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A_α-Spectral Conditions for (a,b,k)-Factor-Critical Graphs and Fractional ID-$[a,b]$-Factor-Critical Covered Graphs

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Abstract In this paper, we first give a sufficient condition for a graph being fractional ID-$[a,b]$-factor-critical covered in terms of its independence number and minimum degree, which partially answers the problem posed by Sizhong Zhou, Hongxia Liu and Yang Xu (2022). Then, an A_α-spectral condition is given to ensure that G is a fractional ID-$[a,b]$-factor-critical covered graph and an (a,b,k)-factor-critical graph, respectively. In fact, (a,b,k)-factor-critical graph is a graph which has an $[a,b]$-factor for $k=0$. Thus, these above results extend the results of Jia Wei and Shenggui Zhang (2023) and Ao Fan, Ruifang Liu and Guoyan Ao (2023) in some sense.

Keywords A_α-spectral radius; (a,b,k)-factor; fractional (g,f)-factor; ID-factor-critical graph; independence number

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1. Introduction

In this paper, we only consider finite, undirected and simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the order and size of G by $|V(G)|=n$ and $|E(G)|=m$, respectively. The complement of G, denoted by G^c, is the graph with the same vertex set where any two non-adjacent vertices in G are adjacent in G^c.

The neighbor set of a vertex $v\in V(G)$ is denoted by $N_G(v)$, and the closed neighbor set is $N_G[v]=\{v\}\cup N_G(v)$. The degree of vertex v is represented by $d_G(v)$, with $\delta(G)$ and $\Delta(G)$ signifying the minimum and maximum degree of G, respectively.

For any two disjoint vertex subsets $S,T\subseteq V(G)$, we define the set of edges between S and T in G as $E_G(S,T)=\{uv: u\in S, v\in T \text{ and } uv\in E(G)\}$, and the number of such edges as $e_G(S,T)=|E_G(S,T)|$. The subgraph induced by S is denoted by $G[S]$.

The join and disjoint union of graphs are represented by ∇ and $\cup$, respectively. We denote the disjoint union of k copies of a complete graph K_n by kK_n.

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The adjacency matrix and the diagonal matrix of vertex degrees of G are represented by $A(G)$ and $D(G)$, respectively. For any real $\alpha \in [0,1]$, following Nikiforov [1], one can define the matrix $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$. It is clear that $A_0(G) = A(G)$, $A_1(G) = D(G)$, and $2A_{1/2}(G) = Q(G)$, where $Q(G)$ is the signless Laplacian matrix. Moreover, the Laplacian matrix $L(G)$ can be expressed as $L(G) = (A_\alpha(G) - A_\beta(G))/(\alpha - \beta)$ for any $\alpha \neq \beta \in [0,1]$.

The largest eigenvalue of $A_\alpha(G)$, akin to the A-spectral radius, is termed the A_α-spectral radius of G, denoted by $\rho_\alpha(G)$. For simplicity, we denote the largest eigenvalue of $A(G)$, the spectral radius of G, by $\rho_0(G)$.

Let g and f be two integer-valued functions on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. Let $h : E(G) \to [0,1]$ be a function satisfying $g(x) \leq d_G^h(x) \leq f(x)$ for all $x \in V(G)$, where $d_G^h(x) = \sum_{e \in E(x)} h(e)$ and $E(x)$ is the set of edges incident to x. We define $F_h = \{e : e \in E(x), h(e) > 0\}$, and term $G[F_h]$ a fractional (g,f)-factor of G with indicator function h, with $G[F_h]$ having vertex set $V(G)$ and edge set F_h.

Li, Yan, and Zhang [2] initially introduced the concept of fractional (g,f)-covered graphs. A graph G is called a fractional (g,f)-covered graph if for any edge $e \in E(G)$, there exists a fractional (g,f)-factor with indicator function h such that $h(e) = 1$. If integers a and b satisfy $1 \leq a \leq b$, and for any $x \in V(G)$, $g(x) = a$ and $f(x) = b$, then the graph is referred to as a fractional $[a,b]$-covered graph.

The concept of fractional ID-$[a,b]$-factor-critical graphs was introduced by Zhou, Sun, and Liu [3]. A graph G is considered fractional ID-$[a,b]$-factor-critical if for any independent set I of G, the graph $G-I$ possesses a fractional $[a,b]$-factor. Furthermore, Zhou et al. [4] defined a fractional ID-$[a,b]$-factor-critical covered graph as one where for any independent set I of G, the graph $G-I$ is a fractional $[a,b]$-covered graph. Additionally, they proposed the following problem:

Problem 1.1 ([4]) Investigate the relationship between other graphic parameters (such as toughness, isolated toughness, binding number, and independence number) and fractional ID-$[a,b]$-factor-critical covered graphs.

