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The t-Tone Coloring of Some Finite Planar Lattices

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Abstract A t-tone coloring of a graph assigns t distinct colors to each vertex with vertices at distance d having fewer than d colors in common. The t-tone chromatic number of a graph is the smallest number of colors used in all t-tone colorings of that graph. In this article, we study t-tone coloring of some finite planar lattices and obtain exact formulas for their t-tone chromatic number.

Keywords t-tone coloring; t-tone chromatic number; planar lattice

MR(2020) Subject Classification 05C10; 05C15

1. Introduction

All graphs considered in this paper are simple and undirected. We write $[1, k]$ as shorthand for $\{1, 2, \ldots, k\}$ and denote by $P_t([k])$ the family of t-element subsets of $[1, k]$. We denote the distance between vertices u and v by $d(u, v)$. Vertices u and v are neighbors if $d(u, v) = 1$ and second-neighbors if $d(u, v) = 2$.

Let G be a graph and t be a positive integer. A function $f : V(G) \to P_t([k])$ is called a proper t-tone k-coloring (or sometimes just a t-tone coloring) of G if $|f(u) \cap f(v)| < d(u, v)$ for all distinct vertices u and v of G. A graph is t-tone k-colorable if it has a proper t-tone k-coloring. The aim of the t-tone coloring problem was primarily to minimize the t-tone chromatic number $\tau_t(G)$, i.e., the smallest positive integer k for which G has a proper t-tone k-coloring.

Given a t-tone coloring f of G, we shall often call $f(v)$ the label associated with the vertex v and the elements of $f(v)$ will be called colors. When the context is clear, we denote the labels $\{a, b\}$, $\{a, b, c\}$ and $\{a, b, c, d\}$ by ab, abc and $abcd$, respectively. Note that for each t, the index $\tau_t(G)$ is monotone, i.e., when H is a subgraph of G, every t-tone k-coloring of G restricts to a t-tone k-coloring of H, so $\tau_t(H) \leq \tau_t(G)$.

The notion of t-tone coloring was initially introduced by Bickle and Phillips [1]. A large number of works on determining the index $\tau_t(G)$ or, in particular, the index $\tau_2(G)$, were established for various graph classes including cycles, theta graphs, complete multipartite graphs, trees [1], Mobius ladders, wheels, fans, products of complete graphs, some products of cycles [2], Sierpinski triangle graphs and Hanoi graphs [3], cactus graphs [4], squares of cycles and trees,

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maximal outerplanar graphs [5], planar graphs [6]. General upper bounds were found in [2,7–10], and lower bounds were studied in [11]. One group [12] studied t-tone coloring of random graphs. Several authors [2,13,14] have studied 2-tone coloring for graph products. For general t, t-tone coloring has been studied for cycles [8,15] and grids [8]. This survey of previous results is adapted from [4].

In this article, we focus on t-tone coloring of some finite planar lattices and obtain exact formulas for their t-tone chromatic number.

2. The t-tone coloring of some finite planar lattices

The lattices we consider here are finite planar graphs which consist periodically of regular polygons. Three typical examples of such lattices are the square lattice $P_{m,n}$, triangular lattice $T_{m,n}$ and hexagonal lattice $H_{m,n}$ which consist of squares, triangles and hexagons, respectively. The other examples we will consider in this section are $G_{m,n}(4,8)$ consisting of regular squares and octagons; $G_{m,n}(4,6,8)$ consisting of regular squares, hexagons and octagons; and $G_{m,n}(4,6,12)$ consisting of regular squares, hexagons and dodecagons, as shown in Figure 1. We note that, in terms of tiling theory, all of the above lattices except $G_{m,n}(4,6,8)$ are Archimedean tiling graphs. Furthermore, it is not difficult to see that $G_{m,n}(4,8)$, $G_{m,n}(4,6,8)$ and $G_{m,n}(4,6,12)$ are all subgraphs of $P_{m,n}$, as shown in Figure 2.

