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The Boundedness of Hardy-Littlewood Maximal Operator Associated with ψ-Rectangles

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Abstract In this paper, the authors establish the boundedness of Hardy-Littlewood maximal operators M^ψ associated with ψ-rectangles on weighted Lebesgue spaces and on two kinds of Lorentz spaces with variable exponent, as well as its corresponding Fefferman-Stein inequalities. All of these generalize the corresponding results in classical case.

Keywords Hardy-Littlewood maximal operator; Lorentz space; variable exponent

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1. Introduction

In various function spaces, the boundedness of operators has always been one of the central problems in harmonic analysis [1–4]. The boundedness of the classical Hardy-Littlewood maximal operator M_{HL} on classical and variable Lebesgue spaces has been systematically discussed. With the establishment of the boundedness of the maximal operator M_{HL} on the space of variable exponent Lebesgue spaces by the early scholars in the 21st century, the whole study on the space of variable exponential functions has been developed rapidly, and plays a crucial role in financial mathematics, image processing and PDE's [5–8].

In the space of variable exponential functions, the study on Lorentz spaces is also a hot topic. The variable exponent Lorentz spaces have been defined in two different ways: $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ (see [9]) and $\mathcal{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ (see [10]), but both types of Lorentz spaces can go back to the spaces $L_p(\mathbb{R}^d)$ in the case of $p(\cdot) = q(\cdot) = p$. In 2008, Ephremidze et al. [9] defined variable Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ and showed that in the case of such space, the local log-condition of exponents was no longer needed for the boundedness of the maximal, fractional and singular integral operators in $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$, instead the exponents $p(\cdot)$ and $q(\cdot)$ should only satisfy some log-type decay conditions. Since the space $L_{p(\cdot)}(\mathbb{R}^d)$ is not translation invariant, the use of non-increasing rearrangements makes it difficult to generalize the constant indices in the definition of the classical Lorentz spaces $L_{p,q}(\mathbb{R}^d)$ to the variable case $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ with $p(\cdot)$ and $q(\cdot) : \mathbb{R}^d \to (0, \infty]$. Therefore, in the Lorentz space introduced by Ephremidze et al. [9], both the exponents $p(\cdot)$

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and $q(\cdot)$ are defined in $(0, \infty]$ instead of the whole $\mathbb{R}^d$. In order to study the validity of the Marcinkiewicz interpolation theorem in the frame of Lebesgue spaces with variable integrability, Kempka et al. [10] carved out an equivalent definition of the classical Lorentz space without using the notion of non-increasing rearrangement, and extended to variable case $\mathcal{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ with $p(\cdot), q(\cdot) : \mathbb{R}^d \to (0, \infty]$ in 2014.

The study of maximal operators on Lorentz spaces with variable exponents has been a matter of interest to scholars. The classical maximal operator only considers the integral averages over squares, and if we take the integral averages over arbitrary rectangles, then the boundedness of the maximal operator in Lebesgue space does not hold. In 2008, in order to extend the convergence of the θ-means of the Fourier transform over cone-like sets, Weisz [11] introduced the modified Hardy-Littlewood maximal operators M^ψ, defined on the ψ-rectangles, on the basis of the cone-like set established by Gát [12] and the operator is bounded in the classical Lebesgue space $L_p(\mathbb{R}^d)$ [11]. In 2016, Szarvas et al. [3] extended the conclusions to the function spaces with variable exponents, and obtained the boundedness of M^ψ on the variable Lebesgue space $L_{p(\cdot)}(\mathbb{R}^d)$.

Based on this, we aim to investigate the boundedness of M^ψ, as well as the Fefferman-Stein inequalities, on several function spaces. This paper is organized as follows.

In Section 2, we present some preliminaries, including the definition of (dyadic) ψ-rectangles and Hardy-Littlewood maximal function associated with ψ-rectangles.

In Section 3, via the Calderón-Zygmund decomposition in the setting of ψ-rectangles, we establish the weighted boundedness of M^ψ on Lebesgue spaces (see Theorem 3.2 below). It is worthy to mention that in this process, we show that for any ψ-rectangle I and $n \in (0, \infty)$, there exists $I' \in \mathcal{I}^\psi$ and $m \in (1, \infty)$ such that $nI \subset I' \subset mI$ (see (3.7) and Remark 3.3 below), which not only plays an important role in the proof of Theorem 3.2, but also has its independent interest.

In Section 4, we exhibit the boundedness of M^ψ on variable Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ (see Theorem 4.4 below), as well as the corresponding Fefferman-Stein vector valued inequality (see Theorem 4.7 below). Note that the variable Lorentz space $L_{p(\cdot),p(\cdot)}(\mathbb{R}^d)$ cannot go back to the variable Lebesgue space $L_{p(\cdot)}(\mathbb{R}^d)$ (see Remark 4.2 below).

In Section 5, we devote to study another variable exponent Lorentz space $\mathcal{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$, which covers the classical variable Lebesgue space as a special case, namely, $\mathcal{L}_{p(\cdot),p(\cdot)}(\mathbb{R}^d) = L_{p(\cdot)}(\mathbb{R}^d)$. In this section, we obtain the boundedness of M^ψ on $\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d)$ (see Theorem 5.4 below).

In Section 6, we briefly recall a more general maximal operator $M^{\psi,\delta}$ which is comparable to M^ψ and hence all the results in Sections 3–5 are also valid for $M^{\psi,\delta}$.

Here and hereafter, we adopt the following notations. Let $\mathbb{Z}$ be the set of all integers and $\mathbb{N} := \{0, 1, 2, \ldots\}$. Denote $f \leq Cg$ by $f \lesssim g$, where C is a positive constant independent of the main parameters and may change from line to line; while $f \sim g$ means $g \lesssim f \lesssim g$. For any Lebesgue measurable set $E \subset \mathbb{R}^n$, we use $|E|$ to denote its Lebesgue measure. For any rectangle $I \subset \mathbb{R}^d$ and number $r \in (0, \infty)$, rI denotes a rectangle with the same center as I but r-times side length. For any $p(\cdot)$, denote by $p'(\cdot)$ its conjugate index, namely, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for any

$x \in \mathbb{R}^d$.

