$f(x)=\sum_{l=-\infty}^{\infty}f(x)\chi_l(x)=\sum_{l=-\infty}^{\infty}f_l(x).$$

By the Minkowski's inequality and (1.7), decompose

$$
\begin{aligned}
&\|[b,\mu^{*,\rho}_{\Omega,\delta}](f)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}\\
&\le \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\Big(\sum_{l=-\infty}^{\infty}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[b,\mu^{*,\rho}_{\Omega,\delta}](f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}\\
&\le \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\Big(\sum_{l=-\infty}^{k-3}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[b,\mu^{*,\rho}_{\Omega,\delta}](f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}+\\
&\quad \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\Big(\sum_{l=k-2}^{k+2}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[b,\mu^{*,\rho}_{\Omega,\delta}](f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}+\\
&\quad \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\Big(\sum_{l=k+3}^{\infty}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[b,\mu^{*,\rho}_{\Omega,\delta}](f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}\\
&= \mathrm{F}_1+\mathrm{F}_2+\mathrm{F}_3.
\end{aligned}
$$

For F_1, just like E_1, we set $k\ge l+3$, $x\in \mathrm{D}_k$ and $z\in\mathrm{D}_l$. By Lemma 2.3, we have

$$
\begin{aligned}
&\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[b,\mu^{*,\rho}_{\Omega,\delta}](f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\
&\le C2^{(k-l)\omega^+\alpha^+}\|[b,\mu^{*,\rho}_{\Omega,\delta}](\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\
&\le C2^{(k-l)\omega^+\alpha^+}\|(b-b_{B_l})\mu^{*,\rho}_{\Omega,\delta}(\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l)\chi_k-\mu^{*,\rho}_{\delta}((b-b_{B_l})\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\
&\le C2^{(k-l)\omega^+\alpha^+}\|(b-b_{B_l})\mu^{*,\rho}_{\Omega,\delta}(\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}+\\
&\quad C2^{(k-l)\omega^+\alpha^+}\|\mu^{*,\rho}_{\Omega,\delta}((b-b_{B_l})\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\
&= \mathrm{M}_1+\mathrm{M}_2.
\end{aligned}
$$

For M_1, by the generalized Hölder's inequality, Lemmas 2.4–2.6, we get

$$
\begin{aligned}
\mathrm{M}_1 \le& C2^{-kn}2^{(k-l)\omega^+\alpha^+}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^1(\mathbb{R}^n)}\|(b-b_{B_l})\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\
\le& C2^{-kn}2^{(k-l)\omega^+\alpha^+}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}\|\chi_l\|_{L^{q'(\cdot)}(\omega^{-q'(\cdot)/q(\cdot)})}(k-l)\|b\|_*\|\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\
\le& C2^{(k-l)(\omega^+\alpha^+-n\delta_2)}(k-l)\|b\|_*\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}.
\end{aligned}
$$

For M_2, by the generalized Hölder's inequality, Lemmas 2.4–2.6, write

$$
\begin{aligned}
\mathrm{M}_2 \le& C2^{-kn}2^{(k-l)\omega^+\alpha^+}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}\|(b-b_{B_l})\chi_l\|_{L^{q'(\cdot)}(\omega^{-q'(\cdot)/q(\cdot)})}\|\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\
\le& C2^{-kn}2^{(k-l)\omega^+\alpha^+}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}\|\chi_l\|_{L^{q'(\cdot)}(\omega^{-q'(\cdot)/q(\cdot)})}\|b\|_*\|\chi_k\|_{L^{q(\cdot)}(\omega_2)}\\
\le& C2^{(k-l)(\omega^+\alpha^+-n\delta_2)}\|b\|_*\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}.
\end{aligned}
$$