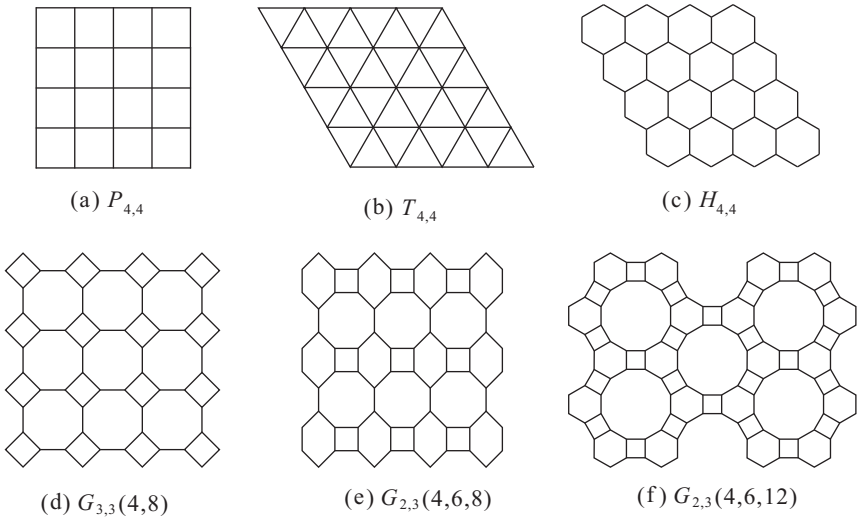
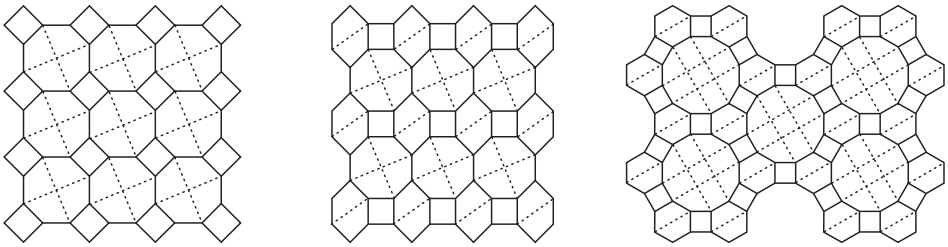


Figure 1 Examples of square lattice $P_{m,n}$, triangular lattice $T_{m,n}$, hexagonal lattice $H_{m,n}$, $G_{m,n}(4,8)$, $G_{m,n}(4,6,8)$ and $G_{m,n}(4,6,12)$

Since the finite planar lattices contain stars, cycles and wheels as their subgraphs, we present the t-tone chromatic number of stars, cycles and wheels as follows.

Lemma 2.1 ([1]) *For the nontrivial star* $K_{1,\Delta}$, $\tau_2(K_{1,\Delta}) = \lceil \frac{5+\sqrt{1+8\Delta}}{2} \rceil$, $\tau_3(K_{1,3}) = 9$ *and* $\tau_4(K_{1,3}) = 13$.



(a) $G_{m,n}(4,8)$ (b) $G_{m,n}(4,6,8)$ (c) $G_{m,n}(4,6,12)$

Figure 2 Examples of three tilings as subgraphs of the square grid $P_{m,n}$

Lemma 2.2 [1,8] *For the cycle C_n, we have:*

$$\tau_2(C_n) = \begin{cases} 6, & \text{if } n \in \{3,4,7\}, \\ 5, & \text{otherwise}; \end{cases}$$

$$\tau_3(C_n) = \begin{cases} 10, & \text{if } n \in \{4,5\}, \\ 9, & \text{if } n \in \{3,7,10,13\}, \\ 8, & \text{otherwise}; \end{cases}$$

$$\tau_4(C_n) = \begin{cases} 14, & \text{if } n = 4, \\ 15, & \text{if } n = 5, \\ 13, & \text{if } n = 7, \\ 12, & \text{otherwise}. \end{cases}$$

Lemma 2.3 *For the 6-wheel W_6, $\tau_3(W_6) = 12$ and $\tau_4(W_6) = 19$.*

Proof Let $V(W_6) = \{v_0, v_1, \ldots, v_6\}$ and v_0 be the center of W_6. Suppose C_6 is the cycle induced by $\{v_1, v_2, \ldots, v_6\}$.