A fractional ID-$[k,k]$-factor-critical covered graph (or simply, a fractional ID-k-factor-critical covered graph) is a special case of a fractional ID-$[a,b]$-factor-critical covered graph where $a = b = k$. An independent set of G is a subset S of $V(G)$ with no two vertices in S being adjacent. If there is no independent set S' of G such that $|S'| > |S|$, then S is deemed a maximum independent set. The cardinality of a maximum independent set of G is known as the independence number of G, denoted by $\gamma(G)$.

In fact, Zhou et al. [5] in 2012 proposed a relationship between the independence number and the minimum degree concerning fractional ID-k-factor-critical covered graphs and provided the following theorem.

Theorem 1.2 ([5]) *Let $k \geq 1$ be a positive integer, and G a graph of order n with minimum*

degree $\delta(G)$. If

$$\gamma(G) \leq \frac{4k(\delta(G)-k+1)}{k^2+6k+1},$$

then G is a fractional ID-k-factor-critical covered graph.

In this article, we partially answer the Problem 1.1 by giving a sufficient condition for a graph being fractional ID-$[a,b]$-factor-critical covered, which is about the independence number and the minimum degree.

Theorem 1.3 *Let a,b be two positive integers with $a \leq b$, and G a graph of order n with minimum degree $\delta(G)$. If*

$$\gamma(G) \leq \frac{4b(\delta(G)-a-1)}{(a+1)^2+4b},$$

then G is a fractional ID-$[a,b]$-factor-critical covered graph.

Based on Theorem 1.3, we provide an A_α-spectral condition to guarantee that G is a fractional ID-$[a,b]$-factor-critical covered graph.

Theorem 1.4 *Let a,b be two positive integers with $a \leq b$, and G a graph of order n with minimum degree $\delta(G)$. If*

$$\rho_\alpha(G^c) \geq \alpha\Delta(G^c) + (1-\alpha)\sqrt{2|E(G^c)|(1-\frac{(a+1)^2+4b}{4b(n-1-\Delta(G^c)-a)+(a+1)^2})},$$

then G is a fractional ID-$[a,b]$-factor-critical covered graph, where $\alpha \in [0,1)$.

It is widely acknowledged that if $h : E(G) \to \{0,1\}$, then a fractional $[a,b]$-factor is equivalent to an $[a,b]$-factor. In other words, an $[a,b]$-factor of G is a spanning subgraph H of G such that for any vertex $v \in V(G)$, the degree condition $a \leq d_H(v) \leq b$ is satisfied, where a and b are positive integers with $a \leq b$. Zhou et al. [6] introduced the concept of a fractional (a,b,k)-factor-critical covered graph. Specifically, a graph G is considered fractional (a,b,k)-factor-critical covered if for any vertex subset U of G with cardinality $|U| = k$, the graph $G-U$ remains a fractional $[a,b]$-covered graph. Furthermore, a graph is deemed an (a,b,k)-factor-critical graph if, upon the removal of any k vertices from G, the resulting subgraph still contains an $[a,b]$-factor. This refined characterization of factor-criticality extends the classical definitions and provides a deeper understanding of the structural properties of graphs in relation to their factorization capabilities.

Theorem 1.5 *Let a,b and k be three positive integers, and G a simple connected graph of order n with minimum degree $\delta(G) \geq a+k$, where $a \leq b$ and $n \geq a+k+1$. Let $\alpha \in [0,1)$. If*

$$\rho_\alpha(G) \geq \sqrt{(n-1)(n-3)+a+1+2k},$$

and $na \equiv 0 \pmod 2$ when $a = b$, then G is an (a,b,k)-factor-critical graph.

2. Preliminaries

Firstly, we give some lemmas that will be used in the next section.

Lemma 2.1 ([2]) *Let G be a graph and $b \geq a \geq 1$ be two integers. Then G is fractional $[a,b]$-covered if and only if for all disjoint subsets $S,T \subseteq V(G)$,*

$$\theta_G(S,T) = b|S| - a|T| + \sum_{x\in T} d_{G-S}(x) \geq \varepsilon(S),$$

where $T = \{x : x \in V(G)\setminus S, d_{G-S}(x) \leq a\}$ and $\varepsilon(S)$ is defined by

$$\varepsilon(S) = \begin{cases} 2, & \text{if } S \text{ is not independent;} \\ 1, & \text{if } S \text{ is independent, and there exists } e = uv \in E(G) \text{ with } u \in S, \\ & v \in T \text{ and } d_{G-S}(v)=a, \text{ or } e_G(S, V(G)\setminus(S\cup T)) \geq 1; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

If G is a simple graph, then a subset S of $V(G)$ is a clique if $G[S]$ is complete. Clearly, S is a clique of G if and only if S is an independent set of G^c. We write $\omega(G)$ for the clique number of G, and it is well known that $\omega(G) = \gamma(G^c)$.

