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Existence, Uniqueness and Stability of Bounded Solutions for Minkowski-Curvature Problems with Asymptotic Boundary Conditions

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Abstract In this article, we show the existence, uniqueness and stability of bounded solutions to the following quasilinear problems with mean curvature operator

$$(\varphi(x'(t)))' = f(t,\ x),\ t \geq t_0, \quad \lim_{t\to\infty} x(t) = \psi_0, \quad \lim_{t\to\infty} x'(t)e^t = 0,$$

where t_0 and ψ_0 are real constants, $\varphi(s) = \frac{s}{\sqrt{1-s^2}}$, $s \in \mathbb{R}$ with $s \in (-1,\ 1)$, $f:\ [t_0,\ \infty)\times\mathbb{R}\to\mathbb{R}$ satisfies the Lipschitz or Osgood-type conditions.

Keywords mean curvature operator; uniqueness; asymptotic boundary conditions; bounded solution

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1. Introduction

In this paper, we are concerned with the existence, uniqueness and stability of solutions for mean curvature problems

$$(\varphi(x'))' = f(t,\ x), \quad t \geq t_0, \tag{1.1}$$

$$\lim_{t\to\infty} x(t) = \psi_0, \quad \lim_{t\to\infty} x'(t)e^t = 0, \tag{1.2}$$

where t_0 and ψ_0 are real constants, $\varphi(s) = \frac{s}{\sqrt{1-s^2}}$ for any $s \in \mathbb{R}$ with $|s| < 1$. If the maximum deviation of the corresponding solution of (1.1) and (1.2) can be controlled within a certain range (such as the same order as or smaller than the disturbance) when there is a small disturbance in the boundary condition $\lim_{t\to\infty} x(t) = \psi_0$, then the solution of (1.1) and (1.2) is said to be stable. Observe that the specific form of asymptotic conditions (1.2) is a recently derived model for arctic gyres with a vanishing azimuthal velocity, and it is semi-infinite because gyres never cross the equator, and also (1.2) expresses the fact that the center of the gyre is a stagnant point, see the detailed discussions in [1–3] and the references within. The Eq. (1.1) is related

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to the mean curvature equation in the Minkowski space. This kind of problems are originated from differential geometry or classical relativity and have been deeply studied, see [4–17] and the references within. Most of these results were obtained on bounded domain.

Let $f:\ \mathbb{R}\to[0,+\infty)$ satisfy local Lipschitz condition. Azzollini [4] applied the critical point theory and the shooting method to show the existence of a positive ground state solution for the Minkowski-curvature problem

$$-\big(t^{N-1}\varphi(x')\big)'=t^{N-1}f(x),\quad t\in(R,\ +\infty),\tag{1.3}$$

$$x(R)=0,\quad x\to 0 \text{ as } t\to+\infty.\tag{1.4}$$

Recently, Cao and Dai [15] used the bifurcation theory to investigate the nonexistence, existence and multiplicity of one-sign or sign-changing solutions to (1.3) and (1.4) if f is replaced with $\lambda a(t)f(x)$, and $a\geq 0$ with $a\not\equiv 0$ satisfies

$$\int_R^{+\infty}t^{N-1}a(t)\mathrm{d}t<+\infty.$$

In [11], the first author of the present paper established the existence of positive solutions to (1.3) and (1.4) for R sufficiently small. We refer to [18–20] for a discussion of the existence of solutions of mean curvature problem in Minkowski spacetime on unbounded domain.

On the other hand, one of our great motivations comes also from Chu and Liang [21] investigating the existence, uniqueness and stability of bounded solutions to a nonlinear second order differential equation

$$x''(t)=f(t,\ x),\quad t\geq t_0$$

with asymptotic conditions (1.2) when f satisfies the assumptions (H1) and (H3)–(H5) (see the below).

What is really interesting is to find the existence, uniqueness and stability of bounded solutions to (1.1) and (1.2). To wit, we shall assume the following hypotheses concerning the nonlinearity f.

(H1) $f:[t_0,\ \infty)\times\mathbb{R}\to\mathbb{R}$ is continuous.

(H2) $\lim_{t\to\infty}e^t f(t,\ x)=0$ uniformly on x.