2. Preliminaries

Let $\psi := (\psi_1, \ldots, \psi_d)$, where for any $j \in \{1, \ldots, d\}$,

$$\begin{cases} \psi_j : (0,\infty) \to (0,\infty), \\ \psi_j \text{ is strictly increasing and continuous function,} \\ \psi_j(1) = 1, \\ \lim\limits_{x\to\infty} \psi_j(x) = \infty, \\ \lim\limits_{x\to 0^+} \psi_j(x) = 0, \\ \psi_1(x) := x, \end{cases} \tag{2.1}$$

and there exist $c_{j,1},\ c_{j,2},\ \xi \in (1,\infty)$ such that

$$c_{j,1}\psi_j(x) \le \psi_j(\xi x) \le c_{j,2}\psi_j(x) \ \text{ for any } x \in (0,\infty) \text{ and } j \in \{1,\ldots,d\}. \tag{2.2}$$

Since $\psi_1(x) := x$, it is obvious that $c_{1,1} = c_{1,2} = \xi$. Also, for any $x \in (0,\infty)$ and $j \in \{1,\ldots,d\}$,

$$c_{j,1}^n \psi_j(x) \le \psi_j(\xi^n x) \le c_{j,2}^n \psi_j(x), \quad \forall\, n \in \mathbb{N}. \tag{2.3}$$

Recall that $\mathcal{I}^\psi$ consists of ψ-rectangles $I = I^1 \times \cdots \times I^d \subset \mathbb{R}^d$, whose sides are parallel to the axes and for any $j = 1,\ldots,d$, $|I^j| = \psi_j(|I^1|)$. Particularly, when $\psi_j = x > 0$ with $j = 1,\ldots,d$, $\mathcal{I}^\psi$ is the collection of cubes. For more details about ψ-rectangles [4]. Note that the Hardy-Littlewood maximal function associated with ψ-rectangles is defined by setting, for any locally integrable function f on $\mathbb{R}^d$,

$$M^\psi f(x) := \sup_{\mathcal{I}^\psi \ni I \ni x} \frac{1}{|I|}\int_I |f(y)|\,\mathrm{d}y, \quad \forall\, x \in \mathbb{R}^d,$$

see, for instance, [11]. Obviously, when $\psi = (x,\ldots,x)$ with $x > 0$, M^ψ goes back to the classical Hardy-Littlewood maximal function

$$M_{HL} f(x) := \sup_{\text{cube } Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y, \quad \forall\, x \in \mathbb{R}^d.$$

Denote by $\Delta_k = \bigcup_{l\in\mathbb{Z}^d} I_k(l)$ the set of dyadic ψ-rectangles [4]

$$I_k(l) = [l_1 2^k, (l_1+1)2^k) \times [l_2\psi_2(2^k), (l_2+1)\psi_2(2^k)) \times \cdots \times [l_d\psi_d(2^k), (l_d+1)\psi_d(2^k)),$$

where $k \in \mathbb{Z}$. Define the maximal function as

$$M_N^\psi f(x) := \sup_{\mathcal{I}_N^\psi \ni I \ni x} \frac{1}{|I|}\int_I |f(y)|\,\mathrm{d}y, \quad \forall\, x \in \mathbb{R}^d \ \text{ and } \ N \in \mathbb{N},$$

where $\mathcal{I}_N^\psi$ denotes those rectangles $I \in \mathcal{I}^\psi$ for which $|I^1| \le 2^N$. Obviously, for any locally integrable function f on $\mathbb{R}^d$,

$$M_N^\psi f \uparrow M^\psi f. \tag{2.4}$$

3. The weighted boundedness of M^ψ on Lebesgue spaces

The following Calderón-Zygmund decomposition comes from [4, Lemma 12] (see also [13, Lemma 3.13]).

Lemma 3.1 ([4]) *Let ψ be as in (2.1) and (2.2). Then for any locally integrable function f on $\mathbb{R}^d$, $t \in (0,\infty)$ and $N \in \mathbb{N}$, there exists a sequence of disjoint ψ-rectangles $I_j \subset \Delta_k$ with $k \le N$ such that*

$$\{M_N^\psi f > t\} \subset \bigcup_j 3I_j$$

and for all j,

$$t \le \frac{C}{|I_j|}\int_{I_j} |f(y)|\,\mathrm{d}y,$$

where C is a positive constant independent of f, t and N.

The weighted boundedness of M^ψ on Lebesgue spaces is follows; see [14, Lemma 1] for its classical case.

Theorem 3.2 *Let $p \in (1,\infty]$ and ϕ be a non-negative function on $\mathbb{R}^d$. Then there exists a positive constant C such that for any locally integrable function f,*

$$\int_{\mathbb{R}^d} (M^\psi f)(x)^p \phi(x)\,\mathrm{d}x \le C\int_{\mathbb{R}^d} |f(x)|^p (M^\psi \phi)(x)\,\mathrm{d}x, \quad \forall\, p \in (1,\infty] \tag{3.1}$$

and

$$\int_{\{x\in\mathbb{R}^d: M^\psi f(x)>t\}} \phi(x)\,\mathrm{d}x \le \frac{C}{t}\int_{\mathbb{R}^d} |f(x)|(M^\psi \phi)(x)\,\mathrm{d}x \tag{3.2}$$

hold true.

