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Mixed Local and Nonlocal Elliptic Problem with Concave-Convex and Discontinuous Nonlinearities

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Abstract This paper investigates the following mixed local and nonlocal elliptic problem featuring concave-convex nonlinearities and a discontinuous right-hand side:

$$\begin{cases} \mathcal{L}(u) = \mathcal{H}(u-\mu)|u|^{p-2}u + \lambda|u|^{q-2}u, & x \in \Omega, \\ u \geq 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a bounded domain, $\mu \geq 0$ and $\lambda > 0$ are real parameters, $\mathcal{H}$ denotes the Heaviside function ($\mathcal{H}(t) = 0$ for $t < 0$, $\mathcal{H}(t) = 1$ for $t > 0$), and the mixed local and nolocal operator is defined as $\mathcal{L}(u) = -\Delta u + (-\Delta)^s u$ with $(-\Delta)^s$ being the restricted fractional Laplace $(0 < s < 1)$. The exponents satisfy $1 < q < 2 < p$. By employing a novel non-smooth variational principle, we establish the existence of an M-solution for this problem and identify a range for the exponent p.

Keywords mixed local and nonlocal operator; discontinuous; non-smooth variational principle; M-solution

MR(2020) Subject Classification 35J15; 35J20

1. Introduction

This paper considers a mixed local and nonlocal elliptic problem with concave-convex and discontinuous nonlinearities of the following type

$$\begin{cases} \mathcal{L}(u) = \mathcal{H}(u-\mu)|u|^{p-2}u + \lambda|u|^{q-2}u, & x \in \Omega, \\ u \geq 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\mathcal{H}$ is the Heaviside function, i.e.,

$$\mathcal{H}(t) = 1 \ \text{ if } t > 0 \ \text{ and } \ \mathcal{H}(t) = 0 \ \text{ if } t < 0,$$

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$\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a bounded domain with a smooth boundary and $\mu \geq 0$ is a positive parameter, $\lambda > 0$, $0 < s < 1$, $1 < q < 2 < p$. Here and throughout the paper, we define the mixed local and nolocal operator as

$$\mathcal{L}(u) = -\Delta u(x) + (-\Delta)^s u(x),$$

where the fractional Laplace $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = a_{N,s}\ \text{P.V.}\ \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y = a_{N,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y,$$

$a_{N,s}$ is given by

$$a_{N,s} = \Big(\int_{\mathbb{R}^N} \frac{1-\cos \xi_1}{|\xi|^{N+2s}}\mathrm{d}\xi \Big)^{-1}.$$

Mixed local and nonlocal operator $\mathcal{L}(u)$ was proposed in the study of stochastic processes, specifically the superposition of two stochastic processes, where $-\Delta$ is the infinitesimal generator of Brownian motion, $(-\Delta)^s$ is the infinitesimal generator of 2s-stable Lévy process. In physics, the operator is widely applied in the theory of optimal searching, biomathematics and animal forging [1, 2]. In mathematics, a significant amount of research is currently being devoted to the study of mixed local and nonlocal elliptic equations [3–7].

In the celebrated paper [8], Ambrosetti et al. first investigated the following semilinear elliptic problems with a supercritical nonlinearity

$$\begin{cases} -\Delta u = u|u|^{p-2} + \mu u|u|^{q-2}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Kouhestani and Moameni revisited problem (1.2) in [9], leveraging a novel non-smooth variational principle introduced by Moameni [10, Theorem 1.2] and [11, Theorem 3.1, Theorem 3.7], through which they established intriguing results on the existence of solutions. Analogous solution existence conclusions were subsequently derived for both the corresponding nonlocal counterpart [12] and the mixed local-nonlocal variant [13] of problem (1.2), via applications of the non-smooth variational principle.

In the realm of mathematical physics, the study of partial differential equations (PDEs) with discontinuous nonlinearities has assumed paramount importance. This significance stems from the fact that numerous free boundary problems can be reformulated as boundary value problems featuring discontinuities in the second term. Ever since Ambrosetti and Badiale [14] initiated the investigation of such problems, they have attracted substantial scholarly attention. Specifically, their pioneering work focused on the following class of elliptic boundary value problems:

$$\begin{cases} \Delta u = f(u) + p(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where f is a discontinuous nonlinearity and p is a given function. Based on Clarkes Dual Variational Principle, Ambrosrtti and Badiale [14, Theorem 1] gave the existence of solution to

problem (1.3). Meanwhile, the following problem was studied by Ambrosetti and Turner [15],

$$\begin{cases} \Delta u = f(u)\mathcal{H}(u-\mu), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $\mathcal{H}$ is also the Heaviside function, $\mu > 0$, f is nondecreasing in u. Employing the dual variational principle, scholars established the existence of positive solutions to a local elliptic problem characterized by discontinuous nonlinearities. For problem (1.4) endowed with free boundary conditions $u = g$ on $\partial\Omega$, Bensid and Bouguima [16, Theorem 1.2] demonstrated that the problem admits at least two positive solutions, with the free boundaries proven to be analytic hypersurfaces. Under identical boundary conditions and the specification $f(u) = \lambda$, Alexander [17, Theorem 2] asserted the existence of a solution to problem (1.4), contingent upon the fulfillment of certain reasonable hypotheses. In a distinct framework, Bandeira et al. [18, Theorem 1.1] showed that when the right-hand side of problem (1.4) assumed the explicit form $\lambda h(x) + \mathcal{H}(u-\mu)|u|^{2^*-2} + \frac{1}{2}(x \cdot \nabla u)$, the problem admitted two nonnegative solutions. The first solution is constructed via a nonsmooth instantiation of Mountain Pass Theorem, while the second arised from a localized application of the Ekeland Variational Principle. Leveraging variational methodologies in tandem with the lower-upper solution technique, Badiale and Tarantello [19, Theorem 3.2] investigated the existence and multiplicity of positive solutions to problem (1.4) under the scenario where $f(u) = \lambda$ was augmented by a term u^{p-1} on the equation's right-hand side. For more recent papers related to elliptic problems with discontinuous nonlinearity, we refer to [20–23] and the references therein.

In the paper [24], Bensid has carried out an investigation into the nonlocal counterpart of the problem (1.4).

$$\begin{cases} (-\Delta)^s u = f(u)\mathcal{H}(u-\mu), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.5}$$

where $\mathcal{H}$ is the Heaviside step function, f is a given function and μ is a positive real parameter, the nonlinear term is discontinuous at $u = \mu$. Employing non-smooth variational techniques, Bensid [24, Theorem 3.1] established the existence of at least two distinct non-zero solutions to problem (1.5). In [25, Theorem 3.1], Achour and Bensid delved into the existence and multiplicity of solutions for a fractional p-Laplace problem featuring discontinuous nonlinearities. For additional findings on fractional Laplace problems with discontinuous nonlinearities, the reader is directed to [26–28] and the references therein.

