Journal of Mathematical Research with Applications
Jan., 2026, Vol. 46, No. 1, pp. 33–39
DOI:10.3770/j.issn:2095-2651.2026.01.004
Http://jmre.dlut.edu.cn

Local Characterizations of Results on the Normal Index of Subgroups in Finite Groups

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Abstract Let G be a finite group and H a subgroup of G. The normal index of H in G is defined as the order of K/H_G, where K is a normal supplement of H in G such that $|K|$ is minimal and $H_G \leq K \trianglelefteq G$. Let p be a prime which divides the order of a group G. In this paper, some characterizations of G being p-solvable or p-supersolvable were obtained by analyzing the normal index of certain subgroups of G. These results can be viewed as local version of recent results in the literature.

Keywords p-solvable group; p-supersolvable group; normal index; maximal subgroup; 2-maximal subgroup

MR(2020) Subject Classification 20D10; 20E28

1. Introduction

In this paper, all groups are assumed to be finite. The symbol G always denotes a group, and $|G|$ represents the order of G. The notation $\pi(G)$ refers to the set of prime divisors of $|G|$, and p generally denotes a fixed prime. For a group G of order $p^n m$, where $(p, m) = 1$, let G_p denote the Sylow p-subgroup of G and $|G|_p = p^n$ the p-part of $|G|$. All other terminology and notation used follow standard conventions, as outlined in [1].

In what follows the term local is meant in the sense that a certain property is generalized in a form referring to a prime, for example, solvable group to p-solvable group, nilpotent group to p-nilpotent group. It is one of the main problems in group theory to determine whether a global property can be described as the conjunction of local properties at each prime. In particular, one may ask whether a local characterization can be given for some fixed prime.

Recall that the Deskins index of a maximal subgroup M in a group G is defined as the order of a chief factor H/K, where H is minimal in the set of normal supplements to M in G. Introduced by Deskins [2], the Deskins index is denoted by $\eta(G : M) = |H/K|$. It was shown in [3] that

Received March 18, 2025; Accepted June 18, 2025
Supported by the National Natural Science Foundation of China (Grant No. 12071092), Guangdong Basic and Applied Basic Research Foundation (Grant No. 2025A1515012072), the Natural Science Research Project of Anhui Educational Committee (Grant No. 2024AH051298) and the Scientific Research Foundation of Bozhou University (Grant No. BYKQ202419).
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$\eta(G:M)$ is uniquely determined by M (see [3, Lemma 1]). Thus, we can select a subgroup $T \trianglelefteq G$ such that T/M_G is a G-chief factor and $\eta(G:M) = |T/M_G|$. Furthermore, $T \cap M/M_G$ is called a c-section of M (see [4]) and it has been proven that every maximal subgroup M of G has a unique c-section up to isomorphism. Let $\mathrm{Sec}(M)$ denote an arbitrary subgroup of G isomorphic to a c-section of M. It follows that $\eta(G:M) = |G:M||T \cap M/M_G| = |G:M| \cdot |\mathrm{Sec}(M)|$.

The Deskins index has been extensively applied by researchers to investigate the solvability and supersolvability of groups, yielding numerous significant results [2, 3, 5–8]. Naturally, one might consider extending the definition of the Deskins index from maximal subgroups to arbitrary subgroups. However, the following example shows that this extension may not be well-defined.

Example 1.1 Let $G = \langle a, e\rangle \times \langle b, d\rangle \times \langle c\rangle \cong S_3 \times S_3 \times C_2$, where $a^2 = e^3 = b^2 = d^3 = c^2 = 1$ and $e^a = e^2$, $d^b = d^2$.

Suppose that $H = \langle ab, ed\rangle \cong S_3$. Then $H_G = 1$. Pick $K_1 = \langle a, e, cb, d\rangle \cong S_3 \times S_3$, $K_2 = \langle a, e, c\rangle \cong S_3 \times C_2$. Then K_1 and K_2 are two minimal normal supplements of H in G. But $|K_1| \neq |K_2|$. Furthermore, $H \cap K_1 = \langle de\rangle \cong C_3$, $H \cap K_2 = 1$, $(H \cap K_1)/H_G \ncong (H \cap K_2)/H_G$.