Proof Without loss of generality, we may assume f is nonnegative. By (2.4), it suffices to show for any $N \in (0,\infty)$,

$$\int_{\mathbb{R}^d} (M_N^\psi f)(x)^p \phi(x)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^d} |f(x)|^p (M^\psi \phi)(x)\,\mathrm{d}x, \quad \forall\, p \in (1,\infty] \tag{3.3}$$

and

$$\int_{\{x\in\mathbb{R}^d: M_N^\psi f(x)>t\}} \phi(x)\,\mathrm{d}x \lesssim \frac{1}{t}\int_{\mathbb{R}^d} |f(x)|(M^\psi \phi)(x)\,\mathrm{d}x, \tag{3.4}$$

which, together with the Fatou lemma and Levi monotonic convergence theorem, imply when $p \in (1,\infty]$,

$$\begin{aligned}\int_{\mathbb{R}^d} (M^\psi f)(x)^p \phi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \lim_{N\to\infty} (M_N^\psi f)(x)^p \phi(x)\,\mathrm{d}x\\ &\le \liminf_{N\to\infty} \int_{\mathbb{R}^d} (M_N^\psi f)(x)^p \phi(x)\,\mathrm{d}x\\ &\lesssim \int_{\mathbb{R}^d} |f(x)|^p (M^\psi \phi)(x)\,\mathrm{d}x,\end{aligned}$$

and when $p = 1$,

$$\int_{\{x\in\mathbb{R}^d: M^\psi f(x)>t\}} \phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \phi(x)\chi_{\{x\in\mathbb{R}^d: M^\psi f(x)>t\}}(x)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \phi(x) \lim_{N\to\infty} \chi_{\{x\in\mathbb{R}^d: M_N^\psi f(x)>t\}}(x)\,\mathrm{d}x \\
&\le \liminf_{N\to\infty} \int_{\mathbb{R}^d} \phi(x)\chi_{\{x\in\mathbb{R}^d: M_N^\psi f(x)>t\}}(x)\,\mathrm{d}x \\
&\lesssim \liminf_{N\to\infty} \int_{\{x\in\mathbb{R}^d: M_N^\psi f(x)>t\}} \phi(x)\,\mathrm{d}x \\
&\lesssim \frac{1}{t}\int_{\mathbb{R}^d} |f(x)|(M^\psi\phi)(x)\,\mathrm{d}x
\end{aligned}
$$

as desired in (3.1) and (3.2).

Note that from the Marcinkiewicz interpolation theorem (see, for instance, [15, Theorem 1.3.2]), (3.3) follows immediately from (3.4) and

$$M_N^\psi : L_\infty(\mathbb{R}^d, M^\psi\phi(x)\mathrm{d}x) \longrightarrow L_\infty(\mathbb{R}^d, \phi(x)\mathrm{d}x). \tag{3.5}$$

To verify (3.4), according to Lemma 3.1, for any $t>0$ there exists a sequence of disjoint ψ-rectangles $\{I_j\}_j \subset \Delta_k$ satisfying

$$\{x\in\mathbb{R}^d : M_N^\psi f(x)>t\} \subset \bigcup_j 3I_j \quad \text{and} \quad t \lesssim \frac{1}{|I_j|}\int_{I_j} |f(y)|\,\mathrm{d}y. \tag{3.6}$$

Note that $I_j \in \Delta_{k_j} \subset \mathcal{I}^\psi$, but $3I_j \in \mathcal{I}^\psi$ does not necessarily hold true.

We claim that there exists $m\in(1,\infty)$ such that for all j,

$$3I_j \subset I_j' \subset mI_j, \tag{3.7}$$

where I_j' is also a ψ-rectangle which has the same center with I_j. Assume the above claim holds true for the moment, define the sets E_j as follows:

$$
\begin{cases}
E_1 = \{x\in\mathbb{R}^d : M_N^\psi f(x)>t\} \cap 3I_1, \\
E_2 = \{x\in\mathbb{R}^d : M_N^\psi f(x)>t\} \cap 3I_2 \setminus E_1, \\
\quad \vdots \\
E_j = \{x\in\mathbb{R}^d : M_N^\psi f(x)>t\} \cap 3I_j \setminus (\cup_{k=1}^{j-1} E_k), \quad \forall\, j\ge 2\cap\mathbb{N}.
\end{cases}
$$

This implies $\{E_j\}_j$ is disjoint and for any j, $E_j \subset 3I_j \subset I_j'$. Furthermore,

$$\{x\in\mathbb{R}^d : M_N^\psi f(x_j)>t\} = \bigcup_j E_j.$$

Also (3.6) tells

$$
\begin{aligned}
\int_{I_j} f(x)(M^\psi\phi)(x)\,\mathrm{d}x &\ge \int_{I_j} f(x)\Big[\frac{1}{|I_j'|}\int_{I_j'} \phi(y)\,\mathrm{d}y\Big]\mathrm{d}x \\
&\ge \int_{I_j} f(x)\Big[\frac{1}{|mI_j|}\int_{E_j} \phi(y)\,\mathrm{d}y\Big]\mathrm{d}x \\
&\gtrsim \Big[\int_{E_j} \phi(y)\,\mathrm{d}y\Big]\cdot\frac{1}{|I_j|}\int_{I_j} f(x)\,\mathrm{d}x \\
&\gtrsim t\int_{E_j} \phi(y)\,\mathrm{d}y.
\end{aligned}
$$

From the above two formulae and the facts that $\{I_j\}_j$ and $\{E_j\}_j$ are disjoint, we deduce that

$$\int_{\{x\in\mathbb{R}^d:\, M_N^\psi f(x)>t\}} \phi(y)\,\mathrm{d}y = \int_{\bigcup_j E_j} \phi(y)\,\mathrm{d}y = \sum_j \int_{E_j} \phi(y)\,\mathrm{d}y \lesssim \sum_j \frac{1}{t}\int_{I_j} f(x)(M^\psi\phi)(x)\,\mathrm{d}x$$
$$\sim \frac{1}{t}\int_{\cup_j I_j} f(x)(M^\psi\phi)(x)\,\mathrm{d}x \lesssim \frac{1}{t}\int_{\mathbb{R}^d} f(x)(M^\psi\phi)(x)\,\mathrm{d}x$$

as desired in (3.4).