(H3) There exists a continuous function $a:[t_0,\ \infty)\to[0,\ \infty)$ satisfying

$$\int_{t_0}^{\infty}sa(s)\mathrm{d}s<\infty,\tag{1.5}$$

such that

$$|f(t,\ x)-f(t,\ y)|\leq a(t)|x-y|,\quad t\geq t_0,\ x,\ y\in\mathbb{R}.\tag{1.6}$$

(H4) There exists a continuous function $a:[t_0,\ \infty)\to[0,\ \infty)$ satisfying (1.5) and a nondecreasing continuous function $g:[0,\ \infty)\to[0,\ \infty)$ such that

$$|f(t,\ x)|\leq a(t)g(|x|),\quad t\geq t_0,\ x\in\mathbb{R}.$$

(H5) Suppose there exists a continuous function $a : [t_0,\ \infty) \to [0,\ \infty)$ satisfying (1.5) such that

$$|f(t,\ x) - f(t,\ y)| \le a(t)G(|x - y|),$$

where G is continuous and nondecreasing, $G(0) = 0$, $G(r) > 0$ for any $r > 0$ and satisfies the Osgood condition

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{1}{G(r)} \mathrm{d}r = \infty. \tag{1.7}$$

Our main results are the following theorems.

Theorem 1.1 *Assume that (H1)–(H3) hold. Then there exists a unique stable bounded solution of* (1.1) *and* (1.2).

Theorem 1.2 *Assume that (H1), (H2) and (H4) hold. Then there exists $T_1 \ge t_0$ such that* (1.1) *and* (1.2) *has at least one bounded solution x on $[T_1,\ \infty)$.*

Theorem 1.3 *Assume that (H1), (H2), (H4) and (H5) hold. Then there exists $T_1 \ge t_0$ such that* (1.1) *and* (1.2) *has a unique solution x on $[T_1,\ \infty)$.*

The remainder of this paper is arranged as follows. In Section 2, we show a preliminary lemma. Section 3 is devoted to proving the existence, uniqueness and stability of bounded solutions of (1.1) and (1.2) under the Lipschitz condition. In Section 4, we give the proofs of Theorems 1.2 and 1.3.

2. Preliminaries

In this section, we will give the equivalent integral expression of (1.1) and (1.2).

Lemma 2.1 *Assume that (H1) and (H2) hold. Then $x \in C^1[t_0,\ \infty)$ is a solution of* (1.1) *and* (1.2) *if and only if $x \in C^1[t_0,\ \infty)$ is a solution of the integral equation*

$$x(t) = \psi_0 + \int_t^{\infty} \varphi^{-1}\Big(\int_s^{\infty} f(\tau,\ x)\mathrm{d}\tau \Big)\mathrm{d}s, \quad t \ge t_0. \tag{2.1}$$

Proof Let $x \in C^1[t_0,\ \infty)$ be a solution of (1.1) and (1.2). Since $\lim_{t\to\infty} x'(t)e^t = 0$ implies that $\lim_{t\to\infty} x'(t) = 0$. So, integrating the equation (1.1) from t to ∞, we show

$$\varphi(-x'(t)) = -\varphi(x'(t)) = \int_t^{\infty} f(s,\ x)\mathrm{d}s, \quad t \ge t_0,$$

and hence

$$-x'(t) = \varphi^{-1}\Big(\int_t^{\infty} f(s,\ x)\mathrm{d}s \Big), \quad t \ge t_0. \tag{2.2}$$

Combining this with $\lim_{t\to\infty} x(t) = \psi_0$, we get that x satisfies (2.1).

On the other hand, let $x \in C^1[t_0,\ \infty)$ be a solution of (2.1), by a series of calculations for (2.1), we obtain $(\varphi(x'(t)))' = f(t,\ x)$. By virtue of (H1), it follows from (H2) that there exists

$T > t_0$ such that, for any $t \geq T$,

$$|f(t,\ x)| < \epsilon e^{-t} \ \text{ uniformly on } x \in \mathbb{R},$$

and so

$$\int_{t_0}^{\infty} f(s,\ x)\mathrm{d}s < \int_{t_0}^{T} f(s,\ x)\mathrm{d}s + \int_{T}^{\infty} \epsilon e^{-s}\mathrm{d}s < \infty. \tag{2.3}$$