Lemma 2.2 ([7]) *Let $\rho_\alpha(G)$ be the A_α-spectral radius of G. Then*

$$\rho_\alpha(G) \leq \alpha\Delta + (1-\alpha)\sqrt{2m(1-\frac{1}{\omega})},$$

where m, Δ and $\omega = \omega(G)$ are respectively the size, the maximum degree and the clique number (i.e., the number of vertices in the largest clique of G) in G.

Recently, Wei and Zhang [8] confirmed a conjecture for the existence of $[a,b]$-factors in graphs, which is posed by Cho et al. [9], and obtained the following lemma.

Lemma 2.3 ([8]) *Let a,b be two positive integers with $a \leq b$, and G a graph of order n and minimum degree $\delta(G) \geq a$, where $n \geq a+1$. If*

$$m \geq \binom{n-1}{2} + \frac{a+1}{2},$$

and $na \equiv 0 \pmod 2$ when $a = b$, then G has an $[a,b]$-factor.

Combining with Lemma 2.3, we give the following lemma.

Lemma 2.4 *Let a,b and k be three integers with $1 \leq a \leq b$, and G a graph of order n and minimum degree $\delta(G) \geq a+k$, where $n \geq a+k+1$. If*

$$m \geq \binom{n-1}{2} + \frac{a+1}{2} + k,$$

and $na \equiv 0 \pmod 2$ when $a = b$, then G is an (a,b,k)-factor-critical graph.

Proof Let U be any subset of $V(G)$ with $|U| = k$ and $H = G - U$. Then $\delta(H) \geq a$ and $|V(H)| \geq a+1$. By $H = G-U$, it is clear that

$$\begin{aligned} e(H) &= e(G) - e(G[U]) - e_G(U, G-U) \\ &\geq e(G) - \binom{k}{2} - k(n-k) \\ &\geq \binom{n-1}{2} + \frac{a+1}{2} + k - \binom{k}{2} - k(n-k) \end{aligned}$$

$$= \binom{n-k-1}{2} + \frac{a+1}{2}.$$

Thus, combining this with $\delta(H) \geq a$, $|V(H)| \geq a+1$, $na \equiv 0 \pmod 2$ when $a = b$ and Lemma 2.3, H has an $[a,b]$-factor. Thus, G is an (a,b,k)-factor-critical graph. □

More recently, Fan, Liu and Ao [10] gave a similar condition in terms of graph size to ensure that G has a fractional $[a,b]$-factor, and posed the following lemma.

Lemma 2.5 ([10]) *Let a,b be two positive integers with $a \leq b$, and G a graph of order n and minimum degree $\delta(G) \geq a$, where $n \geq a+1$. If*

$$m \geq \binom{n-1}{2} + \frac{a+1}{2},$$

and $na \equiv 0 \pmod 2$ when $a = b$, then G has a fractional $[a,b]$-factor.

By a similar analysis as in Lemma 2.4, and by Lemma 2.5, one can also obtain a lemma in terms of graph size to ensure that G is a fractional (a,b,k)-factor-critical graph, and so, we omit its proof.

Lemma 2.6 *Let a,b and k be three integers with $1 \leq a \leq b$, and G be a graph of order n with minimum degree $\delta(G) \geq a+k$, where $n \geq a+k+1$. If*

$$m \geq \binom{n-1}{2} + \frac{a+1}{2} + k,$$

and $na \equiv 0 \pmod 2$ when $a = b$, then G is a fractional (a,b,k)-factor-critical graph.

Lemma 2.7 ([11]) *Let G be a connected graph of order n and minimal degree δ with m edges and $\alpha \in [0,1)$. Then*

$$\rho_\alpha(G) \leq \frac{\delta - 1 + \alpha + \sqrt{(\delta+1-\alpha)^2 + 4(2m - n\delta)(1-\alpha)}}{2}.$$

The equality holds if and only if G is either a regular graph or a bidegreed graph either δ or $n-1$.

Lemma 2.8 ([12,13]) *Let G be a graph with minimum degree $\delta(G)$. Then*

$$\rho_0(G) \leq \frac{\delta(G) - 1 + \sqrt{(\delta(G)+1)^2 + 4(2m - n\delta(G))}}{2}.$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Corollary 2.9 *Let G be a connected graph of order n with m edges and minimal degree δ, and let $\alpha \in [0,1)$. Then*

$$\rho_\alpha(G) \leq \sqrt{2m - n + 1}.$$

The equality holds if and only if $\alpha = 0$ and G is either a 1-regular graph or a bidegreed graph either 1 or $n-1$.