For 3-tone coloring of W_6, we want to color around C_6 in the wheel, using as few colors as possible, we need at least 3+3+2+1 colors, and 3 colors for the center v_0. Thus $\tau_3(W_6) \geq 3+3+2+1+3 = 12$. Next, we construct a 3-tone 12-coloring f of W_6 as follows: $f(v_1) = 123$, $f(v_2) = 456$, $f(v_3) = 178$, $f(v_4) = 249$, $f(v_5) = 358$, $f(v_6) = 679$ and $f(v_0) = (10)(11)(12)$. Therefore, $\tau_3(W_6) = 12$.

Let f be a 4-tone k-coloring of W_6 and $x_i = |\{v_l : i \in f(v_l)\}|$ for each $i \in [1,k]$. It is easy to see that $x_i \leq 3$. Now we will prove the fact that $k \geq 19$.

Firstly, if $x_i \leq 2$ for all $i \in [1,k]$, then nine colors are used twice and six colors are used once since there are 9 second-neighbors in W_6. Thus $\tau_4(W_6) \geq 9+6+4 = 19$. Otherwise, there exists some $a \in [1,k]$ such that $x_a = 3$. Without loss of generality, let $f(v_1) = aa_1a_2a_3$, $f(v_3) = ab_1b_2b_3$ and $f(v_5) = ac_1c_2c_3$. Then any two elements in $\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$ are not equal to each other. In addition, $x_{a_1} + x_{a_2} + x_{a_3} \leq 3$, $x_{b_1} + x_{b_2} + x_{b_3} \leq 3$ and $x_{c_1} + x_{c_2} + x_{c_3} \leq 3$. Finally, it can be checked that at most two colors (say a, b) appear three times. So

$$\begin{aligned} k &\geq 4 + 11 + \lceil \frac{24 - [x_a + x_b + (x_{a_1} + x_{a_2} + x_{a_3}) + (x_{b_1} + x_{b_2} + x_{b_3}) + (x_{c_1} + x_{c_2} + x_{c_3})]}{2} \rceil \\ &\geq 4 + 11 + \lceil \frac{24 - (5+5+5+5+5)}{2} \rceil = 20. \end{aligned}$$

On the other hand, we construct a 4-tone 19-coloring f of W_6 as follows: $f(v_1) = 123(10)$, $f(v_2) = 456(11)$, $f(v_3) = 178(12)$, $f(v_4) = 249(13)$, $f(v_5) = 358(14)$, $f(v_6) = 679(15)$ and $f(v_0) = (16)(17)(18)(19)$. Hence, $\tau_4(W_6) = 19$. □

2.1. 2-tone coloring

We first consider the square lattice $P_{m,n}$ and triangular lattice $T_{m,n}$.

Theorem 2.4 ([2,13]) *If $m, n \geq 2$, then $\tau_2(P_{m,n}) = 6$ and $\tau_2(T_{m,n}) = 7$.*

Theorem 2.5 *If $m, n \geq 2$, then $\tau_2(H_{m,n}) = 6$.*

Proof Firstly, $\tau_2(H_{m,n}) \geq \tau_2(K_{1,3}) = 5$ by Lemma 2.1. Suppose for the contrary that f is a 2-tone 5-coloring of $H_{m,n}$, as shown in Figure 3. Without loss of generality, let $f(v_3) = 12$ and $f(v_2) = 34$, $f(v_4) = 35$, $f(v_9) = 45$. This leads to $\{f(v_8), f(v_{10})\} = \{13, 23\}$. If $f(v_8) = 13$ and $f(v_{10}) = 23$, then $f(v_6) = 15$ and $f(v_{12}) = 24$. Otherwise, v_7 and v_{11} have no proper label. Thus $f(v_1) = 25$ and $f(v_5) = 14$. But now v_0 has no proper label. The case for $f(v_8) = 23$ and $f(v_{10}) = 13$ is similar. Thus $\tau_2(H_{m,n}) \geq 6$. On the other hand, it is easily seen that $H_{m,n}$ is the subgraph of $P_{m,n}$. So $\tau_2(H_{m,n}) \leq \tau_2(P_{m,n}) = 6$ by Theorem 2.4. Therefore, $\tau_2(H_{m,n}) = 6$. □