Thus, it is not difficult to prove $\lim_{t\to\infty} x(t) = \psi_0$. Moreover, it deduces from l'Hospital rule, (H2) and (2.2) and (2.3) that

$$\begin{aligned}\lim_{t\to\infty} x'(t)e^t &= -\lim_{t\to\infty} \varphi^{-1}\Big(\int_t^{\infty} f(s,\ x)\mathrm{d}s\Big)e^t = -\lim_{t\to\infty} \frac{\varphi^{-1}\big(\int_t^{\infty} f(s,\ x)\mathrm{d}s\big)}{e^{-t}} \\ &= -\lim_{t\to\infty} \frac{\int_t^{\infty} f(s,\ x)\mathrm{d}s}{e^{-t}} = -\lim_{t\to\infty} e^t f(t,\ x) = 0.\end{aligned}$$

Thus, x is also a solution of (1.1) and (1.2). □

3. The proof of Theorem 1.1

In this section, we will show that the existence, uniqueness and stability of the solution of (1.1) and (1.2) when f satisfies the Lipschitz-type condition.

Proof of Theorem 1.1 According to Lemma 2.1, we only need to establish the existence, uniqueness and stability of solutions of (2.1). Now it deduces from (1.5) that there exists large enough $T_0 \geq \max\{1,\ t_0\}$ such that

$$\int_{T_0}^{\infty} a(s)\mathrm{d}s \leq \int_{T_0}^{\infty} sa(s)\mathrm{d}s = a_0 < 1.$$

According to (1.6), we have

$$|f(t,\ x)| \leq a(t)|x| + |f(t,\ 0)|, \quad t \geq t_0,\ x \in \mathbb{R}. \tag{3.1}$$

By (H2), we have

$$\lim_{t\to\infty} e^t |f(t,\ 0)| = 0.$$

Assume that T_0 is chosen such that $|f(t,\ 0)| < e^{-t}$ for any $t \geq T_0$, and so

$$\int_t^{\infty} s|f(s,\ 0)|\mathrm{d}s < +\infty, \tag{3.2}$$

for $t \geq T_0$. Now let

$$X = \{x \in C^1[T_0,\ \infty) :\ x \text{ is bounded}\}.$$

Then X is a Banach space endowed with the norm

$$\|x\| = \max\Big\{\sup_{t\geq T_0} |x(t)|,\ \sup_{t\geq T_0} |x'(t)|\Big\}. \tag{3.3}$$

Let us define the operator $\mathcal{F}$ in X,

$$[\mathcal{F}(x)](t) = \psi_0 + \int_t^{\infty} \varphi^{-1}\Big(\int_s^{\infty} f(\tau,\ x)\mathrm{d}\tau\Big)\mathrm{d}s, \quad t \geq T_0. \tag{3.4}$$

According to (3.1), (3.2) and (3.4), for any $x \in X$, $t \geq T_0$, we show

$$\begin{aligned}
|[\mathcal{F}(x)](t)| &\leq |\psi_0| + \Big| \int_t^\infty \varphi^{-1}\Big(\int_s^\infty f(\tau,\ x)\mathrm{d}\tau\Big)\mathrm{d}s\Big| \leq |\psi_0| + \int_t^\infty \int_s^\infty |f(\tau,\ x)|\mathrm{d}\tau\mathrm{d}s \\
&= |\psi_0| + \int_t^\infty \int_t^\tau |f(\tau,\ x)|\mathrm{d}s\mathrm{d}\tau = |\psi_0| + \int_t^\infty (\tau - t)|f(\tau,\ x)|\mathrm{d}\tau \\
&= |\psi_0| + \int_t^\infty (s - t)|f(s,\ x)|\mathrm{d}s \leq |\psi_0| + \int_t^\infty s|f(s,\ x)|\mathrm{d}s \\
&\leq |\psi_0| + \int_t^\infty \big[sa(s)|x(s)| + s|f(s,\ 0)|\big])\mathrm{d}s < +\infty,
\end{aligned}$$

where $\varphi^{-1}(s) = \frac{s}{\sqrt{1+s^2}}$ for $s \in \mathbb{R}$ and $|\varphi^{-1}(s)| \leq |s|$. Then $\mathcal{F} : X \to X$. Moreover, for any $x, y \in X$, we have