What remains is to show (3.7). Let n_1 be the smallest integer for which

$$3 \le \min(c_{i,1};\ \ i=2,\dots,d)^{n_1}.$$

Observe that the left-hand side of (2.3) implies for any j,

$$3\psi_i(|I_j^1|) \le c_{i,1}^{n_1}\psi_i(|I_j^1|) \le \psi_i(\xi^{n_1}|I_j^1|).$$

Let $l = \max\{3,\ \xi^{n_1}\}$. Choose the smallest integer n_2 and a ψ-rectangle I_j' such that

$$l+1 \le \xi^{n_2} \quad\text{and}\quad l|I_j^1| \le |(I_j')^1| \le (l+1)|I_j^1|.$$

By ψ_i strictly increasing and the right-hand side of (2.3),

$$\psi_i((l+1)|I_j^1|) \le \psi_i(\xi^{n_2}|I_j^1|) \le c_{i,2}^{n_2}\psi_i(|I_j^1|) \quad\text{for all } i=2,\dots,d.$$

Let $m = \max\{l+1,\ c_{2,2}^{n_2},\ c_{3,2}^{n_2},\dots,c_{d,2}^{n_2}\}$. Since $|(3I_j)^i| = 3|I_j^i|$ $(i=1,\dots,d)$ and $I_j,\ I_j' \in \mathcal{I}^\psi$, we have

$$3|I_j^i| = 3\psi_i(|I_j^1|) \le \psi_i(\xi^{n_1}|I_j^1|) \le \psi_i(l|I_j^1|) \le \psi_i(|(I_j')^1|) = |(I_j')^i|,$$

which implies $3I_j \subset I_j'$. Also

$$|(I_j')^i| \le \psi_i((l+1)|I_j^1|) \le c_{i,2}^{n_2}\psi_i(|I_j^1|) \le m\psi_i(|I_j^1|)$$

and hence $I_j' \subset mI_j$. Altogether, the above two formulae yield the claim (3.7) and hence complete the proof of (3.4).

Now it turns to show (3.5) is valid. Notice that if there exists some $x_0 \in \mathbb{R}^d$ satisfying $(M^\psi\phi)(x_0) = 0$, then $\phi = 0$ almost everywhere in $\mathbb{R}^d$ and there is nothing to show. So it suffices to assume $M^\psi\phi > 0$ and $(M^\psi\phi)(x) < \infty$ for almost every $x \in \mathbb{R}^d$. Denote $\mathrm{d}\mu(x) = (M^\psi\phi)(x)\mathrm{d}x$ and $\mathrm{d}\nu(x) = \phi(x)\mathrm{d}x$. If $\alpha \ge \|f\|_{L_\infty(\mathbb{R}^d,\,\mu)}$, then

$$\int_{\{x\in\mathbb{R}^d:|f(x)|>\alpha\}} M^\psi\phi(x)\,\mathrm{d}x = 0,$$

which, together with $M^\psi\phi(x) > 0$, implies $|\{x\in\mathbb{R}^d : |f(x)|>\alpha\}| = 0$, namely, $|f(x)| \le \alpha$ for almost every $x \in \mathbb{R}^d$. Hence $(M^\psi f)(x) \le \alpha$ for almost every $x \in \mathbb{R}^d$ and further $\|M^\psi f\|_{L_\infty(\mathbb{R}^d,\,\nu)} \le \alpha$. Combining this with (2.4), we obtain

$$\|M_N^\psi f\|_{L_\infty(\mathbb{R}^d,\,\nu)} \le \|M^\psi f\|_{L_\infty(\mathbb{R}^d,\,\nu)} \le \|f\|_{L_\infty(\mathbb{R}^d,\,\mu)},$$

which is exactly (3.5) and hence complete the proof of Theorem 3.2. □

Remark 3.3 Note that (3.7) is still valid in a general case. Precisely, for any $n \in (0,\infty)$, there exists $m \in (1,\infty)$ such that $nI \subset I' \subset mI$ holds true. This follows immediately from an argument similar to that used in the proof of (3.7), where we only need to replace 3 by any n.

4. The boundedness of M^{ψ} on $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$

In this section, we establish the boundedness of modified maximal operator M^{ψ} on variable Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$. Recall that $\mathcal{P}(\mathbb{R}^d)$ denotes the class of all measurable functions $p(\cdot):\mathbb{R}^d \to (0,\infty]$ such that

$$0 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} p(x) \le p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} p(x) \le \infty.$$

For any given $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, the variable Lebesgue space $L_{p(\cdot)}(\mathbb{R}^d)$ is known as the set of all measurable functions f such that

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^d\setminus\mathbb{R}^d_\infty} |f(x)|^{p(x)}\,\mathrm{d}x + \|f\|_{L_\infty(\mathbb{R}^d_\infty)} < \infty,$$

equipped with the quasi-norm

$$\|f\|_{L_{p(\cdot)}(\mathbb{R}^d)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \le 1\}$$

(see [16]), where $\mathbb{R}^d_\infty := \{x \in \mathbb{R}^d : p(x) = \infty\}$. Clearly, when $p(\cdot) = p$, $L_{p(\cdot)}(\mathbb{R}^d)$ goes back to the classical Lebesgue space $L_p(\mathbb{R}^d)$.