To rigorously establish the existence of solutions to problem (1.1), we commence by introducing fundamental definitions and notations. First, we elaborate on the functional framework associated with the operator $\mathcal{L}$. For an arbitrary parameter $s \in (0,1)$, let us define

$$[u]_s^2 = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

Given that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a amooth boundary, the following function space

is considered:

$$\mathcal{X}^1 = \{u : \mathbb{R}^N \to \mathbb{R} \text{ is Lebesgue measurable} : u|_\Omega \in \mathcal{H}^1(\Omega) \text{ and } [u]_s < \infty\}.$$

For $u \in \mathcal{X}^1$, the norm is defined as follows

$$\|u\|^2_{\mathcal{X}^1} := \|u\|^2_{\mathcal{H}^1(\Omega)} + [u]^2_s.$$

Define working space for the weak solution:

$$\mathcal{X}^1_0 = \{u \in \mathcal{X}^1 : u(x) = 0, x \in \mathbb{R}^N \setminus \Omega\}. \tag{1.6}$$

Any $u \in \mathcal{X}^1_0$ satisfies $u|_\Omega \in \mathcal{H}^1_0(\Omega)$ because Ω has a smooth boundary. Therefore, $\mathcal{X}^1_0$ is a reflexive Banach space equipped with the norm

$$\|u\|^2_{\mathcal{X}^1_0} := \|\nabla u\|^2_{L^2(\Omega)} + [u]^2_s.$$

Also, define space

$$L^p_\mu(\Omega) = \Big\{u \in L^2(\Omega) : \int_\Omega G_\mu(u)\mathrm{d}x < \infty\Big\}$$

equipped with the norm

$$\|u\|_{L^p_\mu(\Omega)} = \Big(\int_\Omega G_\mu(u)\mathrm{d}x\Big)^{\frac{1}{p}}, \tag{1.7}$$

where $G_\mu(t) = \int_0^t g_\mu(s)ds$ and $g_\mu(t) = \mathcal{H}(t-\mu)|t|^{p-2}t + \lambda|t|^{q-2}t$.

Here and throughout the paper, we shall consider the space

$$\mathcal{V} = \mathcal{X}^1_0 \cap L^p_\mu(\Omega).$$

Clearly, $\mathcal{V}$ is a Banach space with the norm

$$\|u\|_{\mathcal{V}} = \|u\|_{\mathcal{X}^1_0} + \|u\|_{L^p_\mu(\Omega)}. \tag{1.8}$$

To simplify notation, denote the right side of problem (1.1) as

$$g_\mu(u) := \mathcal{H}(u-\mu)|u|^{p-2}u + \lambda|u|^{q-2}u.$$

Moreover, we also denote

$$\underline{g}_\mu(u) = \liminf_{v\to u} g(v), \quad \overline{g}_\mu(u) = \limsup_{v\to u} g(v).$$

The following definition is about the M-solution from [29], which is a basic concept of our article.

Definition 1.1 *A function $u \in \mathcal{X}^1_0$ is said to be an M-solution of problem (1.1) if it satisfies the following conditions:*

$$\mathcal{L}(u) = u^*,$$

where $u^*(x) \in [\underline{g}_\mu(u(x)), \overline{g}_\mu(u(x))]$ for a.e., $x \in \Omega$ and

$$[\underline{g}_\mu(t), \overline{g}_\mu(t)] = \begin{cases} \lambda|t|^{q-2}t, & \text{if } t < \mu, \\ [\lambda|t|^{q-2}t, |\mu|^{p-1} + \lambda|t|^{q-2}t], & \text{if } t = \mu, \\ |t|^{p-1} + \lambda|t|^{q-2}t, & \text{if } t > \mu. \end{cases} \tag{1.9}$$

Motivated by the above works, we give the main theorem of this paper.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$ $(N > 2)$ be a bounded domain with a smooth boundary. Assume that $s \in (0,1)$,*

$$1 < q < 2 < p < 2^* = \frac{2N}{N-2}.$$

Then there exists $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*)$, problem (1.1) has an M-solution $u \in \mathcal{V} \cap L^\infty(\Omega)$.

Remark 1.3 It is straightforward that p can take values within the range

$$p > 2_s^* = \frac{2N}{N-2s},$$

this indicates that for the exponent p in the mixed local and nonlocal elliptic problem (1.1), its value range is worse than the exponent 2^* derived solely from the standard Laplace operator $-\Delta$, but can attain values superior to the exponent 2_s^* derived solely from the fractional Laplace operator $(-\Delta)^s$.

Remark 1.4 In [24], Bensid studied the existence and multiplicity of solutions for the following nonlocal elliptic problem with discontinuous nonlinearities

$$\begin{cases} (-\Delta)^s u = f(u)\mathcal{H}(u-\mu), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.10}$$

Regarding the Euler-Lagrange functional $J(u)$ corresponding to problem (1.10) as a locally Lipschitz functional, Bensid employed a non-smooth variant of the mountain pass theorem for locally Lipschitz functionals to identify critical points of $J(u)$. By integrating an application of the Brezis-Lieb Lemma with a control over the minimax level associated with $J(u)$, he established the existence of solutions to problem (1.10). This approach has also been adopted in the investigation of discontinuous problems documented in [25, 27]. In the present paper, a novel non-smooth variational principle introduced in [10, 11] is applied for the first time to the study of problem (1.1).

The main structure of this paper is as follows. In Section 2 we introduce some general facts from convex analysis as well as the corresponding function setting of our problem. In Section 3, we prove Theorem 1.2 by variational principle.

2. Preliminaries

In order to utilize the variational method and prove the existence of solutions to problem (1.1), we need some standard results about non-smooth analysis [10–13, 30].

In this section, $\mathcal{V}$ denotes a locally convex space, $\mathcal{V}^*$ denotes the dual space of $\mathcal{V}$ and $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$. We introduce the following well known definition.

A function $\Psi : \mathcal{V} \to \mathbb{R}$ is known as weakly lower semi-continuous if

$$\Psi(u) \leq \liminf_{n \to \infty} \Psi(u_n),$$

for each $u \in \mathcal{V}$ and any sequence u_n approaching $u \in \mathcal{V}$ in the weak topology on $\mathcal{V}$.