$$\begin{aligned}
\sup_{t\geq T_0} |[\mathcal{F}(x)](t) - [\mathcal{F}(y)](t)| &= \sup_{t\geq T_0} \Big| \int_t^\infty \Big[\varphi^{-1}\Big(\int_s^\infty f(\tau,\ x)\mathrm{d}\tau\Big) - \varphi^{-1}\Big(\int_s^\infty f(\tau,\ y)\mathrm{d}\tau\Big)\Big]\mathrm{d}s\Big| \\
&= \sup_{t\geq T_0} \Big| \int_t^\infty \frac{1}{(1+\xi^2)^{\frac{3}{2}}} \int_s^\infty [f(\tau,\ x) - f(\tau,\ y)]\mathrm{d}\tau\mathrm{d}s\Big| \\
&\leq \sup_{t\geq T_0} \int_t^\infty \int_s^\infty |f(\tau,\ x) - f(\tau,\ y)|\mathrm{d}\tau\mathrm{d}s \\
&= \sup_{t\geq T_0} \int_t^\infty \int_t^\tau |f(\tau,\ x) - f(\tau,\ y)|\mathrm{d}s\mathrm{d}\tau \\
&= \sup_{t\geq T_0} \int_t^\infty (\tau - t)|f(\tau,\ x) - f(\tau,\ y)|\mathrm{d}\tau \\
&= \sup_{t\geq T_0} \int_t^\infty (s - t)|f(s,\ x) - f(s,\ y)|\mathrm{d}s \\
&\leq \sup_{t\geq T_0} \int_t^\infty s|f(s,\ x) - f(s,\ y)|\mathrm{d}s \\
&\leq \int_{T_0}^\infty sa(s)\mathrm{d}s\|x - y\|, \qquad (3.5)
\end{aligned}$$

where ξ intervens between $\int_s^\infty f(\tau,\ x)\mathrm{d}\tau$ and $\int_s^\infty f(\tau,\ y)\mathrm{d}\tau$. Similarly, for any $x, y \in X$, one proves

$$\begin{aligned}
\sup_{t\geq T_0} \big|[\mathcal{F}(x)]'(t) - [\mathcal{F}(y)]'(t)\big| &= \sup_{t\geq T_0} \Big|\varphi^{-1}\Big(\int_t^\infty f(s,\ x)\mathrm{d}s\Big) - \varphi^{-1}\Big(\int_t^\infty f(s,\ y)\mathrm{d}s\Big)\Big| \\
&\leq \sup_{t\geq T_0} \int_t^\infty \big|f(s,\ x) - f(s,\ y)\big|\mathrm{d}\tau \\
&\leq \int_{T_0}^\infty a(s)\mathrm{d}s\|x - y\|. \qquad (3.6)
\end{aligned}$$

So $\mathcal{F}$ is a contraction map on X according to (3.5) and (3.6), and so $\mathcal{F}$ has the unique fixed point $x \in X$, which is a unique bounded solution of (2.1) on $[T_0,\ \infty)$. If $t_0 = T_0$, it has been done. If $t_0 < T_0$, we proceed as in [21, Theorem 2.1] and ignore the details here.

Next we will show the stability of the solution. Let x and $\tilde{x}$ be the solution of (2.1) with

$$\lim_{t\to\infty} x(t) = \psi_0, \quad \lim_{t\to\infty} \tilde{x}(t) = \tilde{\psi}_0,$$

respectively. Together with (1.6), using the similar method to treat (3.5), for any $t \geq T_0$, we have

$$\begin{aligned}|x(t)-\tilde{x}(t)| \leq & |\psi_0-\tilde{\psi}_0|+\sup_{t\geq T_0}\Big|\int_t^\infty \varphi^{-1}\Big(\int_s^\infty f(\tau,\ x)\mathrm{d}\tau\Big)\mathrm{d}s-\int_t^\infty \varphi^{-1}\Big(\int_s^\infty f(\tau,\ \tilde{x})\mathrm{d}\tau\Big)\mathrm{d}s\Big|\\ \leq & |\psi_0-\tilde{\psi}_0|+\sup_{t\geq T_0}\int_t^\infty (s-t)|f(s,\ x)-f(s,\ \tilde{x})|\mathrm{d}s\\ \leq & |\psi_0-\tilde{\psi}_0|+\int_{T_0}^\infty sa(s)\mathrm{d}s\|x-\tilde{x}\|\\ = & |\psi_0-\tilde{\psi}_0|+a_0\|x-\tilde{x}\|. \end{aligned} \tag{3.7}$$