Proof Let $f(\alpha) = \delta - 1 + \alpha + \sqrt{(\delta+1-\alpha)^2 + 4(2m - n\delta)(1-\alpha)}$. By differentiating $f(\alpha)$ with

respect to α, it follows that

$$\frac{\mathrm{d}f}{\mathrm{d}\alpha} = 1 - \frac{\delta - \alpha + 1 + 4m - 2\delta n}{\sqrt{(\delta + 1 - \alpha)^2 + 4(2m - n\delta)(1 - \alpha)}}.$$

Since $1 \leq \delta \leq \frac{2m}{n}$,

$$\delta - \alpha + 1 + 4m - 2\delta n \geq \delta + 1 - \alpha + 2\delta n - 2\delta n = \delta + 1 - \alpha > 0 \tag{2.2}$$

and

$$\begin{aligned}\sqrt{(\delta + 1 - \alpha)^2 + 4(2m - n\delta)(1 - \alpha)} &\geq \sqrt{(\delta + 1 - \alpha)^2 + 4(n\delta - n\delta)(1 - \alpha)} \\ &= \sqrt{(\delta + 1 - \alpha)^2} > 0.\end{aligned} \tag{2.3}$$

Furthermore, from (2.2) and (2.3) we have $(\delta - \alpha + 1 + 4m - 2\delta n)^2 - ((\delta + 1 - \alpha)^2 + 4(2m - n\delta)(1 - \alpha)) = 2m - \delta n + \delta > 0$. Thus,

$$\frac{\delta - \alpha + 1 + 4m - 2\delta n}{\sqrt{(\delta + 1 - \alpha)^2 + 4(2m - n\delta)(1 - \alpha)}} > 1,$$

which indicates that $f(\alpha)$ is monotonically decreasing on the interval $[0, 1)$. Consequently,

$$f(\alpha) \leq f(0) = \delta - 1 + \sqrt{(\delta(G) + 1)^2 + 4(2m - n\delta)}.$$

Set $g(\delta) = \delta - 1 + \sqrt{(\delta(G) + 1)^2 + 4(2m - n\delta)}$. The differentiation of $g(\delta)$ with respect to δ yields:

$$\frac{\mathrm{d}g}{\mathrm{d}\delta} = 1 - \frac{2n - \delta - 1}{\sqrt{(\delta + 1)^2 + 4(2m - n\delta)}}.$$

Since $\delta \leq \frac{2m}{n} \leq n - 1$, $2n - \delta - 1 \geq n > 0$ and

$$\sqrt{(\delta + 1)^2 + 4(2m - n\delta)} \geq \sqrt{(\delta + 1)^2 + 4(2m - 2m)} = \sqrt{(\delta + 1)^2} > 0.$$

Thus,

$$\frac{2n - \delta - 1}{\sqrt{(\delta + 1)^2 + 4(2m - n\delta)}} \geq 1,$$

which further implies that $g(\delta)$ is decreasing on the interval $[1, \frac{2m}{n})$. Therefore, $g(\delta) \leq g(1) = 2\sqrt{2m - n + 1}$.

Overall, we have

$$\begin{aligned}\rho_\alpha(G) &\leq \frac{\delta - 1 + \alpha + \sqrt{(\delta + 1 - \alpha)^2 + 4(2m - n\delta)(1 - \alpha)}}{2} \\ &\leq \frac{\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - n\delta)}}{2} \leq \sqrt{2m - n + 1}.\end{aligned}$$

Then, by Lemmas 2.7 and 2.8, the first equality holds, if and only if, G is either a regular graph or a bidegreed graph either δ or $n-1$. The second and third equalities hold, if and only if, $\alpha = 0$ and $\delta = 1$, respectively. Therefore, the conclusion is established. □

A graph with no cycles is called an acyclic graph. Furthermore, a connected acyclic graph is called a tree.

Lemma 2.10 ([14]) *If $\mathcal{T}$ is a tree of order n and $\alpha \in [0,1]$, then*

$$\rho_\alpha(\mathcal{T}) \leq \frac{\alpha n + \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)}}{2}.$$

Equality holds if and only if $\mathcal{T}$ is the star $K_{1,n-1}$.

Lemma 2.11 ([1]) *If $\alpha \in [0,1]$, then*

$$\rho_\alpha(G) = \max\{\ \rho_\alpha(H) : H \text{ is a component of } G\}.$$

3. Proofs of main results

Based on the previous preparations, we are now ready to prove our main results of the paper.

Proof of Theorem 1.3 Let $I \subseteq V(G)$ be any independent set of G, and set $H = G - I$. Clearly, $\gamma(G) \geq |I| \geq 1$. To complete this theorem, it suffices to show that H is a fractional $[a,b]$-covered graph. Suppose by contradiction that H is not a fractional $[a,b]$-covered graph. Then in light of Lemma 2.1, there exists some subset $S \subseteq V(G)$ such that

$$\theta_H(S,T) = b|S| - a|T| + \sum_{x \in T} d_{H-S}(x) \leq \varepsilon(S) - 1, \tag{3.1}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq a\}$ and $\varepsilon(S)$ is defined in (2.1).