Let $\Psi : \mathcal{V} \to \mathbb{R} \cup \{\infty\}$ be a proper and convex function. Ψ is said to be proper if

$$\mathrm{Dom}(\Psi) := \{u \in \mathcal{V} : \Psi(u) < \infty\} \neq \emptyset.$$

Definition 2.1 *The subdifferential $\partial\Psi$ of Ψ is defined to be the following set-valued operator*

$$\partial\Psi(u) = \{u^* \in \mathcal{V}^* : \langle u^*, v - u\rangle + \Psi(u) \leq \Psi(v), \quad \forall\, v \in \mathcal{V}\},$$

if $u \in Dom(\Psi)$ and $\partial\Psi(u) = \emptyset$ if $u \notin Dom(\Psi)$.

Assume that Ψ is Gâteaux differentiable at u, $D\Psi(u)$ is the Gâteaux derivative of Ψ at u. In this case $\partial\Psi(u) = \{D\Psi(u)\}$. Denote Ψ^* as the Fenchel dual of function Ψ, which is a function on $\mathcal{V}^*$ defined by

$$\Psi^*(u^*) = \sup\{\langle u^*, u\rangle - \Psi(u) : u \in \mathcal{V}\}.$$

Definition 2.2 *Assume that $\mathcal{W}$ is an open subset of $\mathcal{V}$ and $\Phi : \mathcal{W} \to \mathbb{R}$ is a local Lipschitz function.*

(1) The generalized directional derivative of the function Φ at $u \in \mathcal{W}$ in the direction $v \in \mathcal{V}$ is defined by

$$\Phi^0(u; v) = \limsup_{h \to u, t \to 0} \frac{\Phi(h + tv) - \Phi(h)}{t}.$$

(2) The generalized derivative (also known as the Clarke subdifferential) $\partial_c\Phi(u)$ of Φ at a point $u \in \mathcal{W}$ is the subset of the dual space defined as

$$\partial_c\Phi(u) = \{u^* \in \mathcal{V}^* : \Phi^0(u; v) \geq \langle u^*, v\rangle,\ \forall\, v \in \mathcal{V}\}. \tag{2.1}$$

We now recall the following definition of a critical point of $I = \Psi - \Phi$.

Definition 2.3 *Suppose $\mathcal{V}$ is a real Banach space. Let $I = \Psi - \Phi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$, where $\Phi : \mathcal{V} \to \mathbb{R}$ is locally Lipschitz and $\Psi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. A point $u \in \mathcal{V}$ is a critical point of $I = \Psi - \Phi$ if $u \in Dom(\Psi)$ and for any $v \in \mathcal{V}$,*

$$\Psi(v) - \Psi(u) \geq \Phi^0(u; v - u).$$

Definition 2.4 *An operator $M : \mathcal{V} \to 2^{\mathcal{V}^*}$ is said to be paramonotone if M is monotone and for $u^* \in M(u)$, $v^* \in M(v)$, the identity*

$$\langle v^* - u^*, v - u\rangle = 0,$$

implies that $u^*, v^* \in M(u) \cap M(v)$.

Definition 2.5 *Let $\Psi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semi continuous function, $\mathcal{K}$ be a closed convex subset of $\mathcal{V}$. The restriction of Ψ to $\mathcal{K}$ is denoted by $\Psi_{\mathcal{K}}$ and defined by*

$$\Psi_{\mathcal{K}}(u) = \begin{cases} \Psi(u), & u \in \mathcal{K}, \\ +\infty, & \text{otherwise.} \end{cases}$$

We shall now recall the following variational principle established in [11, Theorem 3.7].

Lemma 2.6 *Let $\Psi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function,*

$\mathcal{K}$ be a closed convex subset of V. Assume that $\Phi : \mathcal{V} \to \mathbb{R}$ is locally Lipschitz. Suppose that the following two assertions hold:

(1) The functional $I = \Psi_{\mathcal{K}} - \Phi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ has a critical point at $u_0 \in \mathcal{K}$ as in Definition 2.3;

(2) If u_0 belongs to the boundary of $\mathcal{K}$, then at least one of the following conditions holds:

(2-1) There exist $u_0 \in \mathcal{K}$ and a paramonotone operator $M : \mathcal{V} \to 2^{\mathcal{V}^*}$ with $M(u_0)$ being a singleton such that

$$\{\partial_c \Phi(u_0) + M(u_0)\} \cap \{\partial \Psi(v_0) + M(v_0)\} \neq \emptyset.$$

(2-2) There exist $u_0 \in \mathcal{K}$ and a paramonotone operator $M : \mathcal{V} \to 2^{\mathcal{V}^*}$ with $M(0) = 0$ such that

$$\{\partial_c \Phi(u_0)\} \cap \{\partial \Psi(v_0) + M(v_0 - u_0)\} \neq \emptyset.$$

Then

$$u_0 \in \partial_c \Phi(u_0) \cap \partial \Psi \neq \emptyset.$$

The Palais-Smale compactness condition for functions of the type $I = \Psi - \Phi$ reads as follows in [31, Definition 3.2].

Lemma 2.7 *We say that the functional $I = \Psi - \Phi$ satisfies the Palais-Smale (PS) compactness condition at level $c \in \mathbb{R}$ if every sequence $\{u_n\} \subset \mathcal{V}$ such that*

(1) $I(u_n) \to c \in \mathbb{R}$,

(2) $\Psi(v) - \Psi(u_n) \geq \Phi^0(u_n, v - u_n) - \epsilon_n \|v - u_n\|_{\mathcal{V}}$, $\forall\, v \in \mathcal{V}$,

where $\epsilon_n \to 0$, possesses a convergent subsequence.

The following lemma about the critical points of functions of the form $I = \Psi - \Phi$ was proved in [31, Corollary 3.2].

Lemma 2.8 (Mountain Pass Theorem) *Suppose that $I = \Psi - \Phi : \mathcal{V} \to (-\infty, +\infty]$, satisfies PS and the Mountain Pass Geometry (MPG):*

(1) $I(0) = 0$;

(2) There exists $e \in \mathcal{V}$ such that $I(e) \leq 0$;

(3) There exists some $\xi > 0$ such that $0 < \xi < \|e\|$ and for every $u \in \mathcal{V}$ with $\|u\| = \xi$ one has $I(u) > 0$.

Then I has a critical value $c > 0$ which is characterized by

$$c = \inf_{\omega \in \Gamma} \sup_{t \in [0,1]} I[\omega(t)],$$

where

$$\Gamma = \{\omega \in C([0,1], \mathcal{V}) : \omega(0) = 0, \omega(1) = e\}.$$

3. The main result

In this section, we prove the existence of solutions to problem (1.1) by Lemma 2.6.