Recently, Li, Zhang, and Yin [9, 10] introduced the following concept to further advance the discussion in this area:

Definition 1.2 *Let H be a proper subgroup of a group G and K a normal subgroup of G containing H_G. Then $|K/H_G|$ is called the normal index of H in G, denoted by $\eta^*(G:H)$, if K satisfies*

(1) $G = HK$;

(2) $|K| \le |T|$ for any normal subgroup T of G such that $H_G \le T$ and $G = HT$.

It is not difficult to see that for a maximal subgroup M of a group G, the relation $\eta^*(G:M) = \eta(G:M)$ holds.

By applying the concept of normal index, Li et al. [8–10] obtained significant results concerning the solvability and supersolvability of groups. In this paper, we will present local versions of the main results stated above, expressed in terms involving a prime.

2. Preliminaries

In this section, we present several fundamental results. First, from [10, Lemma 2.2], we have the following:

Lemma 2.1 *Let H be a subgroup of G and N a normal subgroup of G contained in H. Then $\eta^*(G:H)_p = \eta^*(G/N:H/N)_p$.*

Following [9], a group G is called an NIT-group if $\eta^*(G:H) = \eta^*(G:K)\eta^*(K:H)$ for any subgroup H and any subgroup K of G such that $H \le K \le G$. Let p be a prime in $\pi(G)$. We call a group G is an NIT_p-group if $\eta^*(G:H)_p = \eta^*(G:K)_p\eta^*(K:H)_p$ for all subgroups $H \le K \le G$.

Lemma 2.2 *Let G be an NIT_p-group and $T \trianglelefteq G$. Then*

(1) G/T is also an NIT_p-group;

(2) Every subgroup of G is an NIT_p-gorup.

Proof We only prove (2) since (1) is easily proved by Lemma 2.1.

Let $N \leq G$ and $H \leq K \leq N$. Since G is an NIT_p-group, we have

$$\eta^*(G : H)_p = \eta^*(G : N)_p \eta^*(N : H)_p,$$

$$\eta^*(G : K)_p = \eta^*(G : N)_p \eta^*(N : K)_p,$$

$$\eta^*(G : H)_p = \eta^*(G : K)_p \eta^*(K : H)_p.$$

So $\eta^*(G : H)_p = \eta^*(G : N)_p \eta^*(N : K)_p \eta^*(K : H)_p = \eta^*(G : N)_p \eta^*(N : H)_p$, and then $\eta^*(N : H)_p = \eta^*(N : K)_p \eta^*(K : H)_p$, N is an NIT_p-group. □

Lemma 2.3 *Let G be a group and $H \leq M \lessdot G$. Then $\eta(G : M) \,\big|\eta^*(G : H)$.*

Proof Assume that $\eta^*(G : H) = |K/H_G|$, where K is a normal supplement of H in G containing H_G such that $|K|$ is minimal. Now, let N/M_G be a minimal normal subgroup of G/M_G such that $N/M_G \leq KM_G/M_G$. Then, $G = NM$. Indeed, N is a minimal normal supplement of M in G containing M_G, which implies that $\eta(G : M) = \eta(G/M_G : M/M_G) = |N/M_G|$. On the other hand, since $H_G \leq K \cap M_G$, we have $\eta(G : M) = |N/M_G| \,\big|\, |KM_G/M_G| = |K/K \cap M_G| = |(K/H_G)/(K \cap M_G/H_G)| \,\big|\, |K/H_G| = \eta^*(G : H)$, as desired. □

Now by Lemma 2.3 and [9, Lemma 4.2], we have

Lemma 2.4 *Let G be a p-solvable group and $H \lessdot M \lessdot G$. Then $\eta(G : M)_p | \eta^*(G : H)_p$ and $\eta(M : H)_p | \eta^*(G : H)_p$.*

3. Main results

The first result serves as a localized version of [9, Theorem 3.4].

Theorem 3.1 *Let G be a group and $p \in \pi(G)$. If G is an NIT_p-group, then the following statements hold.*

(1) G is a p-supersolvable group;

(2) $\eta^*(G : H)_p = |G : H|_p$ for every subgroup H of G.