By the definition of $\varepsilon(S)$, it is easy to verify $\varepsilon(S) \leq |S|$. Now, we first assert that $|T| \geq 1$. Since if not, $T = \emptyset$ and from (3.1) we have $\varepsilon(S) - 1 \geq \theta_H(S,T) = b|S| - a|T| + \sum_{x \in T} d_{H-S}(x) = b|S| \geq |S| \geq \varepsilon(S)$, a contradiction. Thus, there exists at least a vertex, say $v_0 \in T$, such that

$$d_{H-S}(v_0) = \min\{d_{H-S}(x), x \in T\} = d.$$

Furthermore, it is easy to see that $\delta(G) \leq d_G(v_0) \leq d_{G-I-S}(v_0) + |I| + |S| = d + |I| + |S|$, which implies $|S| \geq \delta(G) - |I| - d$. On the other hand, by the definition of T, we have $0 \leq d \leq a$. Here we conclude that $0 \leq d < a$, since if $d = a$, then from (3.1) we obtain $\varepsilon(S) - 1 \geq \theta_H(S,T) = b|S| - a|T| + \sum_{x \in T} d_{H-S}(x) = b|S| - a|T| + a|T| \geq |S| \geq \varepsilon(S)$, a contradiction. Hence, $0 \leq d < a$.

Now we consider all the vertices of T. Let $T_1 = G[T]$. By $|T| \geq 1$, there exists at least a vertex, say $x_1 \in V(T_1)$, such that $d_{T_1}(x_1) = \delta(T_1)$. Set $N_1 = N_{T_1}[x_1]$ and $n_1 = |N_1|$, it is clear that $N_1 \subseteq T$. If $N_1 = T$, then T is a clique and all vertices in N_1 have degree $n_1 - 1$ in T_1. Combining with $d_{T_1}(x) \leq d_{H-S}(x)$ for any $x \in T$ and $\gamma(G) \geq |I| \geq 1$, we have

$$\begin{aligned}
1 \geq \theta_H(S,T) &= b|S| - a|T| + \sum_{x \in T} d_{H-S}(x) \\
&\geq b(\delta(G) - |I| - d) - an_1 + n_1(n_1 - 1) \\
&\quad \text{(since } d_{T_1}(x) \leq d_{H-S}(x),\ N_1 = T \text{ and } |S| \geq \delta(G) - |I| - d) \\
&= b(\delta(G) - |I| - d) + (n_1^2 - (a+1)n_1) \\
&= b(\delta(G) - |I| - d) + ((n_1 - \frac{a+1}{2})^2 - \frac{(a+1)^2}{4}) \\
&\geq b(\delta(G) - |I| - d) - \frac{(a+1)^2}{4}
\end{aligned}$$

$$\begin{aligned}
&\geq b(\delta(G)-\gamma(G)-d)-\frac{(a+1)^2}{4}\gamma(G) \quad (\text{since } \gamma(G)\geq |I|\geq 1)\\
&= b(\delta(G)-d)-\frac{(a+1)^2+4b}{4}\gamma(G)\\
&\geq b(\delta(G)-d)-\frac{(a+1)^2+4b}{4}\frac{4b(\delta(G)-a-1)}{(a+1)^2+4b}\\
&= b(a-d+1)\geq 2,
\end{aligned}$$

which leads to a contradiction. Otherwise, $N_1 \subset T$, let $T_2 = G[T]-N_1$. And then, there exists at least a vertex $x_2 \in V(T_2)$, such that $d_{T_2}(x_2)=\delta(T_2)$. Set $N_2 = N_{T_2}[x_2]$ and $n_2 = |N_2|$, it is clear that $N_2 \subseteq V(T_2)$. If $N_2 = V(T_2)$, then by the choice of x_2 one can deduce that all vertices in N_2 have degree at least n_2-1 in T_2. Hence, similar to the above analysis, we arrive at a contradiction. Otherwise, $N_2 \subset V(T_2)$, let $T_3 = G[T]-(N_1\cup N_2)$. Continuing this procedure, we construct a sequence below:

(i) $T_1 = G[T]$ and $T_2 = G[T]-N_1$;

(ii) $T_i = G[T]-\bigcup_{1\leq j\leq i-1} N_j$;

(iii) $N_i = N_{T_i}[x_i]$, where x_i is a vertex in T_i such that $d_{T_i}(x_i)=\delta(T_i)$, $i=2,3,\ldots$.