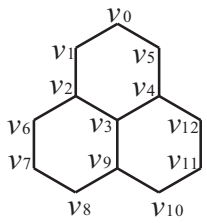


Figure 3 Non-2-tone 5-colorability of $H_{m,n}$

Theorem 2.6 *If $m, n \geq 1$, then $\tau_2(G_{m,n}(4, 8)) = 6$, $\tau_2(G_{m,n}(4, 6, 8)) = 6$ and $\tau_2(G_{m,n}(4, 6, 12)) = 6$.*

Proof Lemma 2.2 implies that $\tau_2(G_{m,n}(4, 8)) \geq \tau_2(C_4) = 6$, $\tau_2(G_{m,n}(4, 6, 8)) \geq \tau_2(C_4) = 6$ and $\tau_2(G_{m,n}(4, 6, 12)) \geq \tau_2(C_4) = 6$. On the other hand, by Theorem 2.4, $\tau_2(G_{m,n}(4, 8)) \leq \tau_2(P_{m,n}) = 6$, $\tau_2(G_{m,n}(4, 6, 8)) \leq \tau_2(P_{m,n}) = 6$ and $\tau_2(G_{m,n}(4, 6, 12)) \leq \tau_2(P_{m,n}) = 6$ since $G_{m,n}(4, 8)$, $G_{m,n}(4, 6, 8)$ and $G_{m,n}(4, 6, 12)$ are all subgraphs of $P_{m,n}$. Therefore, $\tau_2(G_{m,n}(4, 8)) = 6$, $\tau_2(G_{m,n}(4, 6, 8)) = 6$ and $\tau_2(G_{m,n}(4, 6, 12)) = 6$. □

2.2. 3-tone coloring

In this section, we consider the 3-tone coloring of $P_{m,n}$, $H_{m,n}$, $T_{m,n}$, $G_{m,n}(4, 8)$, $G_{m,n}(4, 6, 8)$ and $G_{m,n}(4, 6, 12)$.

Theorem 2.7 ([8]) *If $m, n \geq 2$, then $\tau_3(P_{m,n}) = 10$.*

Theorem 2.8 *If $m, n \geq 2$, then $\tau_3(H_{m,n}) = 9$.*

Proof Firstly, $\tau_3(H_{m,n}) \geq \tau_3(K_{1,3}) = 9$ by Lemma 2.1. Next, we construct a 3-tone 9-coloring

of $H_{m,n}$ by piecing together copies of $H_{2,2}$ with the coloring shown in Figure 4. Therefore, $\tau_3(H_{m,n}) = 9$. □

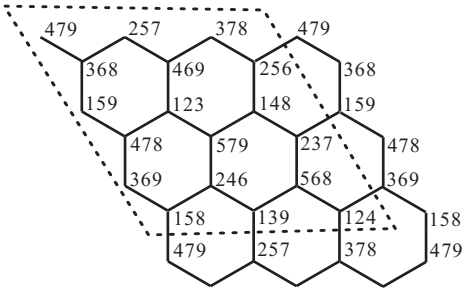


Figure 4 A 3-tone 9-coloring of $H_{2,2}$

Theorem 2.9 *If $m, n \geq 2$, then $\tau_3(T_{m,n}) = 12$.*

Proof Firstly, $\tau_3(T_{m,n}) \geq \tau_3(W_6) = 12$ by Lemma 2.3. Next, we construct a 3-tone 12-coloring of $T_{m,n}$ by piecing together copies of $T_{3,3}$ with the coloring shown in Figure 5. Therefore, $\tau_3(T_{m,n}) = 12$. □