Accordingly,

$$\begin{aligned}|x'(t)-\tilde{x}'(t)| \leq & \sup_{t\geq T_0}\Big|\varphi^{-1}\Big(\int_t^\infty f(s,\ x)\mathrm{d}s\Big)-\varphi^{-1}\Big(\int_t^\infty f(s,\ \tilde{x})\mathrm{d}s\Big)\Big|\\ \leq & \sup_{t\geq T_0}\int_t^\infty \big|f(s,\ x)-f(s,\ \tilde{x})\big|\mathrm{d}s \leq \sup_{t\geq T_0}\int_t^\infty a(s)\big|x(s)-\tilde{x}(s)\big|\mathrm{d}s\\ \leq & \int_{T_0}^\infty a(s)\mathrm{d}s\|x-\tilde{x}\| \leq a_0\|x-\tilde{x}\|. \end{aligned} \tag{3.8}$$

(3.7) and (3.8) yield that

$$\|x-\tilde{x}\| \leq \frac{1}{1-a_0}|\psi_0-\tilde{\psi}_0| \tag{3.9}$$

and

$$\big|x(T_0)-\tilde{x}(T_0)\big|+\big|x'(T_0)-\tilde{x}'(T_0)\big| \leq \frac{1+a_0}{1-a_0}|\psi_0-\tilde{\psi}_0|. \tag{3.10}$$

If $T_0=t_0$, the proof is completed. If $t_0<T_0$, the stability of the solution will be proved in the following.

Since $(\varphi(x'))'=\frac{x''}{[1-(x')^2]^{\frac{3}{2}}}$, the Eq. (1.1) can be written into

$$\frac{x''}{[1-(x')^2]^{\frac{3}{2}}}=f(t,\ x),$$

and hence

$$x''(t)=\big[1-(x'(t))^2\big]^{\frac{3}{2}}f(t,\ x).$$

Set

$$s=T_0-t,\ y(s)=x(T_0-s),\ z(s)=y'(s),\quad 0\leq s\leq T_0-t_0.$$

Then $y'(s)=-x'(T_0-s)$,

$$\begin{aligned}y''(s)=x''(T_0-s)&=\big[1-\big(x'(T_0-s)\big)^2\big]^{\frac{3}{2}}f\big(T_0-s,\ x(T_0-s)\big)\\ &=\big[1-\big(z(s)\big)^2\big]^{\frac{3}{2}}f\big(T_0-s,\ y(s)\big)\end{aligned} \tag{3.11}$$

and

$$y(0)=x(T_0),\quad y'(0)=-x'(T_0). \tag{3.12}$$

Combining this with (1.5), we consider the system

$$\begin{pmatrix} y'(s) \\ z'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a(T_0 - s) & 0 \end{pmatrix} \begin{pmatrix} y(s) \\ z(s) \end{pmatrix}. \tag{3.13}$$

Denote by

$$A(s) = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}$$

the fundamental matrix solution of the system (3.13). Obviously, (3.11) and (3.12) can also be written equivalently

$$\begin{aligned} \begin{pmatrix} y'(s) \\ z'(s) \end{pmatrix} =& \begin{pmatrix} 0 & 1 \\ -a(T_0 - s) & 0 \end{pmatrix} \begin{pmatrix} y(s) \\ z(s) \end{pmatrix} + \\ & \begin{pmatrix} 0 \\ \left[1 - \big(z(s)\big)^2\right]^{\frac{3}{2}} f\big(T_0 - s,\ y(s)\big) + a(T_0 - s)y(s) \end{pmatrix}, \\ \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} =& \begin{pmatrix} x(T_0) \\ -x'(T_0) \end{pmatrix}. \end{aligned}$$