Proof (1) We prove this result by induction on $|G|$. First, assume that G is a simple group. If G is abelian, it is obviously p-supersolvable. Therefore, we may assume that G is a non-abelian simple group. By hypothesis, for any maximal subgroup M of G, we have $\eta^*(G : 1)_p = \eta^*(G : M)_p \eta^*(M : 1)_p$. Since G is simple, $\eta^*(G : M)_p = |G : M|_p = |G|_p$, which implies that M is a p'-group. By our choice of M, G itself must then be a p'-group, so G is p-supersolvable. Next, assume G is not simple. Let N be a minimal normal subgroup of G. By Lemma 2.2, both G/N and N are NIT_p-groups. By the inductive hypothesis, G/N and N are p-supersolvable. If N is a p'-group or $|N| = p$, clearly, G is p-supersolvable. Therefore, we may assume that N is an elementary abelain p-group with $|N| > p$. Moreover, since the class of p-supersolvable groups is a saturated formation, we may assume that N is unique. Let $T \lessdot N$. If $T \neq 1$, then $T_G = 1$. By hypothesis, $\eta^*(G:1)_p = \eta^*(G:T)_p\eta^*(T:1)_p$, which implies $\eta^*(G:T)_p = |G:T|_p$. Let $K \trianglelefteq G$ such that $G = TK$ and $\eta^*(G:T)_p = |K/T_G|_p = |K|_p = |G:T|_p$. Since N is unique, we have $N \leq K$ and then $G = K$. Thus $\eta^*(G:T)_p = |K|_p = |G|_p = |G:T|_p$, implying $T = 1$ and $|N| = p$, which is a contradiction.

(2) Suppose that G is an NIT_p-group, and let H be a subgroup of G. Then, by definition, $\eta^*(G:1)_p = \eta^*(G:H)_p\eta^*(H:1)_p$. It is straightforward to verify that $\eta^*(G:1) = |G|$ and $\eta^*(H:1) = |H|$. Consequently, $\eta^*(G:H)_p = (|G|/|H|)_p = |G:H|_p$. □

The following can be regarded as a local version of [9, Lemma 4.4 and Theorem 4.6], respectively.

Theorem 3.2 *Let G be a p-solvable group and $p \in \pi(G)$. If G has a p-supersolvable subgroup H such that $\eta^*(G:H)_p = 1$ or p, then G/H_G is p-supersolvable.*

Proof Let $T \trianglelefteq G$ such that $\eta^*(G:H)_p = |T/H_G|_p$. Then $G = HT$ and it follows that $G/H_G = (H/H_G)(T/H_G)$. Moreover, $(G/H_G)/(T/H_G) \cong G/T \cong H/H\cap T$, which is p-supersolvable. If $\eta^*(G:H)_p = |T/H_G|_p = 1$, it is clear that G/H_G is p-supersolvable. Therefore, $\eta^*(G:H)_p = |T/H_G|_p = p$. Since G is p-solvable, every G/H_G-chief factor below T/H_G is either a cyclic group of order p or a p'-group. This ensures that G/H_G is p-supersolvable. □

Theorem 3.3 *Let G be a group and $p \in \pi(G)$. Suppose that G has a maximal subgroup M such that $\eta^*(G:L)_p = |G:L|_p$ for each maximal subgroup L of M. Then $Z(G/M_G) = 1$.*

Proof Assume that $M_G \neq 1$. Then we may take a minimal normal subgroup N of G such that $N \leq M_G$. Consider the quotient group G/N. By Lemma 2.1, it is clear that G/N satisfies our hypothesis. By induction on $|G|$, we have $Z((G/N)/(M/N)_{G/N}) = Z(G/M_G) = 1$. Thus we may assume that $M_G = 1$. If $Z(G) > 1$, we can choose a minimal normal subgroup N of G such that $N \leq Z(G)$, then $N \nleq M$. Since $G = MN$ and N is central, it follows that $G = MN = M \times N$. Consequently, $M = M_G$, which is a contradiction. □

The following results can be regarded as the local versions of [8, Theorem 3.12], [3, Theorem 6], and [9, Theorem 4.9], respectively.