Using $p(1+\epsilon)\ge 1$, we also have

$$
\begin{aligned}
\mathrm{F}_1 \le& C\|b\|_*\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\epsilon^{\theta}\sum_{k=-\infty}^{k_0}\Big(\sum_{l=-\infty}^{k-3}2^{(k-l)(\omega^+\alpha^+-n\delta_2)}(k-l)\times\\
&\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}
\end{aligned}
$$

$$\le C\|b\|_*\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\epsilon^\theta\sum_{k=-\infty}^{k_0}\Big(\sum_{l=-\infty}^{k-3}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|^{p(1+\epsilon)}_{L^{q(\cdot)}(\omega_2)}\Big)\times$$
$$\Big(\sum_{l=-\infty}^{k-3}2^{(k-l)(\omega^+\alpha^+-n\delta_2)p'(1+\epsilon)}(k-l)^{p'(1+\epsilon)}\Big)^{\frac{p(1+\epsilon)}{p'(1+\epsilon)}}\Big)^{\frac{1}{p(1+\epsilon)}}$$
$$\le C\|b\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}.$$

For F_2, by the $(L^{p(\cdot)}(\omega),L^{p(\cdot)}(\omega))$-boundedness of the $[b,\mu^{*,\rho}_{\Omega,\delta}]$ and Lemma 2.3, write

$$\sum_{l=k-2}^{k+2}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[b,\mu^{*,\rho}_{\Omega,\delta}](f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}$$
$$\le\sum_{l=k-2}^{k-1}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[b,\mu^{*,\rho}_{\Omega,\delta}](f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}+\sum_{l=k}^{k+1}\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[b,\mu^{*,\rho}_{\Omega,\delta}](f_l)\chi_k\|+$$
$$\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[b,\mu^{*,\rho}_{\Omega,\delta}](f_{k+2})\chi_k\|_{L^{q(\cdot)}(\omega_2)}$$
$$\le C\sum_{l=k-2}^{k-1}2^{(k-l)\omega_1^+\alpha^+}\|[b,\mu^{*,\rho}_{\Omega,\delta}](\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}+$$
$$C\sum_{l=k}^{k+1}\|[b,\mu^{*,\rho}_{\Omega,\delta}](\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}+$$
$$C2^{-2\omega_1^-\alpha^-}\|[b,\mu^{*,\rho}_{\Omega,\delta}](\omega_1(B_{k+2})^{\frac{\alpha(\cdot)}{n}}f_{k+2})\chi_k\|_{L^{q(\cdot)}(\omega_2)}$$
$$\le C\sum_{l=k-2}^{k-1}2^{(k-l)\omega_1^+\alpha^+}\|b\|_*\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}+C\sum_{l=k}^{k+2}\|b\|_*\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}.$$

By this, we can get

$$F_2\le C\|b\|_*\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\epsilon^\theta\sum_{k=-\infty}^{k_0}\Big(\sum_{l=k-2}^{k-1}2^{(k-l)\omega_1^+\alpha^+}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}+$$
$$C\|b\|_*\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\epsilon^\theta\sum_{k=-\infty}^{k_0}\Big(\sum_{l=k}^{k+2}\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}\Big)^{p(1+\epsilon)}\Big)^{\frac{1}{p(1+\epsilon)}}$$
$$\le C\|b\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)}.$$

For F_3, we set that $l\ge k+3$, $x\in D_k$ and $z\in D_l$. Then we have

$$\|\omega_1(B_k)^{\frac{\alpha(\cdot)}{n}}[B,\mu^{*,\rho}_{\Omega,\delta}](f_l)\chi_k\|_{L^{q(\cdot)}(\omega_2)}\le C2^{(k-l)(\omega^-\alpha^-+n\delta_1)}(k-l)\|b\|_*\|\omega_1(B_l)^{\frac{\alpha(\cdot)}{n}}f_l\|_{L^{q(\cdot)}(\omega_2)}.$$

With an argument similar to that used in the estimate for F_1, it is easy to obtain

$$F_3\le C\|b\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p),\theta,q(\cdot)}(\omega_1,\omega_2)},$$

which, combining the estimates for F_1 and F_2, yields the desired result. Hence, we complete the proof of Theorem 1.11. □

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