Different from $\mathcal{P}(\mathbb{R}^d)$, we follow [9, Chapter 2] to use $\mathcal{P}(0,\infty)$ to denote the collection of all measurable functions $p(\cdot):(0,\infty)\to(0,\infty]$. For any $p(\cdot)\in\mathcal{P}(0,\infty)$, define

$$\tilde{p_-} := \operatorname*{ess\,inf}_{0<t<\infty} p(t) \quad \text{and} \quad \tilde{p_+} := \operatorname*{ess\,sup}_{0<t<\infty} p(t).$$

For any $a \in [0,\infty)$, set $\mathcal{P}_a = \{p(\cdot) \in \mathcal{P}(0,\infty) : a < \tilde{p_-} \le \tilde{p_+} < \infty\}$ and $\mathbb{P}(0,\infty)$ to be the collection of all $p(\cdot) \in L_\infty(0,\infty)$ such that

$$p(0) := \lim_{t\to0} p(t) \quad \text{and} \quad p_\infty := \lim_{t\to\infty} p(t)$$

exist and satisfy

$$\begin{cases} |p(t)-p(0)| \le \frac{C}{-\ln t}, & \text{when } t \in (0,1/2]; \\ |p(t)-p_\infty| \le \frac{C}{\ln(e+t)}, & \text{when } t \in (0,\infty), \end{cases}$$

where the positive constant C is independent of t. Similarly, define

$$\mathbb{P}_a(0,\infty) := \mathbb{P}(0,\infty) \cap \mathcal{P}_a(0,\infty).$$

Recall that the non-increasing rearrangement of a measurable function f on $\mathbb{R}^d$ is defined by setting

$$f^*(t) = \inf\{s > 0 : |\{x \in \mathbb{R}^d : |f(x)| > s\}| \le t\}, \quad \forall\, t \in (0,\infty). \tag{4.1}$$

It is well known that [15, Page 54]

$$(f+g)^*(t) \le f^*(\frac{t}{2}) + g^*(\frac{t}{2}), \quad \forall\, t \in (0,\infty). \tag{4.2}$$

The following definition of variable exponent Lorentz space comes from [9, Definition 2.3].

Definition 4.1 ([9]) *Let $p(\cdot)$ and $q(\cdot) \in \mathcal{P}_0(0,\infty)$. Then the variable exponent Lorentz space $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ is defined to be the set of all measurable functions f on $\mathbb{R}^d$ such that*

$$\Im_{p(\cdot),q(\cdot)}(f) := \int_0^\infty t^{\frac{q(t)}{p(t)}-1}|f^*(t)|^{q(t)}\,\mathrm{d}t < \infty,$$

equipped with the norm

$$\|f\|_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)} = \inf\{\lambda > 0 : \Im_{p(\cdot),q(\cdot)}(f/\lambda) \le 1\} = \|t^{\frac{1}{p(t)}-\frac{1}{q(t)}}f^*(t)\|_{L_{q(\cdot)}(0,\infty)}.$$

Remark 4.2 The spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ coincide with usual Lorentz spaces $L_{p,q}(\mathbb{R}^d)$ if $p(\cdot) = p$ and $q(\cdot) = q$ are constants. However, it is well-known that when $p(\cdot) \in \mathcal{P}(0,\infty)$,

$$L_{p(\cdot),p(\cdot)}(\mathbb{R}^d) \neq L_{p(\cdot)}(\mathbb{R}^d).$$

See [10, Remark 2.6] for more details.

For any $p(\cdot)$ and $q(\cdot) \in \mathcal{P}_0(0,\infty)$ and any measurable function f on $\mathbb{R}^d$, define

$$\|f\|^*_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)} := \|t^{\frac{1}{p(t)}-\frac{1}{q(t)}}f^{**}(t)\|_{L_{q(\cdot)}(0,\infty)},$$

where for any $t \in (0,\infty)$,

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s.$$

Clearly, $f^* \le f^{**}$ shows $\|f\|_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)} \le \|f\|^*_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)}$. Also, [9, Theorem 2.4] tells the following lemma.

Lemma 4.3 ([9]) *Let $p(\cdot) \in \mathbb{P}_0(0,\infty)$ and $q(\cdot) \in \mathbb{P}_1(0,\infty)$. Then there exists a positive constant C such that for any measurable function f on $\mathbb{R}^d$,*

$$\|f\|^*_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)} \le C\|f\|_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)}$$

holds if and only if $p(0) > 1$ and $p_\infty > 1$.

Theorem 4.4 *Let $p(\cdot)$ and $q(\cdot) \in \mathbb{P}_1(0,\infty)$. Then M^ψ is bounded on $L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$.*

Proof Note that it suffices to show for any locally integrable function f on $\mathbb{R}^d$,

$$(M^\psi f)^*(t) \lesssim f^{**}(t), \quad \forall\, t \in (0,\infty). \tag{4.3}$$

Assume for the moment that (4.3) holds true, then by Definition 4.1 and Lemma 4.3,

$$\begin{aligned}\|M^\psi f\|_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)} &= \|t^{\frac{1}{p(t)}-\frac{1}{q(t)}}(M^\psi f)^*(t)\|_{L_{q(\cdot)}(0,\infty)}\\ &\lesssim \|t^{\frac{1}{p(t)}-\frac{1}{q(t)}}f^{**}(t)\|_{L_{q(\cdot)}(0,\infty)}\\ &\lesssim \|f\|_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)}\end{aligned}$$

as desired.