We repeat the process until $i=r+1$, where $r+1$ such that $V(T_r)=N_r$, i.e., $V(T_{r+1})=\emptyset$. Set $n_i=|N_i|$. Then it follows from the above definition that $\{x_1,x_2,\ldots,x_r\}\subseteq T$ is an independent set of G. And one can easily deduce that

$$\begin{aligned}
&\gamma(G[T])\geq r\geq 1,\\
&|T| = \sum_{1\leq i\leq r} n_i.
\end{aligned} \tag{3.2}$$

Since x_i is a vertex in T_i such that $d_{T_i}(x_i)=\delta(T_i)$, one can obtain that all vertices in N_i have degree at least n_i-1 in T_i, i.e., $d_{T_i}(x)\geq n_i-1$ for any $x\in N_i$ and $i=1,2,3,\ldots,r$. Thus, one can deduce that

$$\sum_{1\leq i\leq r}\Big(\sum_{x\in N_i} d_{T_i}(x)\Big)\geq \sum_{1\leq i\leq r}(n_i^2-n_i). \tag{3.3}$$

Then it follows from (3.3) that

$$\begin{aligned}
d_{H-S}(T) &\geq \sum_{1\leq i\leq r}\Big(\sum_{x\in N_i} d_{T_i}(x)\Big)+\sum_{1\leq i<j\leq r} e_G(N_i,N_j)\\
&\geq \sum_{1\leq i\leq r}(n_i^2-n_i)+\sum_{1\leq i<j\leq r} e_G(N_i,N_j)\\
&\geq \sum_{1\leq i\leq r}(n_i^2-n_i).
\end{aligned} \tag{3.4}$$

It is easy to see that $\gamma(G)\geq\gamma(G[T])\geq r$ and $\gamma(G)\geq|I|$. Then combining this with (2.1), (3.1), (3.2), (3.4), $0\leq d<a$ and $|S|\geq\delta(G)-|I|-d$, one can deduce that

$$\begin{aligned}
1\geq\theta_H(S,T) &= b|S|-a|T|+\sum_{x\in T} d_{H-S}(x)\\
&\geq b(\delta(G)-|I|-d)-a\Big(\sum_{1\leq i\leq r} n_i\Big)+\sum_{1\leq i\leq r}(n_i^2-n_i)
\end{aligned}$$

$$\begin{aligned}
&\text{(since (3.2), (3.4) and } |S| \geq \delta(G) - |I| - d) \\
&= b(\delta(G) - |I| - d) + \sum_{1\leq i\leq r} (n_i^2 - (a+1)n_i) \\
&= b(\delta(G) - |I| - d) + \sum_{1\leq i\leq r} ((n_i - \frac{a+1}{2})^2 - \frac{(a+1)^2}{4}) \\
&\geq b(\delta(G) - |I| - d) - \frac{(a+1)^2}{4}r \\
&\geq b(\delta(G) - \gamma(G) - d) - \frac{(a+1)^2}{4}\gamma(G) \quad (\text{since } \gamma(G) \geq |I| \text{ and } \gamma(G) \geq r) \\
&= b(\delta(G) - d) - \frac{(a+1)^2 + 4b}{4}\gamma(G) \\
&\geq b(\delta(G) - d) - \frac{(a+1)^2 + 4b}{4}\frac{4b(\delta(G) - a - 1)}{(a+1)^2 + 4b} \\
&= b(a - d + 1) \geq 2,
\end{aligned}$$

a contradiction.

Overall, H is a fractional $[a,b]$-covered graph. Consequently, G is a fractional ID-$[a,b]$-factor-critical covered graph. □

Remark 3.1 We aim to demonstrate that the condition $\gamma(G) \leq \frac{4b(\delta(G)-a-1)}{(a+1)^2+4b}$ in Theorem 1.3 is optimal in a certain sense. Specifically, we intend to show that the expression $\frac{4b(\delta(G)-a-1)}{(a+1)^2+4b}$ cannot be incremented by 1 without losing its validity.

Let $\sigma = \lfloor \frac{4b(\delta(G)-a-1)}{(a+1)^2+4b} \rfloor \geq 2$. Consider a graph $G = (\sigma+1)K_{a+1}$. It is evident that

$$\frac{4b(\delta(G) - a - 1)}{(a+1)^2 + 4b} < \gamma(G) = \sigma + 1 = \lfloor \frac{4b(\delta(G) - a - 1)}{(a+1)^2 + 4b} \rfloor + 1 \leq \frac{4b(\delta(G) - a - 1)}{(a+1)^2 + 4b} + 1.$$

Now, let us construct an independent set I by selecting a vertex from each K_{a+1}, denoted by x_i, for $1 \leq i \leq \sigma + 1$. Define $I = \{x_1, x_2, \ldots, x_{\sigma+1}\}$ and $H = G - I = (\sigma+1)K_a$. Set $S = \emptyset$ and $T = V(H)$. One can deduce that

$$\begin{aligned}
\theta_H(S,T) &= b|S| - a|T| + d_{H-S}(T) = -a(\sigma+1) + (a-1)a(\sigma+1) \\
&= -(\sigma+1)a < 0.
\end{aligned}$$

According to Lemma 2.1, H is not a fractional $[a,b]$-covered graph. Therefore, G is not a fractional ID-$[a,b]$-factor-critical covered graph.