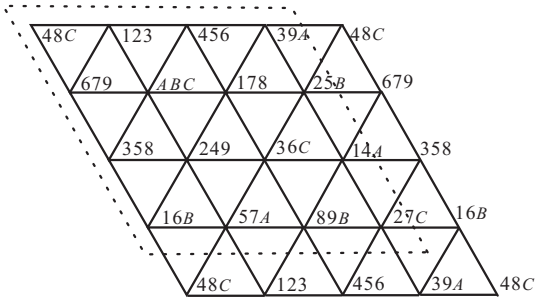


Figure 5 A 3-tone 12-coloring of $T_{3,3}$, where $A = 10$, $B = 11$, $C = 12$

Theorem 2.10 *If $m, n \geq 1$, then $\tau_3(G_{m,n}(4,8)) = 10$, $\tau_3(G_{m,n}(4,6,8)) = 10$ and $\tau_3(G_{m,n}(4,6,12)) = 10$.*

Proof Firstly, $\tau_3(G_{m,n}(4,8)) \geq \tau_3(C_4) = 10$, $\tau_3(G_{m,n}(4,6,8)) \geq \tau_3(C_4) = 10$ and $\tau_3(G_{m,n}(4,6,12)) \geq \tau_3(C_4) = 10$ since it contains C_4. Next, by Theorem 2.7, $\tau_3(G_{m,n}(4,8)) \leq \tau_3(P_{m,n}) = 10$, $\tau_3(G_{m,n}(4,6,8)) \leq \tau_3(P_{m,n}) = 10$ and $\tau_3(G_{m,n}(4,6,12)) \leq \tau_3(P_{m,n}) = 10$ since $G_{m,n}(4,8)$, $G_{m,n}(4,6,8)$ and $G_{m,n}(4,6,12)$ are all subgraphs of $P_{m,n}$. Therefore, $\tau_3(G_{m,n}(4,8)) = 10$, $\tau_3(G_{m,n}(4,6,8)) = 10$ and $\tau_3(G_{m,n}(4,6,12)) = 10$. □

2.3. 4-tone coloring

In this section, we consider the 4-tone coloring of $P_{m,n}$, $H_{m,n}$, $T_{m,n}$, $G_{m,n}(4,8)$, $G_{m,n}(4,6,8)$ and $G_{m,n}(4,6,12)$. Recall that the square of G, denoted G^2, is the graph with $V(G^2) = V(G)$ and edge set $E(G^2) = \{uv : d(u,v) \leq 2\}$.

Theorem 2.11 ([8]) *If $m, n \geq 2$, then $\tau_4(P_{m,n}) = 14$.*

Theorem 2.12 *If $m, n \geq 2$, then $\tau_4(H_{m,n}) = 13$.*

Proof Firstly, $\tau_3(H_{m,n}) \geq \tau_4(K_{1,3}) = 13$ by Lemma 2.1. Let $f_1 : V(H_{m,n}) \to P_3([9])$ be a 3-tone 9-coloring of $H_{m,n}$ by piecing together copies of $H_{5,5}$ with the coloring shown in Figure 6 and $f_2 : V(H^2_{m,n}) \to \{10, 11, 12, 13\}$ be a 4-coloring of $H^2_{m,n}$ by piecing together copies of $H^2_{5,5}$ with the coloring shown in Figure 7. Define $f(u) = f_1(u) \cup f_2(u)$ for all $u \in V(H_{m,n})$.