Now we focus on proving (4.3). Without loss of generality, we may suppose $f^{**}(t) < \infty$ for any $t \in (0,\infty)$, otherwise there is nothing to show. By [11, Theorem 1], we know that M^ψ is weak-type $(1,1)$, namely, for any $f \in L_1(\mathbb{R}^d)$ and $t \in (0,\infty)$,

$$|\{x \in \mathbb{R}^d : (M^\psi f)(x) > t\}| \lesssim \frac{\|f\|_{L_1(\mathbb{R}^d)}}{t},$$

which, together with (4.1) implies, for any $t \in (0, \infty)$,

$$t(M^{\psi} f)^{*}(t) \lesssim \|f\|_{L_1(\mathbb{R}^d)}. \tag{4.4}$$

For such $f \in L_1(\mathbb{R}^d)$, [17, Theorem 6.2 in Chapter 2] tells for any ε and $t \in (0, \infty)$, there exist functions $g_t \in L_1(\mathbb{R}^d)$ and $h_t \in L_\infty(\mathbb{R}^d)$ such that

$$f = g_t + h_t \quad \text{and} \quad \|g_t\|_{L_1(\mathbb{R}^d)} + t\|h_t\|_{L_\infty(\mathbb{R}^d)} \le tf^{**}(t) + \varepsilon,$$

which, together with (4.2), (4.4) and the obvious fact that M^{ψ} is bounded from $L_\infty(\mathbb{R}^d)$ to $L_\infty(\mathbb{R}^d)$, implies for any $s \in (0, \infty)$,

$$\begin{aligned}(M^{\psi} f)^{*}(s) &\le (M^{\psi} g_t)^{*}(\frac{s}{2}) + (M^{\psi} h_t)^{*}(\frac{s}{2}) \lesssim \frac{\|g_t\|_{L_1(\mathbb{R}^d)}}{s} + \|h_t\|_{L_\infty(\mathbb{R}^d)} \\ &= \frac{1}{s}(\|g_t\|_{L_1(\mathbb{R}^d)} + s\|h_t\|_{L_\infty(\mathbb{R}^d)}) \lesssim f^{**}(s) + \frac{\varepsilon}{s}.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain (4.3) and hence complete the proof of Theorem 4.4. □

Given a nonnegative function w on $\mathbb{R}^d$ and a set $E \subset \mathbb{R}^d$, let

$$w(E) := \int_E w(x)\,\mathrm{d}x.$$

The weighted Lebesgue space $L_p(\mathbb{R}^d; w)$ is defined to be the collection of all measurable functions f such that

$$\|f\|_{L_p(\mathbb{R}^d;w)} := \Big(\int_{\mathbb{R}^d} |f(x)|^p w(x)\,\mathrm{d}x\Big)^{1/p} < \infty.$$

We refer the readers to [15] for more details about $L_p(\mathbb{R}^d; w)$.

Let $\mathcal{B}$ be a collection of open sets in $\mathbb{R}^d$. Recall that the maximal operator associated with $\mathcal{B}$ is defined by setting

$$M_{\mathcal{B}} f(x) := \sup_{\mathcal{B} \ni B \ni x} \frac{1}{|B|}\int_B |f(y)|\,\mathrm{d}y, \quad \forall\, x \in \bigcup_{B \in \mathcal{B}} B.$$

See, for instance, [18, Page 28]. A nonnegative function w belongs to $A_{p,\mathcal{B}}$ for some $p \in (1, \infty)$, if there exists a positive constant C such that for every $B \in \mathcal{B}$,

$$\Big(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,\mathrm{d}x\Big)^{p-1} \le C,$$

while $p = 1$, $w \in A_{1,\mathcal{B}}$ means $M_{\mathcal{B}} w \le Cw(x)$ for almost every $x \in \mathbb{R}^d$ with C independent of x.

Now we recall the definition of Muckenhoupt basis from [18, Definition 3.1].

Definition 4.5 ([18]) *A collection of open sets $\mathcal{B}$ in $\mathbb{R}^d$ is called a Muckenhoupt basis if for any $p \in (1, \infty)$ and $w \in A_{p,\mathcal{B}}$, there exists a positive constant C such that*

$$\int_{\mathbb{R}^d} (M_{\mathcal{B}} f(x))^p w(x)\,\mathrm{d}x \le C \int_{\mathbb{R}^d} |f(x)|^p w(x)\,\mathrm{d}x$$

holds for every $f \in L_p(\mathbb{R}^d; w)$.

Given a Banach function space $\mathbb{X}$, for every $p \in (0, \infty)$, the scale of spaces $\mathbb{X}^p$ is defined as

$$\mathbb{X}^p := \{f \text{ measurable in } \mathbb{R}^d : |f|^p \in \mathbb{X}\}$$

with the quasi-norm $\|f\|_{\mathbb{X}^p} = \||f|^p\|_{\mathbb{X}}^{\frac{1}{p}}$. For more details about $\mathbb{X}^p$ (see [18, Page 67]).

The following lemma is a special case of [18, Corollary 4.8] in setting of Muckenhoupt basis $\mathcal{B} = \mathcal{I}^\psi$.

Lemma 4.6 ([18]) *Let $\mathcal{F}$ be a family of pairs (f, g) with non-negative and measurable functions which are not identically zero. Suppose that for some $p_0 \in [1, \infty)$ and every $w_0 \in A_{p_0,\mathcal{I}^\psi}$,*

$$\int_{\mathbb{R}^d} f(x)^{p_0} w_0(x)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^d} g(x)^{p_0} w_0(x)\,\mathrm{d}x, \quad \forall\,(f, g) \in \mathcal{F}, \tag{4.5}$$

where the implict positive constant is independent of (f, g). If $\mathbb{X}$ is a Banach function space such that M^ψ is bounded on the associate dual space of $\mathbb{X}$, then for any $p \in (1, \infty)$, there exists a positive constant C such that

$$\|f\|_{\mathbb{X}^p} \le C\|g\|_{\mathbb{X}^p}, \quad \forall\,(f, g) \in \mathcal{F}.$$

Furthermore, for any $q \in (1, \infty)$ and $\{(f_i, g_i)\}_{i\in\mathbb{N}} \subset \mathcal{F}$,

$$\Big\|\Big(\sum_{i\in\mathbb{N}} f_i^q\Big)^{1/q}\Big\|_{\mathbb{X}^p} \le C\Big\|\Big(\sum_{i\in\mathbb{N}} g_i^q\Big)^{1/q}\Big\|_{\mathbb{X}^p}.$$

Further, we give the following Fefferman-Stein vector valued inequality for M^ψ.