For every $u \in \mathcal{V}$, define

$$\mathcal{K} := \{u \in \mathcal{V} : u \geq 0, \|u\|_{L^\infty(\Omega)} \leq r\}.$$

Obviously, $\mathcal{K}$ is a convex weakly closed subset of $\mathcal{V}$. Note that the duality pairing between $\mathcal{V}$ and its dual $\mathcal{V}^*$ is defined by

$$\langle u, u^* \rangle = \int_\Omega u(x)u^*(x)\mathrm{d}x, \quad \text{for all} \quad u \in \mathcal{V}, \ u \in \mathcal{V}^*.$$

To apply the non-smooth variational principle (Lemma 2.6), we first define the Euler-Lagrange functional associated with problem (1.1) as

$$I_{\mathcal{K}}(u) = \Psi_{\mathcal{K}}(u) - \Phi(u). \tag{3.1}$$

Define $\Psi(u) : \mathcal{V} \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{2}(\|\nabla u\|^2_{L^2(\Omega)} + [u]^2_s), \quad u \in \mathcal{V}.$$

The restriction of Ψ to $\mathcal{K}$ is defined by

$$\Psi_{\mathcal{K}}(u) = \begin{cases} \frac{1}{2}(\|\nabla u\|^2_{L^2(\Omega)} + [u]^2_s), & u \in \mathcal{K}, \\ \infty, & \text{otherwise}, \end{cases} \tag{3.2}$$

and define $\Phi(u) : \mathcal{V} \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega G_\mu(u)\mathrm{d}x, \quad u \in \mathcal{V}. \tag{3.3}$$

Definition 3.1 *We say that a function $u \in \mathcal{X}^1_0 \cap L^p_\mu(\Omega)$ is a weak solution of problem (1.1) if there exists $\eta^p_\mu \in [\underline{g}_\mu(u), \overline{g}_\mu(u)]$ (see (1.9) for more details) and every test function $\phi \in C^\infty_0(\Omega)$ satisfies*

$$\int_\Omega \nabla u \cdot \nabla \phi \mathrm{d}x + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u(x) - u(y)][\phi(x) - \phi(y)]}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_\Omega \eta^p_\mu \phi \mathrm{d}x.$$

Let $\Psi_{\mathcal{K}}$ be defined as (3.2). Then $\Psi_{\mathcal{K}} \in C^1(\mathcal{K}, \mathbb{R})$ and

$$\langle D\Psi_{\mathcal{K}}(u), v\rangle = \int_\Omega \nabla u(x) \cdot \nabla v(x)\mathrm{d}x + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

A detailed version of Theorem 1.2 is given as follows.

Theorem 3.2 *Suppose that the functional $I_{\mathcal{K}}(u)$ defined as (3.1) has a critical point $u_0 \in \mathcal{V}$ as in Definition 2.3 and there exists $v_0 \in \mathcal{K}$, $u^*_0 \in [\underline{g}_\mu(u_0), \overline{g}_\mu(u_0)]$ such that*

$$\mathcal{L}(v_0) = u^*_0,$$

in the weak sense, i.e.,

$$\int_\Omega \nabla v_0 \cdot \nabla \phi \mathrm{d}x + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[v_0(x) - v_0(y)][\phi(x) - \phi(y)]}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_\Omega u^*_0 \phi \mathrm{d}x, \quad \forall\, \phi \in \mathcal{V}.$$

Then u_0 is an M-solution of problem (1.1), $I_{\mathcal{K}}(u_0) = c$, and the critical value c is characterized by

$$c = \inf_{\omega \in \Gamma} \sup_{t \in [0,1]} I[\omega(t)],$$

where

$$\Gamma = \{\omega \in C([0,1], \mathcal{V}) : \omega(0) = 0 \neq \omega(1)\}.$$

The next lemma is important in our arguments.

Lemma 3.3 *$\Phi : \mathcal{V} \to \mathbb{R}$ defined as (3.3) is locally Lipschitz functional and satisfies $\partial_c \Phi(u) \subset [\underline{g}_\mu(u), \overline{g}_\mu(u)]$ a.e. in Ω.*

We investigate some key facts of $\mathcal{X}_0^1$ defined as (1.6) in the following embedding proposition.

Proposition 3.4 *The embedding $\mathcal{X}_0^1 \hookrightarrow L^{2^*}(\Omega)$ is continuous, where $2^* = \frac{2N}{N-2}$.*

Proof By the Sobolev embedding Theorem, we know that

$$\|u\|^2_{L^{2^*}(\Omega)} \leq \mathcal{S}\|\nabla u\|^2_{L^2(\Omega)},$$

where

$$\mathcal{S}^{-1} = \inf_{v \in \mathcal{H}_0^1(\Omega)\setminus\{0\}} \frac{\|v\|^2_{\mathcal{H}_0^1(\Omega)}}{\|v\|^2_{L^{2^*}(\Omega)}}.$$

It follows that

$$\|u\|^2_{L^{2^*}(\Omega)} \leq C\mathcal{S}\|\nabla u\|^2_{L^2(\Omega)} \leq C\mathcal{S}\|u\|^2_{\mathcal{X}_0^1}. \tag{3.4}$$

Thus the embedding $\mathcal{X}_0^1 \hookrightarrow L^{2^*}(\Omega)$ is continuous. □

Lemma 3.5 *Let v_1, v_2 be nonnegative function in $\mathbb{R}^N$, $\epsilon \in \mathbb{R}^+$. Then*

$$(v_1 - v_2)(v_1^\epsilon - v_2^\epsilon) \geq \frac{4\epsilon}{(\epsilon+1)^2}(v_1^{\frac{\epsilon+1}{2}} - v_2^{\frac{\epsilon+1}{2}})^2.$$

Lemma 3.6 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary, $2sm > N$, $f \in L^m(\Omega)$ be a locally measurable function and u a weak solution of the problem*

$$\begin{cases} -\Delta u + (-\Delta)^s u = f(u), & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.5}$$

Then there exists a constant $C > 0$, depending on N, Ω, s and m such that

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^m(\Omega)}.$$

Proof Let u be a weak solution of the problem (3.5), in the weak sense, i.e.,

$$\int_\Omega \nabla u \cdot \nabla \varphi \mathrm{d}x + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u(x) - u(y)][\varphi(x) - \varphi(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_\Omega \xi\varphi \mathrm{d}x, \quad \forall\, \varphi \in C_0^\infty(\Omega),$$

where

$$\xi(x) \in [\underline{f}(u(x)), \overline{f}(u(x))] \quad \text{a.e. } x \in \mathbb{R}^N$$

and

$$\underline{f}(t) = \liminf_{s \to t} f(s), \quad \overline{f}(t) = \limsup_{s \to t} f(s).$$