Proof of Theorem 1.4 Assume that G is not a fractional ID-$[a,b]$-factor-critical covered graph, then

$$\rho_\alpha(G^c) < \alpha\Delta(G^c) + (1-\alpha)\sqrt{2|E(G^c)|(1 - \frac{(a+1)^2 + 4b}{4b(n-1-\Delta(G^c)-a) + (a+1)^2})}.$$

Combining with Lemma 2.2, one can obtain that

$$(1-\alpha)\sqrt{2|E(G^c)|(1 - \frac{(a+1)^2 + 4b}{4b(n-1-\Delta(G^c)-a) + (a+1)^2})} \leq (1-\alpha)\sqrt{2|E(G^c)|(1 - \frac{1}{\omega(G^c)})},$$

i.e.,

$$\omega(G^c) \geq \frac{4b(\delta(G)-a)+(a+1)^2}{(a+1)^2+4b} = \frac{4b(\delta(G)-a-1)}{(a+1)^2+4b}+1.$$

Note that $\omega(G^c)=\gamma(G)$. Hence,

$$\gamma(G) \geq \frac{4b(\delta(G)-a-1)}{(a+1)^2+4b}+1,$$

which contradicts Theorem 1.3. Therefore, the proof is completed. □

Proof of Theorem 1.5 If $\rho_\alpha(G) \geq \sqrt{(n-1)(n-3)+a+1+2k}$, then it follows from Corollary 2.9 that

$$\sqrt{(n-1)(n-3)+a+1+2k} \leq \rho_\alpha(G) \leq \frac{2m(1-\alpha)}{n-1}+\alpha n-1 \quad \text{for} \quad \alpha \in [0,1),$$

which implies that $m \geq \binom{n-1}{2}+\frac{a+1}{2}+k$. Then by Lemma 2.4 and $na \equiv 0 \pmod 2$ when $a=b$, we have that G is an (a,b,k)-factor-critical graph. □

4. Concluding remarks

It is well known that if G is an (a,b,k)-factor-critical graph, then G is a fractional (a,b,k)-factor-critical graph. So by a similar analysis as in Theorem 1.5, and together with Lemma 2.6 one can also obtain the following corollary.

Corollary 4.1 *Let a,b and k be three integers with $1 \leq a \leq b$, and G a simple connected graph of order n and minimum degree $\delta(G) \geq a+k$, where $n \geq a+k+1$. Let $\alpha \in [0,1)$. If*

$$\rho_\alpha(G) \geq \sqrt{(n-1)(n-3)+a+1+2k},$$

and $na \equiv 0 \pmod 2$ when $a=b$, then G is a fractional (a,b,k)-factor-critical graph.

Note that for $\alpha=0$, Wei et al. [8] obtained the following theorem.

Theorem 4.2 ([8]) *Let a,b be two positive integers with $a \leq b$, and G a graph of order n and minimum degree $\delta(G) \geq a$, where $n \geq a+1$. If*

$$\rho_0(G) > \rho_0(K_{a-1}\nabla(K_{n-a} \cup K_1)),$$

and $na \equiv 0 \pmod 2$ when $a=b$, then G has an $[a,b]$-factor.

Fan et al. [10] gave the following condition for a graph to admit a fractional $[a,b]$-factor.

Theorem 4.3 ([10]) *Let a,b be two positive integers with $a \leq b$, and G a graph of order $n \geq a+1$. If*

$$\rho_0(G) \geq \rho_0(K_{a-1}\nabla(K_{n-a} \cup K_1)),$$

and $na \equiv 0 \pmod 2$ when $a=b$, then G has a fractional $[a,b]$-factor unless $G \cong K_{a-1}\nabla(K_{n-a} \cup K_1)$.

In this paper, we also extend the results of Theorems 4.2 and 4.3 by giving the following theorems.

Theorem 4.4 *Let a,b,k be three integers with $1 \le a \le b$ and $k \ge 0$. And let G be a graph of order $n \ge a+k+1$. If*

$$\rho_0(G) \ge \rho_0(K_{a+k-1}\nabla(K_{n-k-a} \cup K_1)),$$

and $na \equiv 0 \pmod 2$ when $a = b$, then G is an (a,b,k)-factor-critical graph unless $G \cong K_{a-1}$ $\nabla(K_{n-a} \cup K_1)$.

Proof First, we assert that $\delta(G) \ge a+k$, since if not, then we obtain that G must be a proper subgraph of $K_{a+k-1}\nabla(K_{n-k-a} \cup K_1)$ by $\delta(G) \le a+k-1$. So $\rho_0(G) \le \rho_0(K_{a+k-1}\nabla$ $(K_{n-k-a} \cup K_1))$. Together with $\rho_0(G) \ge \rho_0(K_{a+k-1}\nabla(K_{n-k-a} \cup K_1))$, one can obtain that $G \cong K_{a+k-1}\nabla(K_{n-k-a} \cup K_1)$, a contradiction.