Theorem 3.4 *Let G be a group and $p \in \pi(G)$. Then G is p-supersolvable if and only if G is p-solvable and $\eta(G:M)_p = 1$ or p for every maximal subgroup M of G.*

Proof By [8, Corollary 3.11], it suffices to show that there is no $M \lessdot G$ such that $\eta(G:M)_p = p$ and $|G:M|_p = 1$. Suppose, for contradiction, that $\eta(G:M)_p = |N/M_G|_p = p$ for some normal subgroup N of G. Then $G = MN$ and $|N/M_G| = p$ since G is p-solvable. Note that $\eta(G:M)_p = |G:M|_p|N/M_G \cap M/M_G|_p = |N/M_G \cap M/M_G| = p$. This implies that $N/M_G \leq M/M_G$, which in turn means $G = M$, a contradiction. □

Theorem 3.5 *Let G be a group and M be a maximal subgroup of G. Then G is p-supersolvable* *if M satisfies one of the following conditions:*

(1) M is p-supersolvable, $F_p(G) \nleq M$ and $\eta(G:M)_p = p$ or 1;

(2) M is p-supersolvable and $F_p(G) \nleq M$ and M has a maximal subgroup L such that $\eta^*(G:L)_p = p$.

Proof Obviously, G is p-solvable in both cases (1) and (2). We focus on case (1), as case (2) follows directly from case (1) and Lemma 2.4.

If $O_{p'}(G) = 1$, then $F_p(G) = F(G)$ and the proof is similar to that of [9, Theorem 4.9]. If $O_{p'}(G) > 1$ and $O_{p'}(G) \nleq M$, then $G/O_{p'}(G) \cong M/M \cap O_{p'}(G)$ is p-supersolvable, and thus G is also p-supersolvable.

Now assume $1 < O_{p'}(G) \leq M$. By Lemma 2.1, $G/O_{p'}(G)$ satisfies the hypothesis. By induction, $G/O_{p'}(G)$ is p-supersolvable. Therefore, G itself is p-supersolvable. □

Theorem 3.6 *Let G be a p-solvable group and M be a maximal subgroup of G. Then G is p-supersolvable if M satisfies one of the following conditions:*

(1) M is p-supersolvable, $M_G = 1$ and M has a maximal subgroup L such that $\eta^*(G:L)_p = 1$ or p;

(2) $\eta^*(G:L)_p = 1$ or p for every maximal subgroup L of M;

(3) M is p-supersolvable, $M_G = 1$ and $\eta(G:M)_p = p$ or 1;

(4) M is cyclic and $\eta(G:M)_p = p$ or 1;

(5) M has a cyclic maximal subgroup L such that $\eta^*(G:L)_p = p$ or 1.

Proof (1) Assume that $\eta^*(G:L)_p = |T|_p = p^n$, where $n \in \{0,1\}$ and T is a normal supplement of L in G such that $|T|$ is minimal. Then $G = LT = MT$. Let N be a minimal normal subgroup of G contained in T. Then $G = MN$, and $G/N = MN/N \cong M/M\cap N$, which is p-supersolvable. Furthermore, N is either an elementary abelian p-group or a p'-group. If $n = 0$ or N is a p'-group, it is clear that G is p-supersolvable. Now assume that $n = 1$ and N is a cyclic group of order p since $|T|_p = p$. It follows from $M \cong G/N$ that G is p-supersolvable.

(2) By Lemma 2.4, we have $\eta(M:L)_p = p$ or 1 for every $L \lessdot M$. Therefore, M is p-supersolvable by Theorem 3.4. Let N be a minimal normal subgroup of G. We now claim by induction that G/N is p-supersolvable. If $N \nleq M$, then $G = MN$, and $G/N = MN/N \cong M/M\cap N$, which is p-supersolvable. Hence, we may assume $N \leq M$. By induction on $|G|$ and Lemma 2.1, we conclude that G/N is p-supersolvable, as desired. So we may assume that N is unique. Since G is p-solvable, we can further assume N is an elementary abelian p-group with $|N| > p$. On the other hand, if $M_G > 1$, then $N \leq M$ and $N \nleq \Phi(M)$. Otherwise, $N \leq \Phi(G)$, which implies G is p-supersolvable, as desired. Now let L be a maximal subgroup of M such that $N \nleq L$. Then $L_G = 1$, and there exists some $T \trianglelefteq G$ such that $\eta^*(G:L)_p = |T|_p = p$ or 1. If $\eta^*(G:L)_p = 1$, clearly, $N \leq T$ is a p'-group, which is a contradiction. Thus $\eta^*(G:L)_p = |T|_p = p$. It leads to $|N| = p$, another contradiction. Hence, $M_G = 1$. Finally, by case (1), G is p-supersolvable.