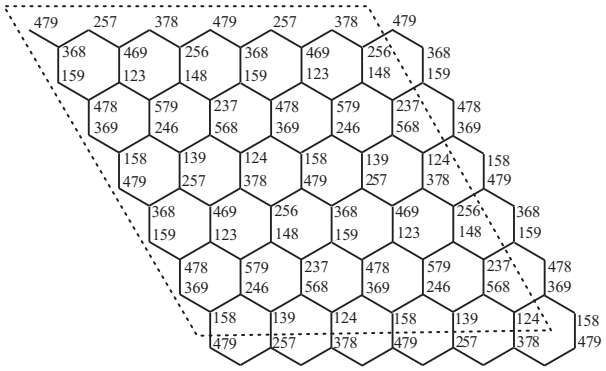


Figure 6 A 3-tone 9-coloring of $H_{5,5}$

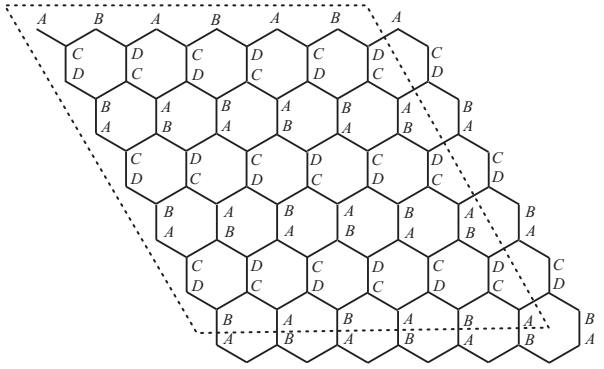


Figure 7 A 4-coloring of $H^2_{5,5}$, where $A = 10$, $B = 11$, $C = 12$, $D = 13$

Next, we will prove f is a 4-tone 13-coloring of $H_{m,n}$. Firstly, if $d(u, v) = 1$, then $|f(u) \cap f(v)| < 1$ since $|f_1(u) \cap f_1(v)| < 1$ and $f_2(u) \neq f_2(v)$. If $d(u, v) = 2$, then $|f_1(u) \cap f_1(v)| < 2$ and $f_2(u) \neq f_2(v)$. Thus $|f(u) \cap f(v)| < 2$. For $d(u, v) = 3$, according to the coloring we defined, we always have $|f_1(u) \cap f_1(v)| < 2$, $f_2(u) = f_2(v)$ or $|f_1(u) \cap f_1(v)| < 3$, $f_2(u) \neq f_2(v)$. Therefore, we have $|f(u) \cap f(v)| < 3$. For $d(u, v) = 4$, we have $|f_1(u) \cap f_1(v)| < 3$. So $|f(u) \cap f(v)| < 4$. Therefore, $\tau_4(H_{m,n}) = 13$. □

Theorem 2.13 *If $m, n \geq 2$, then $\tau_4(T_{m,n}) = 19$.*

Proof Firstly, $\tau_4(T_{m,n}) \geq \tau_4(W_6) = 19$ by Lemma 2.3. Let $f_1 : V(T_{m,n}) \to P_3([12])$ be a 3-tone 12-coloring of $T_{m,n}$ by piecing together copies of $T_{6,6}$ with the coloring shown in Figure 8 and $f_2 : V(T^2_{m,n}) \to \{13, 14, \ldots, 19\}$ be a 7-coloring of $T^2_{m,n}$ by piecing together copies of $T^2_{6,6}$ with the coloring shown in Figure 9. Define $f(u) = f_1(u) \cup f_2(u)$ for all $u \in V(T_{m,n})$.