Clearly, K_{n-1} is also a proper subgraph of $K_{a+k-1}\nabla(K_{n-k-a} \cup K_1)$. So $n-2 = \rho_0(K_{n-1}) < \rho_0(K_{a+k-1}\nabla(K_{n-k-a} \cup K_1))$. Then, it follows from Lemma 2.8 that

$$\begin{aligned} n-2 &< \rho_0(K_{a+k-1}\nabla(K_{n-k-a} \cup K_1)) \le \rho_0(G) \\ &\le \frac{(a+k)-1+\sqrt{((a+k)+1)^2+4(2m-n(a+k))}}{2}, \end{aligned}$$

which implies that $m > \binom{n-1}{2} + \frac{a+k}{2}$, that is, $m > \binom{n-1}{2} + \frac{a+1}{2} + k$. Furthermore, by Lemma 2.4 and $na \equiv 0 \pmod 2$ when $a = b$, G is an (a,b,k)-factor-critical graph.

Note that the minimum degree of $K_{a+k-1}\nabla(K_{n-k-a} \cup K_1)$ is $a+k-1$. If we choose a vertex subset $U \subseteq V(K_{a+k-1})$ such that $|U| = k$, then it is easy to see that after deleting the vertices in U the new graph $K_{a-1}\nabla(K_{n-k-a} \cup K_1)$ we obtained has no $[a,b]$-factor. Thus, $K_{a+k-1}\nabla(K_{n-k-a} \cup K_1)$ is not an (a,b,k)-factor-critical graph. Hence, we complete the proof. □

Theorem 4.5 *Let a,b and k be three integers with $1 \le a \le b$, and G a graph of order $n \ge a+k+1$. If*

$$\rho_0(G) \ge \rho_0(K_{a+k-1}\nabla(K_{n-k-a} \cup K_1)),$$

and $na \equiv 0 \pmod 2$ when $a = b$, then G is a fractional (a,b,k)-factor-critical graph unless $G \cong K_{a+k-1}$ $\nabla(K_{n-k-a} \cup K_1)$.

The proof is similar to that of Theorem 4.4, so we omit it here.

Remark 4.6 We shall give a simple example to verify the condition in Theorem 1.4 by proving that if G is not a fractional ID-$[a,b]$-factor-critical covered graph, then

$$\rho_\alpha(G^c) < \alpha\Delta(G^c) + (1-\alpha)\sqrt{2|E(G^c)|(1 - \frac{(a+1)^2+4b}{4b(n-1-\Delta(G^c)-a)+(a+1)^2})}.$$

Set $G_1 = K_{a-1}\nabla(K_{n-a} \cup K_1)$. It is easy to check that the complement of G_1 is $G_1^c = (a-1)K_1 \cup ((n-a)K_1\nabla K_1)$.

By Lemmas 2.9 and 2.10, one can obtain that

$$\rho_\alpha(G_1^c) = \frac{\alpha(n-a+1)+\sqrt{\alpha^2(n-a+1)^2+4((n-a+1)-1)(1-2\alpha)}}{2}.$$

And then by Lemma 2.1, it is easy to see that G_1 is not a fractional $[a,b]$-covered graph, so it is

not a fractional ID-$[a,b]$-factor-critical covered graph. Now we will verify

$$\rho_\alpha(G_1^c) < \alpha\Delta(G^c) + (1-\alpha)\sqrt{2|E(G^c)|(1-\frac{(a+1)^2+4b}{4b(n-1-\Delta(G^c)-a)+(a+1)^2})},$$

when $\alpha \in [\frac{1}{2}, 1)$. For $\alpha \in [\frac{1}{2}, 1)$, we have

$$\begin{aligned}
&\alpha(n-a+1) + \sqrt{\alpha^2(n-a+1)^2 + 4((n-a+1)-1)(1-2\alpha)} - \\
&\quad 2\alpha\Delta(G^c) - 2(1-\alpha)\sqrt{2|E(G^c)|(1-\frac{(a+1)^2+4b}{4b(n-1-\Delta(G^c)-a)+(a+1)^2})} \\
&= \alpha(-n+a+1) + \sqrt{\alpha^2(n-a+1)^2 + 4((n-a+1)-1)(1-2\alpha)} - \\
&\quad 2(1-\alpha)\sqrt{2|E(G^c)|(1-\frac{(a+1)^2+4b}{4b(n-1-\Delta(G^c)-a)+(a+1)^2})} \\
&< \alpha(-n+a+1) + \sqrt{\alpha^2(n-a+1)^2} = 0,
\end{aligned}$$

that is, $\rho_\alpha(G_1^c) < \alpha\Delta(G^c) + (1-\alpha)\sqrt{2|E(G^c)|(1-\frac{(a+1)^2+4b}{4b(n-1-\Delta(G^c)-a)+(a+1)^2})}$.

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