(3) It is a corollary of Theorem 3.2.

(4) By (3), it suffices to show that $M_G = 1$. If $M_G > 1$, obviously, G/M_G satisfies our

hypothesis. By induction, G/M_G is p-supersolvable. Since M is cyclic, it follows that G is p-supersolvable, as desired.

(5) By Theorem 3.2, G/L_G is p-supersolvable. So G is p-supersolvable since L is cyclic. □

Corollary 3.7 *Let G be a p-solvable group for which $p \in \pi(G)$. Then G is p-supersolvable if $\eta^*(G:L)_p = p$ or 1 for every 2-maximal subgroup L of G.*

A 2-maximal subgroup H of a group G is called c-2-maximal if $H \lessdot M \lessdot G$ and $|G:M|$ is composite. We now examine the following result, which serves as a local version of [10, Theorem 3.8], which is a more detailed description of Corollary 3.7.

Theorem 3.8 *Let G be a p-solvable group, where $p \in \pi(G)$. Then G is p-supersolvable if $\eta^*(G:L)_p = 1$ or p for every c-2-maximal subgroup L of G.*

Proof We proceed by induction on $|G|$. Let N be a minimal normal subgroup of G. Clearly, G/N satisfies the hypothesis, so G/N is p-supersolvable by induction. Now assume that N is unique and $N \nleq \Phi(G)$. Since G is p-solvable, we may further assume that N is an elementary abelian p-group with $|N| > p$. Hence, there exists a maximal subgroup M of G such that $G = MN$ and $M_G = 1$. Clearly, $N \cap M \trianglelefteq G$, and by our choice of N, we have $N \cap M = 1$. Moreover, $\eta(G:M)_p = |G:M| = |N| > p$. If $|G:M| = r$, where r is a prime, then $|N| = p$, contradicting the assumption that $|N| > p$. This implies that $|G:M|$ is a composite. Thus, for every $L \lessdot M$, L is a c-2-maximal subgroup of G. By hypothesis, $\eta^*(G:L)_p = 1$ or p. However, by Lemma 2.4, $\eta(G:M)_p = |N| = 1$ or p, which contradicts $|N| > p$. □

Remark 3.9 It is important to note that in Theorems 3.2, 3.4, 3.6 and 3.8, we assume that the group G is p-solvable. This assumption is indeed necessary, as demonstrated by the alternating group A_5, which serves as a counterexample when G is not p-solvable. Furthermore, this observation indicates that while focusing on a fixed prime divisor can yield several structural characterizations of groups, the local approach may encounter limitations in certain contexts.

In [9, Theorem 3.5], the authors proved that G is solvable under the assumption that $\eta^*(G:H) = \eta^*(G:M)\eta^*(M:H)$ for every maximal subgroup M of G and every maximal subgroup H of M. This naturally leads to the following question:

Question 3.10 Let G be a group and $p \in \pi(G)$. If for every maximal subgroup M of G and every maximal subgroup H of M, $\eta^*(G:H)_p = \eta^*(G:M)_p\eta^*(M:H)_p$, is G a p-solvable group?

Noting that our Theorem 3.1 is a partial extension of [9, Theorem 3.4], a natural question arises: Can we obtain a complete local characterization? In other words, is it possible to provide an affirmative answer to the following question:

Question 3.11 Let G be a group and $p \in \pi(G)$. Are the following statements equivalent?

(1) G is an NIT_p-group;

(2) G is a p-supersolvable NIT_p-group;

(3) $\eta^*(G:H)_p = |G:H|_p$ for any subgroup H of G.

Acknowledgements The authors would like to thank the referees for their valuable suggestions and comments.

Conflict of Interest The authors declare no conflict of interest.

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