By the variation of constants formula, we have

$$\begin{aligned} \begin{pmatrix} y(s) \\ z(s) \end{pmatrix} =& A(s) \begin{pmatrix} x(T_0) \\ -x'(T_0) \end{pmatrix} + \\ & \int_0^s A(s - \tau) \begin{pmatrix} 0 \\ \left[1 - \big(z(\tau)\big)^2\right]^{\frac{3}{2}} f\big(T_0 - \tau,\ y(\tau)\big) + a(T_0 - \tau)y(\tau) \end{pmatrix} \mathrm{d}\tau, \end{aligned}$$

then

$$\begin{aligned} y(s) =& a_{11}(s)x(T_0) - a_{12}(s)x'(T_0) + \int_0^s a_{12}(s - \tau)\left[1 - \big(z(\tau)\big)^2\right]^{\frac{3}{2}} f\big(T_0 - \tau,\ y(\tau)\big)\mathrm{d}\tau + \\ & \int_0^s a_{12}(s - \tau)a(T_0 - \tau)y(\tau)\mathrm{d}\tau. \end{aligned}$$

Similarly, let

$$\tilde{y}(s) = \tilde{x}(T_0 - s),\ \tilde{z}(s) = \tilde{y}'(s), \quad 0 \le s \le T_0 - t_0.$$

We can derive

$$\begin{aligned} y(s) - \tilde{y}(s) =& a_{11}(s)\big[x(T_0) - \tilde{x}(T_0)\big] - a_{12}(s)\big[x'(T_0) - \tilde{x}'(T_0)\big] + \\ & \int_0^s a_{12}(s - \tau)\big[\big(1 - \big(z(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0 - \tau,\ y(\tau)\big) - \\ & \big(1 - \big(\tilde{z}(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0 - \tau,\ \tilde{y}(\tau)\big)\big]\mathrm{d}\tau + \\ & \int_0^s a_{12}(s - \tau)a(T_0 - \tau)\big[y(\tau) - \tilde{y}(\tau)\big]\mathrm{d}\tau. \end{aligned} \tag{3.14}$$

Let

$$a_{11}^* = \max_{s \in [0, T_0 - t_0]} \big|a_{11}(s)\big|, \quad a_{12}^* = \max_{s \in [0, T_0 - t_0]} \big|a_{12}(s)\big|, \quad a_1^* = \max\{a_{11}^*,\ a_{12}^*\}. \tag{3.15}$$

For any $0 \le s \le T_0 - t_0$, (3.14) becomes

$$\begin{aligned}\big|y(s)-\tilde{y}(s)\big| \le & a_{11}^*\big|x(T_0)-\tilde{x}(T_0)\big| + a_{12}^*\big|x'(T_0)-\tilde{x}'(T_0)\big| + \\ & \int_0^s a_{12}^*\big|\big(1-\big(z(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0-\tau,\ y(\tau)\big) - \big(1-\big(\tilde{z}(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0-\tau,\ \tilde{y}(\tau)\big)\big|\mathrm{d}\tau + \\ & \int_0^s a_{12}^* a(T_0-\tau)\big|y(\tau)-\tilde{y}(\tau)\big|\mathrm{d}\tau. \end{aligned} \tag{3.16}$$

Note that it follows from the mean value theorem that there exists z^* intervening z with $\tilde{z}$ such that

$$\big|\big(1-\big(z(\tau)\big)^2\big)^{\frac{3}{2}} - \big(1-\big(\tilde{z}(\tau)\big)^2\big)^{\frac{3}{2}}\big| = \big|-3z^*(1-z^{*2})^{\frac{1}{2}}(z-\tilde{z})\big| \le 3|z-\tilde{z}|. \tag{3.17}$$