Theorem 4.7 *Suppose that for some $p_0 \in [1, \infty)$ the family $\mathcal{F}$ is such that for all $w_0 \in A_{p_0,\mathcal{I}^\psi}$, inequality (4.5) holds. Let $p(\cdot)$ and $q(\cdot) \in \mathbb{P}_1(0, \infty)$. For any $r \in (1, \infty)$, there exists a positive constant C such that for any locally integrable function sequence $\{f_i\}_{i\in\mathbb{N}}$ with $\{(M^\psi f_i, |f_i|)\}_{i\in\mathbb{N}} \subset \mathcal{F}$,*

$$\Big\|\Big(\sum_{i\in\mathbb{N}}(M^\psi(f_i))^r\Big)^{1/r}\Big\|_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)} \le C\Big\|\Big(\sum_{i\in\mathbb{N}}|f_i|^r\Big)^{1/r}\Big\|_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)}.$$

Proof Denote by $\mathcal{I}$ the set of all rectangles in $\mathbb{R}^d$ whose sides are parallel to the coordinate axes. By [18, Page 29], $\mathcal{I}$ constitutes a Muckenhoupt basis in $\mathbb{R}^d$. Clearly, $\mathcal{I}^\psi \subset \mathcal{I}$ and hence $M^\psi \lesssim M_{\mathcal{I}}$. Then Definition 4.5 implies for any $p \in (1, \infty)$, $w \in A_{p,\mathcal{I}^\psi}$ and $f \in L_p(\mathbb{R}^d; w)$,

$$\begin{aligned}\int_{\mathbb{R}^d}(M^\psi f)(x)^p w(x)\,\mathrm{d}x &\lesssim \int_{\mathbb{R}^d}(M_{\mathcal{I}} f)(x)^p w(x)\,\mathrm{d}x\\ &\lesssim \int_{\mathbb{R}^d}|f(x)|^p w(x)\,\mathrm{d}x < \infty,\end{aligned}$$

which means $\mathcal{I}^\psi$ constitutes a Muckenhoupt basis. We choose the family $\mathcal{F}$ consisting of pairs of the form $\{(M^\psi f, |f|)\}$.

Choose $s_0 \in (0, 1)$ such that $s_0 p(\cdot)$ and $s_0 q(\cdot) \in \mathbb{P}_1(0, \infty)$, then by [9, Theorem 2.8], we know that $L_{s_0 p(\cdot),s_0 q(\cdot)}(\mathbb{R}^d)$ is a Banach function space. According to [9, Lemma 2.7],

$$(L_{s_0 p(\cdot),s_0 q(\cdot)}(\mathbb{R}^d))' = L_{s_0 p'(\cdot),s_0 q'(\cdot)}(\mathbb{R}^d).$$

By an argument similar to that used in [19, Page 241], it is not hard to verify that if follows at once that $s_0 p'(\cdot)$ and $s_0 q'(\cdot) \in \mathbb{P}_1(0, \infty)$. Then M^ψ is bounded on the associated dual space of $L_{s_0 p(\cdot),s_0 q(\cdot)}(\mathbb{R}^d)$. Combining this with Lemma 4.6 and the fact that $\{(M^\psi f_i, |f_i|)\}_{i\in\mathbb{N}} \subset \mathcal{F}$, we arrive at

$$\Big\|\Big(\sum_{i\in\mathbb{N}}(M^{\psi}(f_i))^r\Big)^{1/r}\Big\|_{(L_{s_0p(\cdot),s_0q(\cdot)}(\mathbb{R}^d))^{\frac{1}{s_0}}}\lesssim\Big\|\Big(\sum_{i\in\mathbb{N}}|f_i|^r\Big)^{1/r}\Big\|_{(L_{s_0p(\cdot),s_0q(\cdot)}(\mathbb{R}^d))^{\frac{1}{s_0}}},$$

which together with

$$(L_{s_0p(\cdot),s_0q(\cdot)}(\mathbb{R}^d))^{\frac{1}{s_0}}=L_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$$

(see [20, Lemma 2.3]), implies the desired result and hence completes the proof of Theorem 4.7. □

5. The boundedness of M^{ψ} on $\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d)$

In this section, we focus on another kind of variable exponent Lorentz space, denoted by $\mathcal{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$, which was introduced in [10] and studied intensively in [21–24]. Note that

$$\mathcal{L}_{p(\cdot),p(\cdot)}(\mathbb{R}^d)=L_{p(\cdot)}(\mathbb{R}^d)$$

with $p(\cdot)\in\mathcal{P}(\mathbb{R}^d)$, which is different from $L_{p(\cdot),p(\cdot)}(\mathbb{R}^d)$ as mentioned in Remark 4.2.

First we focus on one special case that $q(\cdot)$ is constant, namely $\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d)$ (see, for instance, [10, Definition 2.2]).

Definition 5.1 ([10]) *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^d)$. Then the variable Lorentz space $\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d)$ is defined to be the set of all measurable functions f such that*

$$\infty>\|f\|_{\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d)}:=\begin{cases}(\int_0^\infty\lambda^q\|\chi_{\{x\in\mathbb{R}^d:|f(x)|>\lambda\}}\|^q_{L_{p(\cdot)}(\mathbb{R}^d)}\,\mathrm{d}\lambda)^{1/q}, & \text{if } q<\infty,\\ \sup_{0<\lambda<\infty}\lambda\|\chi_{\{x\in\mathbb{R}^d:|f(x)|>\lambda\}}\|_{L_{p(\cdot)}(\mathbb{R}^d)}, & \text{if } q=\infty.\end{cases}$$

Remark 5.2 It is well-known that $\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d)=L_{p,q}(\mathbb{R}^d)$ when $p(\cdot)=p\in(0,\infty)$, and $\mathcal{L}_{\infty,q}(\mathbb{R}^d)=L_\infty(\mathbb{R}^d)$ when $p(\cdot)=\infty$ and $q\in(0,\infty)$ (see [10, Page 942]).