For each $h \in \mathbb{N}$ and $\beta > 1$, consider

$$A_h = \{x \in \mathbb{R}^N : 0 \le u^{\beta-1} \le h\}, \quad B_h = \{x \in \mathbb{R}^N : u^{\beta-1} > h\}$$

and

$$u_h(x) = \begin{cases} u \cdot u^{2(\beta-1)}, & x \in A_h, \\ h^2 u, & x \in B_h. \end{cases}$$

Notice that

$$\nabla u_h(x) = \begin{cases} (2\beta-1)|u|^{2(\beta-1)}\nabla u, & x \in A_h, \\ h^2 \nabla u, & x \in B_h. \end{cases}$$

Taking u_h as test function of problem (3.5), we have

$$\int_\Omega \nabla u \cdot \nabla u_h \mathrm{d}x + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u(x)-u(y)][u_h(x)-u_h(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_\Omega \xi u_h \mathrm{d}x. \tag{3.6}$$

It is easy to see that

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla u_h \mathrm{d}x =& \int_{\Omega \cap A_h} \nabla u \cdot \nabla u_h \mathrm{d}x + \int_{\Omega \cap B_h} \nabla u \cdot \nabla u_h \mathrm{d}x \\ =& (2\beta-1) \int_{\Omega \cap A_h} |u|^{2(\beta-1)} |\nabla u|^2 \mathrm{d}x + h^2 \int_{\Omega \cap B_h} |\nabla u|^2 \mathrm{d}x \ge 0. \end{aligned} \tag{3.7}$$

Upon coherent combination of (3.6) with (3.7), we conclusively derive that

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u(x)-u(y)][u_h(x)-u_h(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \le \int_\Omega \xi u_h \mathrm{d}x. \tag{3.8}$$

Define

$$\begin{aligned} \Omega_1 &= \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : 0 \le u(x)^{\beta-1} \le h, 0 \le u(y)^{\beta-1} \le h\}, \\ \Omega_2 &= \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : u(x)^{\beta-1} > h, 0 \le u(y)^{\beta-1} \le h\}, \\ \Omega_3 &= \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : 0 \le u(x)^{\beta-1} \le h, u(y)^{\beta-1} > h\}, \\ \Omega_4 &= \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : u(x)^{\beta-1} > h, u(y)^{\beta-1} > h\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u(x)-u(y)][u_h(x)-u_h(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &= \Big(\iint_{\Omega_1} + \iint_{\Omega_2} + \iint_{\Omega_3} + \iint_{\Omega_4} \Big) \frac{[u(x)-u(y)][u_h(x)-u_h(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &= \iint_{\Omega_1} \frac{[u(x)-u(y)][u^{2\beta-1}(x)-u^{2\beta-1}(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \\ &\quad \iint_{\Omega_2} \frac{[u(x)-u(y)][h^2 u(x)-u^{2\beta-1}(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \\ &\quad \iint_{\Omega_3} \frac{[u(x)-u(y)][u^{2\beta-1}(x)-h^2 u(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + h^2 \iint_{\Omega_4} \frac{[u(x)-u(y)]^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.9}$$

By Lemma 3.5, we obtain

$$\iint_{\Omega_1} \frac{|u^\beta(x)-u^\beta(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \le \frac{\beta^2}{2\beta-1} \iint_{\Omega_1} \frac{[u(x)-u(y)][u^{2\beta-1}(x)-u^{2\beta-1}(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y$$

$$\leq \beta^2 \iint_{\Omega_1} \frac{[u(x)-u(y)][u^{2\beta-1}(x)-u^{2\beta-1}(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y. \tag{3.10}$$

Define

$$w_h = \begin{cases} uu^{\beta-1}, & x \in A_h, \\ hu, & x \in B_h. \end{cases}$$

Then

$$\begin{aligned} &\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|w_h(x)-w_h(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &= \iint_{\Omega_1} \frac{[u^\beta(x)-u^\beta(y)]^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \iint_{\Omega_2} \frac{[hu(x)-u^\beta(y)]^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \\ &\quad \iint_{\Omega_3} \frac{[u^\beta(x)-hu(y)]^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + h^2 \iint_{\Omega_4} \frac{[u(x)-u(y)]^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.11}$$

Drawing from (3.9)–(3.11), we deduce that

$$\begin{aligned} &\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|w_h(x)-w_h(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{[u(x)-u(y)][u_h(x)-u_h(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &\leq (\beta^2-1)\iint_{\Omega_1} \frac{[u(x)-u(y)][u^{2\beta-1}(x)-u^{2\beta-1}(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \\ &\quad \iint_{\Omega_2} \frac{u(x)u(y)[h-u^{\beta-1}(y)]^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \iint_{\Omega_3} \frac{u(x)u(y)[h-u^{\beta-1}(x)]^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.12}$$

Note that $[u(x)-u(y)][u_h(x)-u_h(y)] = [u(x)-u(y)][h^2u(x)-u^{2\beta-1}(y)] > 0$, a.e. $(x,y)\in\Omega_2$. It can be readily verified that

$$u(x)u(y)[h-u^{\beta-1}(y)]^2 < [u(x)-u(y)][h^2u(x)-u^{2\beta-1}(y)]. \tag{3.13}$$

Similarly, for a.e. $(x,y)\in\Omega_3$, we can still find that

$$u(x)u(y)[h-u^{\beta-1}(x)]^2 < [u(x)-u(y)][u_h(x)-u_h(y)] = [u(x)-u(y)][u^{2\beta-1}(x)-h^2u(y)]. \tag{3.14}$$

By combining (3.13) and (3.14), we can obtain

$$\begin{aligned} &\iint_{\Omega_2} \frac{u(x)u(y)[h-u^{\beta-1}(y)]^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \iint_{\Omega_3} \frac{u(x)u(y)[h-u^{\beta-1}(x)]^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &< \iint_{\Omega_2} \frac{[u(x)-u(y)][h^2u(x)-u^{2\beta-1}(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \\ &\quad \iint_{\Omega_3} \frac{[u(x)-u(y)][u^{2\beta-1}(x)-h^2u(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.15}$$