Then, from (1.6) and (3.17), we obtain

$$\begin{aligned} & \big|\big(1-\big(z(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0-\tau,\ y(\tau)\big) - \big(1-\big(\tilde{z}(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0-\tau,\ \tilde{y}(\tau)\big)\big| \\ & = \big|\big(1-\big(z(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0-\tau,\ y(\tau)\big) - \big(1-\big(z(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0-\tau,\ \tilde{y}(\tau)\big) + \\ & \quad \big(1-\big(z(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0-\tau,\ \tilde{y}(\tau)\big) - \big(1-\big(\tilde{z}(\tau)\big)^2\big)^{\frac{3}{2}} f\big(T_0-\tau,\ \tilde{y}(\tau)\big)\big| \\ & \le \big(1-\big(z(\tau)\big)^2\big)^{\frac{3}{2}}\big|f\big(T_0-\tau,\ y(\tau)\big) - f\big(T_0-\tau,\ \tilde{y}(\tau)\big)\big| + \\ & \quad \big|\big[\big(1-\big(z(\tau)\big)^2\big)^{\frac{3}{2}} - \big(1-\big(\tilde{z}(\tau)\big)^2\big)^{\frac{3}{2}}\big] f\big(T_0-\tau,\ \tilde{y}(\tau)\big)\big| \\ & \le \big|f\big(T_0-\tau,\ y(\tau)\big) - f\big(T_0-\tau,\ \tilde{y}(\tau)\big)\big| + \\ & \quad \big|\big[(1-\big(z(\tau)\big)^2)^{\frac{3}{2}} - \big(1-\big(\tilde{z}(\tau)\big)^2\big)^{\frac{3}{2}}\big] f\big(T_0-\tau,\ \tilde{y}(\tau)\big)\big| \\ & \le a(T_0-\tau)|y-\tilde{y}| + 3M|z-\tilde{z}|. \end{aligned} \tag{3.18}$$

Indeed, by (H1), for any $\tau \in [0,\ T_0-t_0]$, there exists some constant $M>0$ such that

$$|f(T_0-\tau,\ \tilde{y}(\tau))| \le M.$$

From (1.6), (3.8), (3.9), (3.15), (3.16) and (3.18), we obtain

$$\begin{aligned}\big|y(s)-\tilde{y}(s)\big| \le & a_1^*\big\{\big|x(T_0)-\tilde{x}(T_0)\big| + \big|x'(T_0)-\tilde{x}'(T_0)\big|\big\} + \\ & 2\int_0^s a_{12}^* a(T_0-\tau)\big|y(\tau)-\tilde{y}(\tau)\big|\mathrm{d}\tau + \\ & a_1^*\frac{3M(T_0-t_0)a_0}{1-a_0}|\psi_0-\tilde{\psi}_0|. \end{aligned} \tag{3.19}$$

By the virtue of (3.10), (3.19) can be turned into

$$\big|y(s)-\tilde{y}(s)\big| \le \frac{a_1^*\big[1+a_0+3M(T_0-t_0)a_0\big]}{1-a_0}|\psi_0-\tilde{\psi}_0| + 2\int_0^s a_{12}^* a(T_0-\tau)|y(\tau)-\tilde{y}(\tau)|\mathrm{d}\tau.$$

Using the Bellman-Gronwall inequality, we have

$$\begin{aligned}\big|y(s)-\tilde{y}(s)\big| & \le \frac{a_1^*\big[1+a_0+3M(T_0-t_0)a_0\big]}{1-a_0}|\psi_0-\tilde{\psi}_0|e^{(2a_{12}^*\int_0^{T_0-t_0} a(T_0-\tau)\mathrm{d}\tau)} \\ & \le \frac{a_1^*\big[1+a_0+3M(T_0-t_0)a_0\big]}{1-a_0}|\psi_0-\tilde{\psi}_0|e^{(2a_{12}^*\int_{t_0}^{T_0} a(\tau)\mathrm{d}\tau)}. \end{aligned}$$

Take $s = T_0 - t$,

$$\big|x(s) - \tilde{x}(s)\big| \le \frac{a_1^*\big[1 + a_0 + 3M(T_0 - t_0)a_0\big]}{1 - a_0}\big|\psi_0 - \tilde{\psi}_0\big|e^{(2a_{12}^*\int_{t_0}^{T_0} a(\tau)\mathrm{d}\tau)}. \tag{3.20}$$