The following lemma comes from [10, Theorem 4.1 and Remark 4.2].

Lemma 5.3 ([10]) *Let $q\in(0,\infty]$ and $\theta\in(0,1)$. Assume $p(\cdot)$ and $\widetilde{p}(\cdot)\in\mathcal{P}(\mathbb{R}^d)$ satisfying $p_+<\infty$ and $\frac{1}{\widetilde{p}(\cdot)}=\frac{1-\theta}{p(\cdot)}$. Then*

$$(L_{p(\cdot)}(\mathbb{R}^d),L_\infty(\mathbb{R}^d))_{\theta,q}=\mathcal{L}_{\widetilde{p}(\cdot),q}(\mathbb{R}^d).$$

Recall that $p(\cdot)\in\mathcal{P}(\mathbb{R}^d)$ is said to be locally log-Hölder continuous (denoted as $p(\cdot)\in LH_0(\mathbb{R}^d)$), if there exists a constant C such that

$$|p(x)-p(y)|\le\frac{C}{\log(e+1/|x-y|)},\quad\forall\,x,\,y\in\mathbb{R}^d,$$

while $p(\cdot)\in\mathcal{P}(\mathbb{R}^d)$ is said to be log-Hölder continuous at infinity (denoted as $p(\cdot)\in LH_\infty(\mathbb{R}^d)$), if there exist constants C and p_∞ such that

$$|p(x)-p_\infty|\le\frac{C}{\log(e+|x|)},\quad\forall\,x\in\mathbb{R}^d.$$

Let $LH(\mathbb{R}^d):=LH_0(\mathbb{R}^d)\cap LH_\infty(\mathbb{R}^d)$.

Theorem 5.4 *Let $q\in(0,\infty]$ and $p(\cdot)\in\mathcal{P}(\mathbb{R}^d)$ such that $1/p(\cdot)\in LH(\mathbb{R}^d)$ and $p_->1$. Then*

M^{ψ} is bounded from $\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d)$ to $\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d)$.

Proof By Lemma 5.3, we know that

$$\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d) = (L_{\frac{p(\cdot)}{1-\theta}}(\mathbb{R}^d), L_{\infty}(\mathbb{R}^d))_{\theta,q} \quad \text{with} \quad \theta \in (0,1).$$

This, together with the boundedness of M^{ψ} on $L_{\frac{p(\cdot)}{1-\theta}}(\mathbb{R}^d)$ (see [3, Theorem 5.1]) and the trivial boundedness of M^{ψ} on $L_{\infty}(\mathbb{R}^d)$, we immediately conclude that M^{ψ} is bounded on $\mathcal{L}_{p(\cdot),q}(\mathbb{R}^d)$ and complete the proof of Theorem 5.4. □

We mention that the variable Lorentz space $\mathcal{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ with two variable exponents is introduced in [10, Definition 2.3] precisely for any $p(\cdot)$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ satisfying $0 < p_- \leq p_+ \leq \infty$ and $0 < q_- \leq q_+ \leq \infty$. Then the variable Lorentz space $\mathcal{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ is defined to be the collection of all measurable functions f such that

$$\|f\|_{\mathcal{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)} := \inf\{\lambda > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(2^k \chi_{\{x\in\mathbb{R}^d : |f(x)/\lambda| > 2^k\}}) \leq 1\} < \infty,$$

where

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}((f_k)_k) := \sum_{k\in\mathbb{Z}} \inf\{\lambda > 0 : \varrho_{p(\cdot)}(\frac{f_k}{\lambda_k^{1/q(\cdot)}}) \leq 1\}$$

with the convention $\lambda^{1/\infty} = 1$; see also [10, 25].

Remark 5.5 We mention that the boundedness of the classical Hardy-Littlewood maximal function on $\mathcal{L}_{p(\cdot),\, q(\cdot)}(\mathbb{R}^d)$ is still an open problem. And we only discuss the case, when $q(\cdot) = q$ is a constant function.

6. A more general maximal operator $M^{\psi,\delta}$

Let $\delta = (\delta_1, \ldots, \delta_d)$ with $\delta_1 = 1$ and $\delta_j \geq 1$. Recall the cone-like set with respect to the first dimension is defined as

$$\Omega^d_{\psi,\delta} := \{y = (y_1, \ldots, y_d) \in \mathbb{R}^d_+ : \delta_j^{-1}\psi_j(y_1) \leq y_j \leq \delta_j \psi_j(y_1) \ \text{ with } \ j \in \{1, \ldots, d\}\}.$$

When we say rectangle $I \in \mathcal{I}^{\psi,\delta}$, it means $(|I^1|, \ldots, |I^d|) \in \Omega^d_{\psi,\delta}$ with their sides parallel to the axes. If $\delta = 1$, then $\mathcal{I}^{\psi,\delta} = \mathcal{I}^{\psi}$. This cone-like set was first introduced and investigated by Gát [12] in 2007.

Recall that the Hardy-Littlewood maximal function on a cone-like set is defined by setting

$$M^{\psi,\delta} f(x) := \sup_{\mathcal{I}^{\psi,\delta} \ni I \ni x} \frac{1}{|I|} \int_I |f(y)|\, \mathrm{d}y, \quad \forall\, x \in \mathbb{R}^d.$$

If $\delta = 1$, then we denote the maximal function by $M^{\psi} f$. Moreover, Weisz proved in [11, Page 45] that $M^{\psi,\delta} f \sim M^{\psi} f$, which implies that the boundedness for M^{ψ} obtained in Sections 3–5 is also valid for $M^{\psi,\delta}$.

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