Recall (3.10), (3.12) and (3.15), it admits of easy verification that

$$\begin{aligned} &\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|w_h(x)-w_h(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &\leq \beta^2 \iint_{\Omega_1} \frac{[u(x)-u(y)][u^{2\beta-1}(x)-u^{2\beta-1}(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \\ &\quad 2\iint_{\Omega_2} \frac{[u(x)-u(y)][h^2u(x)-u^{2\beta-1}(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \end{aligned}$$

$$2\iint_{\Omega_3}\frac{[u(x)-u(y)][u^{2\beta-1}(x)-h^2u(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\iint_{\Omega_4}\frac{h^2[u(x)-u(y)]^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$\leq 2\beta^2\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{[u(x)-u(y)][u_h(x)-u_h(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

Since (3.8) holds, it follows that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|w_h(x)-w_h(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\leq 2\beta^2\int_\Omega \xi u_h\mathrm{d}x\leq 2\beta^2\int_\Omega |f|u_h\mathrm{d}x.$$

By applying the Sobolev embedding theorem and the Hölder inequality, we derive the following result

$$\|w_h\|^2_{L^{2_s^*}(\Omega)}\leq 2\beta^2\mathcal{S}_1\int_\Omega |f|u_h\mathrm{d}x\leq 2\beta^2\mathcal{S}_1\|f\|_{L^m(\Omega)}\|u_h\|_{L^{m'}(\Omega)},$$

where $\frac{1}{m}+\frac{1}{m'}=1$ and $\mathcal{S}_1$ satisfies

$$\mathcal{S}_1^{-1}=\inf_{v\in\mathcal{X}_0^1\setminus\{0\}}\frac{\|v\|^2_{\mathcal{X}_0^1}}{\|v\|^2_{L^{2_s^*}(\Omega)}}.$$

For sufficiently large $h>0$, it is clear that $u_h=u^{2\beta-1}$, $w_h=u^\beta$. Moreover, by invoking the embedding relations of L^p spaces and the monotonicity of norms in L^p spaces, we deduce that

$$\|u\|^{2\beta}_{L^{2_s^*\beta}(\Omega)}\leq 2\beta^2\mathcal{S}_1\|f\|_{L^m(\Omega)}\|u\|^{2\beta-1}_{L^{m'(2\beta-1)}(\Omega)}\leq \beta^2C_0\mathcal{S}_1\|f\|_{L^m(\Omega)}\|u\|^{2\beta}_{L^{2m'\beta}(\Omega)}.$$

Thus,

$$\|u\|_{L^{2_s^*\beta}(\Omega)}\leq \beta^{\frac{1}{\beta}}\left(C_0\mathcal{S}_1\right)^{\frac{1}{2\beta}}\|f\|^{\frac{1}{2\beta}}_{L^m(\Omega)}\|u\|_{L^{2m'\beta}(\Omega)}. \tag{3.16}$$

Since $\frac{N}{N-2s}>m'$, set $\sigma=\frac{N}{(N-2s)m'}$. When $\beta=\sigma$ in (3.16), we have $2m'\beta=2_s^*$ and

$$\|u\|_{L^{2_s^*\sigma}(\Omega)}\leq \sigma^{\frac{1}{\sigma}}\left(C_0\mathcal{S}_1\right)^{\frac{1}{2\sigma}}\|f\|^{\frac{1}{2\sigma}}_{L^m(\Omega)}\|u\|_{L^{2_s^*}(\Omega)}. \tag{3.17}$$

When $\beta=2\sigma$ in (3.16), we have $2m'\beta=2_s^*\sigma$ and

$$\|u\|_{L^{2_s^*\sigma^2}(\Omega)}\leq \sigma^{\frac{2}{\sigma^2}}\left(C_0\mathcal{S}_1\right)^{\frac{1}{2\sigma^2}}\|f\|^{\frac{1}{2\sigma^2}}_{L^m(\Omega)}\|u\|_{L^{2_s^*\sigma}(\Omega)}. \tag{3.18}$$

By combining (3.17) and (3.18), we derive that

$$\|u\|_{L^{2_s^*\sigma^2}(\Omega)}\leq \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}}\left(C_0\mathcal{S}_1\right)^{\frac{1}{2\sigma}+\frac{1}{2\sigma^2}}\|f\|^{\frac{1}{2\sigma}+\frac{1}{2\sigma^2}}_{L^m(\Omega)}\|u\|_{L^{2_s^*}(\Omega)}.$$

Arguing by iteration, β in (3.16) as σ^j, we can show that

$$\|u\|_{L^{2_s^*\sigma^j}(\Omega)}\leq \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}+\cdots+\frac{j}{\sigma^j}}\left(C_0\mathcal{S}_1\right)^{\frac{1}{2\sigma}+\frac{1}{2\sigma^2}+\cdots+\frac{1}{2\sigma^j}}\|f\|^{\frac{1}{2\sigma}+\frac{1}{2\sigma^2}+\cdots+\frac{1}{2\sigma^j}}_{L^m(\Omega)}\|u\|_{L^{2_s^*}(\Omega)}.$$

Once that

$$\sum_{j=1}^{\infty}\frac{j}{\sigma^j}=\frac{1}{(\sigma-1)}\quad\sum_{j=1}^{\infty}\frac{1}{2\sigma^j}=\frac{1}{2(\sigma-1)}$$

and

$$\sigma=\frac{N(m-1)}{(N-2s)m},$$

as $j\to\infty$, we have

$$\|u\|_{L^\infty(\Omega)}\leq C\|f\|_{L^m(\Omega)}.\quad\square$$

The following lemma gives some important properties of $\Psi_{\mathcal{K}}$, which ensures that we can apply Lemma 2.6.

Lemma 3.7 *The functional $\Psi_{\mathcal{K}}$ defined in (3.2) is convex and weakly lower semi-continuous.*

Proof According to the proof of [32, Lemma 2.6], we have known that $\Psi_{\mathcal{K}}$ is convex and weakly lower semi-continuous. □

Lemma 3.8 *The functional $I_{\mathcal{K}}(u)$ defined in (3.1) satisfies PS and MPG.*

Proof We first sketch that $I_{\mathcal{K}}(u)$ satisfies the PS (Lemma 2.7). For all $u \in \mathcal{V}$ and $c \in \mathbb{R}$, let $\{u_n\} \in \mathcal{K}$ be a sequence such that

$$I_{\mathcal{K}}(u_n) \to c,$$

which means that for large enough n, we have

$$\frac{1}{2}\|u_n\|^2_{\mathcal{X}_0^1} - \int_\Omega G_\mu(u_n)\mathrm{d}x \le c. \tag{3.19}$$