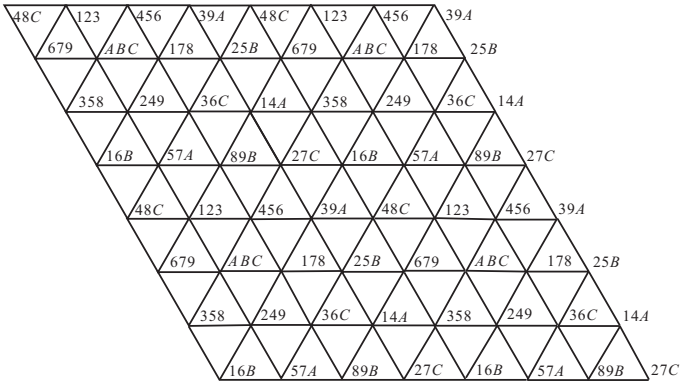


Figure 8 A 3-tone 12-coloring of $T_{6,6}$, where $A = 10$, $B = 11$, $C = 12$

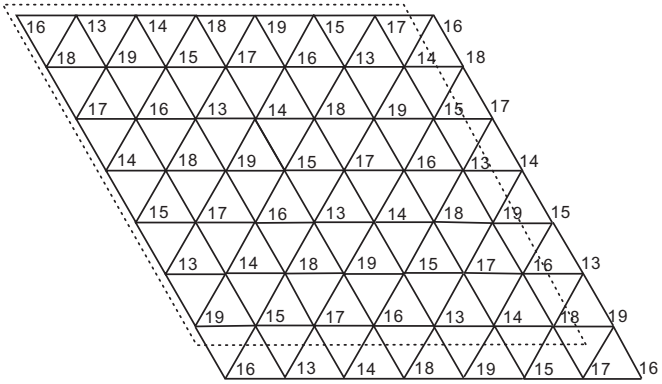


Figure 9 A 7-coloring of $T^2_{6,6}$

Next, we show that f is a 4-tone 19-coloring of $T_{m,n}$. Firstly, if $d(u, v) = 1$, then $|f(u) \cap f(v)| < 1$ since $|f_1(u) \cap f_1(v)| < 1$ and $f_2(u) \neq f_2(v)$. For $d(u, v) = 2$, according to the coloring defined, we have $|f_1(u) \cap f_1(v)| < 2$ and $f_2(u) \neq f_2(v)$. Thus $|f(u) \cap f(v)| < 2$. For $d(u, v) = 3$, we have $|f_1(u) \cap f_1(v)| < 2$. Therefore, we always have $|f(u) \cap f(v)| < 3$. For $d(u, v) = 4$, note that $f_2(u) \neq f_2(v)$. So $|f(u) \cap f(v)| < 4$. Therefore, $\tau_4(T_{m,n}) = 19$. □

Theorem 2.14 *If $m, n \geq 1$, then $\tau_4(G_{m,n}(4, 8)) = 14$, $\tau_4(G_{m,n}(4, 6, 8)) = 14$ and $\tau_4(G_{m,n}(4, 6, 12)) = 14$.*

Proof Firstly, $\tau_4(G_{m,n}(4, 8)) \geq \tau_4(C_4) = 14$, $\tau_4(G_{m,n}(4, 6, 8)) \geq \tau_4(C_4) = 14$ and $\tau_4(G_{m,n}(4, 6, 12)) \geq \tau_4(C_4) = 14$ since they all contain C_4. Next, by Theorem 2.11, $\tau_4(G_{m,n}(4, 8)) \leq \tau_4(P_{m,n}) = 14$, $\tau_4(G_{m,n}(4, 6, 8)) \leq \tau_4(P_{m,n}) = 14$ and $\tau_4(G_{m,n}(4, 6, 12)) \leq \tau_4(P_{m,n}) = 14$ since $G_{m,n}(4, 8)$, $G_{m,n}(4, 6, 8)$ and $G_{m,n}(4, 6, 12)$ are all subgraphs of $P_{m,n}$. Therefore, $\tau_4(G_{m,n}(4, 8)) = 14$, $\tau_4(G_{m,n}(4, 6, 8)) = 14$ and $\tau_4(G_{m,n}(4, 6, 12)) = 14$. □

3. Concluding remarks

In this paper, we study the t-tone coloring of some finite planar lattices for $t = 2, 3$ and 4. It was conjectured that $\tau_2(G) \leq 6$ when G is a cubic graph and does not contain $K_4 - e$ (see [1]).

The conjecture was refuted [7] by demonstrating that it fails for the Heawood graph, which does not contain $K_4 - e$ (and indeed has girth 6). But based on the results we obtained, we find that the conjecture holds for $H_{m,n}$, $G_{m,n}(4,8)$, $G_{m,n}(4,6,8)$ and $G_{m,n}(4,6,12)$.

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Conflict of Interest The author declares no conflict of interest.

Availability of Data and Material Not applicable.

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