According to Lemma 2.7, it is also clear that if $\epsilon_n \to 0$ as $n \to \infty$, we obtain

$$\Psi_{\mathcal{K}}(v) - \Psi_{\mathcal{K}}(u_n) \ge \Phi^0(u_n; v - u_n) - \epsilon_n\|v - u_n\|_{\mathcal{V}}, \quad \forall\, v \in \mathcal{V},$$

together with the definition of the generalized derivative to local Lipschitz function Φ (see (2.1) and (3.3) for more details) and Lemma 3.3, we have

$$\Psi_{\mathcal{K}}(v) - \Psi_{\mathcal{K}}(u_n) \ge \langle u_n^*, v - u_n\rangle - \epsilon_n\|v - u_n\|_{\mathcal{V}}, \tag{3.20}$$

where $u_n^* \in \partial_c\Phi(u_n) \subset [\underline{g}_\mu(u_n), \overline{g}_\mu(u_n)]$ and

$$\langle u_n^*, v - u_n\rangle = \int_\Omega u_n^* v\mathrm{d}x - \int_\Omega G_\mu(u_n)\mathrm{d}x. \tag{3.21}$$

In particular, for $v = tu_n$ in (3.20), we have

$$\frac{1-t^2}{2}\|u_n\|^2_{\mathcal{X}_0^1} + (t-1)\int_\Omega G_\mu(u_n)\mathrm{d}x \le \epsilon_n(t-1)\|u_n\|_{\mathcal{V}}. \tag{3.22}$$

Adding (3.19) and (3.22), together with the definition of norm $\|\cdot\|_{\mathcal{V}}$, we obtain

$$\|u_n\|^2_{\mathcal{X}_0^1} \le C_1\|u_n\|_{\mathcal{V}}, \tag{3.23}$$

where C_1 is a positive constant. It is easy to check that

$$\begin{aligned}
G_\mu(t) =& \int_0^t \mathcal{H}(s-\mu)|s|^{p-2}s + \lambda|s|^{q-2}s\mathrm{d}s \\
=& \int_0^\mu \mathcal{H}(s-\mu)|s|^{p-2}s + \lambda|s|^{q-2}s\mathrm{d}s + \int_\mu^t \left[\mathcal{H}(s-\mu)|s|^{p-2}s + \lambda|s|^{q-2}s\right]\mathrm{d}s \\
=& \int_0^\mu \lambda|s|^{q-2}s\mathrm{d}s + \int_\mu^t \left[|s|^{p-2}s + \lambda|s|^{q-2}s\right]\mathrm{d}s \\
=& \frac{1}{p}\left[|t|^p - \mu^p\right] + \frac{\lambda}{q}|t|^q \le \frac{1}{p}|t|^p + \frac{\lambda}{q}|t|^q.
\end{aligned}$$

Thus,

$$\begin{aligned}\|u\|_{L^p_\mu(\Omega)} =&\Big(\int_\Omega G_\mu(u)\mathrm{d}x\Big)^{\frac{1}{p}} \le \Big(\frac{1}{p}\int_\Omega |u|^p\mathrm{d}x + \frac{\lambda}{q}\int_\Omega |u|^q\mathrm{d}x\Big)^{\frac{1}{p}}\\ =&\left(\frac{1}{p}\right)^{\frac{1}{p}}\|u\|_{L^p(\Omega)} + \left(\frac{\lambda}{q}\right)^{\frac{1}{p}}\|u\|^{\frac{q}{p}}_{L^q(\Omega)} \le C\|u\|_{L^p(\Omega)}.\end{aligned}$$

Note that $p < 2^*$, together with (3.4), we deduce that

$$\|u\|_{L^p_\mu(\Omega)} \le C\|u\|_{\mathcal{X}^1_0},$$

which means that

$$\|u\|_{\mathcal{X}^1_0} \le \|u\|_{\mathcal{V}} \le C\|u\|_{\mathcal{X}^1_0}. \tag{3.24}$$

Consider (3.23) and (3.24), we know that $\{u_n\}$ is bounded in $\mathcal{X}^1_0$. In the Sobolev space $\mathcal{X}^1_0$, after passing to a sequence if it is necessary, there exists $u_0 \in \mathcal{X}^1_0$ such that

$$\begin{aligned}&u_n \rightharpoonup u_0 \quad \text{weakly in} \quad \mathcal{X}^1_0,\\ &u_n \to u_0 \quad \text{a.e.} \quad x \in \Omega.\end{aligned}$$

According to the boundedness of $\{u_n\}$ in $\mathcal{X}^1_0$, together with (3.24), we have that $\{u_n\}$ is strongly convergent to u_0 in $L^p_\mu(\Omega)$. Let $v = u_0$ in (3.20), we have

$$\frac{1}{2}(\|u_0\|^2_{\mathcal{X}^1_0} - \|u_n\|^2_{\mathcal{X}^1_0}) - \int_\Omega u^*_n(u_0 - u_n)\mathrm{d}x \ge -\epsilon_n\|u_0 - u_n\|_{\mathcal{V}}. \tag{3.25}$$

As $n \to \infty$ in (3.25), we obtain

$$\frac{1}{2}\Big(\limsup_{n\to\infty}\|u_n\|^2_{\mathcal{X}^1_0} - \|u_0\|^2_{\mathcal{X}^1_0}\Big) \le 0.$$

Therefore, $u_n \to u_0$ strongly in $\mathcal{V}$, which completes the proof of PS.

The mountain pass geometry (MPG, Lemma 2.8) for the functional $I_{\mathcal{K}}(u)$ is now validated.

Observe that $I_{\mathcal{K}}(0) = 0$. Next we demonstrate that there exists $0 \neq e \in \mathcal{K}$ such that $I_{\mathcal{K}}(e) \le 0$. In fact, $\delta u \in \mathcal{K}$ for any $\delta \in (0, 1]$ and $u \in \mathcal{K} \setminus \{0\}$. So, we can deduce that

$$\begin{aligned}I_{\mathcal{K}}(\delta u) =&\frac{1}{2}\|\delta\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}[\delta u]^2_s - \int_\Omega G_\mu(\delta u)\mathrm{d}x\\ \le&\frac{\delta^2}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{\delta^2}{2}[u]^2_s - \frac{\lambda\delta^q}{q}\int_\Omega |u(x)|^q\mathrm{d}x\\ =&\delta^q\Big\{\frac{\delta^{2-q}}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{\delta^{2-q}}{2}[u]^2_s - \frac{\lambda}{q}\int_\Omega |u(x)|^q\mathrm{d}x\Big\}.\end{aligned}$$

This fact, combined with $\lambda > 0$ and $u \in \mathcal{K} \setminus \{0\}$, allows us to easily see that $I_{\mathcal{K}}(\delta u) < 0$ for sufficiently small δ. Next, consider $\bar{u} \in \mathcal{K}$ with $\|\bar{u}\|_{\mathcal{V}} = \rho > 0$ and

$$I_{\mathcal{K}}(\bar{u}) = \frac{1}{2}\|\bar{u}\|^2_{\mathcal{X}^1_0} - \int_\Omega G_\mu(\bar{u})\mathrm{d}x.$$

Observing the definition of norm in (1.7) and (1.8), we obtain

$$\int_\Omega G_\mu(\bar{u})\mathrm{d}x \le C_0\|\bar{u}\|^p_{\mathcal{V}}.$$

Thus, for $p > 2$ and a sufficiently small $\rho > 0$, it can be shown that

$$\begin{aligned} I_{\mathcal{K}}(\bar{u}) =& \frac{1}{2}\|\bar{u}\|^2_{\mathcal{X}_0^1} - \int_\Omega G_\mu(\bar{u})\mathrm{d}x \geq \frac{1}{2}\|\bar{u}\|^2_{\mathcal{X}_0^1} - C_0\|\bar{u}\|^p_{\mathcal{V}} \\ \geq& \frac{1}{2C}\|\bar{u}\|^2_{\mathcal{V}} - C_0\|\bar{u}\|^p_{\mathcal{V}} = \frac{1}{2C}\rho^2 - C_0\rho^p > 0, \end{aligned}$$

which means that MPG holds. Using Lemma 2.8, $I_{\mathcal{K}}(u)$ has a critical point $u_0 \in \mathcal{V}$ with $I_{\mathcal{K}}(u_0) = c > 0$, the critical value c being given by

$$c = \inf_{\omega\in\Gamma}\sup_{t\in[0,1]} I[\omega(t)],$$

where

$$\Gamma = \{\omega \in C([0,1], \mathcal{V}) : \omega(0) = 0 \neq \omega(1)\}. \ \square$$

Lemma 3.9 *For each $u \in \mathcal{K}$, there exists $v \in \mathcal{K}$ which solves*

$$\begin{cases} \mathcal{L}(v) = u^*, & x \in \Omega, \\ v \geq 0, & x \in \Omega, \\ v = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

in the weak sense, where $u^* \in [\underline{g}_\mu(u), \overline{g}_\mu(u)]$ (see (1.9) for more details about $[\underline{g}_\mu(u), \overline{g}_\mu(u)]$).

Proof By standard methods we know that there exists $v \in \mathcal{X}_0^1 \cap L^\infty(\Omega)$ such that

$$\begin{cases} \mathcal{L}(v) = u^*, & x \in \Omega, \\ v \geq 0, & x \in \Omega, \\ v = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

We shall prove $v \in \mathcal{K}$. Let

$$[\underline{f}(u), \overline{f}(u)] := [\underline{g}_\mu(u), \overline{g}_\mu(u)].$$

Clearly, for $f \in L^m(\Omega)$ with $m > \frac{N}{2s}$, by leveraging the membership $u \in \mathcal{K}$ and invoking Lemma 3.6, one can readily deduce that

$$\begin{aligned} \|v\|_{L^\infty(\Omega)} \leq& C\|\mathcal{H}(u-\mu)|u|^{p-2}u + \lambda|u|^{q-2}u\|_{L^m(\Omega)} \\ \leq& C\|\mathcal{H}(u-\mu)|u|^{p-2}u\|_{L^m(\Omega)} + C\|\lambda|u|^{q-2}u\|_{L^m(\Omega)} \\ \leq& C\|u\|^{p-1}_{L^{m(p-1)}(\Omega)} + C\lambda\|u\|^{q-1}_{L^{m(q-1)}(\Omega)} \\ \leq& C\|u\|^{p-1}_{L^\infty(\Omega)} + C\lambda\|u\|^{q-1}_{L^\infty(\Omega)} \\ \leq& Cr^{p-1} + C\lambda r^{q-1}, \end{aligned}$$

where C is a positive constant. We can choose $\lambda^* > 0$ small enough such that, for each $\lambda \in (0, \lambda^*)$,

$$Cr^{p-1} + C\lambda r^{q-1} \leq r,$$

which means that $v \in \mathcal{K}$. $\square$

Now we give the proof of the main theorem.

Proof of Theorem 3.2 By Lemmas 3.8 and 2.8, there exists a critical point $u_0 \in \mathcal{K}$ of $I_{\mathcal{K}}(u)$

with $u_0 \neq 0$. Recall Definition 2.3, we have

$$\Psi_{\mathcal{K}}(v) - \Psi_{\mathcal{K}}(u_0) \geq \Phi^0(u_0; v - u_0), \quad \forall\, v \in \mathcal{V},$$

together with the definition of the property of local Lipschitz function Φ (see (2.1) and (3.3) for more details) and Lemma 3.3, we obtain

$$\Psi_{\mathcal{K}}(v) - \Psi_{\mathcal{K}}(u_0) \geq \langle u_0^*, v - u_0 \rangle, \tag{3.26}$$

where $u_0^* \in \partial_c \Phi(u_0) \subset [\underline{g}_\mu(u_0), \overline{g}_\mu(u_0)]$. Also, by Lemma 3.9, there exists $(v_0 - u_0) \in \mathcal{K}$ such that

$$\begin{aligned} &\int_\Omega \nabla u_0 \cdot \nabla (v_0 - u_0) \mathrm{d}x + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u_0(x) - u_0(y)][(v_0 - u_0)(x) - (v_0 - u_0)(y)]}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &= \int_\Omega u_0^* (v_0 - u_0) \mathrm{d}x. \end{aligned} \tag{3.27}$$

By setting $v = v_0$ in (3.26), together with (3.21) and (3.27), we obtain

$$\begin{aligned} \frac{1}{2}\|v_0\|^2_{\mathcal{X}_0^1} - \frac{1}{2}\|u_0\|^2_{\mathcal{X}_0^1} &\geq \int_\Omega u_0^* v_0 \mathrm{d}x - \int_\Omega G_\mu(u_0) \mathrm{d}x \\ &= \int_\Omega \nabla v_0 \cdot \nabla (v_0 - u_0) \mathrm{d}x + \\ &\quad \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[v_0(x) - v_0(y)][(v_0 - u_0)(x) - (v_0 - u_0)(y)]}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Therefore,

$$\frac{1}{2}(\|\nabla u_0 - \nabla v_0\|^2_{L^2(\Omega)} + [v_0 - u_0]_s^2) \leq 0,$$

which implies that $u_0 = v_0$, and we have that u_0 is an M-solution of problem (1.1). □

Finally, the proof of Theorem 1.2 is a combination of Theorem 3.2 and Lemma 3.8.

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