Similarly, we can also deduce that

$$\big|x'(t) - \tilde{x}'(t)\big| \le \frac{a_2^*\big[1 + a_0 + 3M(T_0 - t_0)a_0\big]}{1 - a_0}\big|\psi_0 - \tilde{\psi}_0\big|e^{(2a_{22}^*\int_{t_0}^{T_0} a(\tau)\mathrm{d}\tau)}, \tag{3.21}$$

where

$$a_{21}^* = \max_{s\in[0,T_0-t_0]}\big|a_{21}(s)\big|, \quad a_{22}^* = \max_{s\in[0,T_0-t_0]}\big|a_{22}(s)\big|, \quad a_2^* = \max\big\{a_{21}^*,\ a_{22}^*\big\}.$$

Consequently, it deduces from (3.9), (3.20) and (3.21) that the solution is stable in the norm (3.3) with respect to ψ_0. □

4. The proofs of Theorems 1.2 and 1.3

When the nonlinearities f satisfy the Osgood condition, we will show the proof of the existence and uniqueness of bounded solutions for (1.1) and (1.2) in this section.

Proof of Theorem 1.2 For any given positive constant $\rho < 1$, it follows from (H4) that we can choose large enough T_1 such that

$$g\big(|\psi_0| + \rho\big)\int_{T_1}^{\infty} a(s)\mathrm{d}s \le g\big(|\psi_0| + \rho\big)\int_{T_1}^{\infty} sa(s)\mathrm{d}s \le \rho. \tag{4.1}$$

Let

$$E = \Big\{x \in C^1[T_1,\ \infty) :\ \lim_{t\to\infty} x(t) = \psi_0\Big\}$$

be the Banach space under the norm (3.3). Clearly, $E \subset X$ when $T_0 = T_1$ are sufficiently large. Set

$$\Omega = \{x \in E :\ |\psi_0| - \rho \le x(t) \le |\psi_0| + \rho,\ t \ge T_1\}.$$

For any $x \in \Omega$, we have

$$\begin{aligned}
\lim_{t\to\infty}\big|[\mathcal{F}(x)](t) - \psi_0\big| &= \lim_{t\to\infty}\Big|\int_t^{\infty}\varphi^{-1}\Big(\int_s^{\infty} f(\tau,\ x)\mathrm{d}\tau\Big)\mathrm{d}s\Big| \le \lim_{t\to\infty}\int_t^{\infty}\int_s^{\infty}|f(\tau,\ x)|\mathrm{d}\tau\mathrm{d}s\\
&= \lim_{t\to\infty}\int_t^{\infty}\int_t^{\tau}|f(\tau,\ x)|\mathrm{d}s\mathrm{d}\tau = \lim_{t\to\infty}\int_t^{\infty}(\tau - t)|f(\tau,\ x)|\mathrm{d}\tau\\
&= \lim_{t\to\infty}\int_t^{\infty}(s - t)|f(s,\ x)|\mathrm{d}s \le \lim_{t\to\infty}\int_t^{\infty}s|f(s,\ x)|\mathrm{d}s = 0,
\end{aligned}$$

which implies $\mathcal{F} : \Omega \to E$. Furthermore, for any $t \ge T_1$, $x \in \Omega$, it can be seen from (H4), (4.1) and monotonicity of g that

$$\begin{aligned}
\big|[\mathcal{F}(x)](t) - \psi_0\big| &= \Big|\int_t^{\infty}\varphi^{-1}\Big(\int_s^{\infty} f(\tau,\ x)\mathrm{d}\tau\Big)\mathrm{d}s\Big| \le \int_t^{\infty}\int_s^{\infty}|f(\tau,\ x)|\mathrm{d}\tau\mathrm{d}s\\
&\le \int_t^{\infty}(s - t)|f(s,\ x)|\mathrm{d}s \le \int_t^{\infty}sa(s)g(|x(s)|)\mathrm{d}s\\
&\le g(|\psi_0| + \rho)\int_t^{\infty}sa(s)\mathrm{d}s \le \rho
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\big|[\mathcal{F}(x)]'(t)\big| &\le \Big|\varphi^{-1}\Big(\int_t^\infty f(s,\ x)\mathrm{d}s\Big)\Big| \le \int_t^\infty |f(s,\ x)|\mathrm{d}s\\ &\le \int_t^\infty a(s)g(|x(s)|)\mathrm{d}s \le g\big(|\psi_0|+\rho\big)\int_{T_1}^\infty a(s)\mathrm{d}s \le \rho. \end{aligned}\tag{4.3}$$

Thus, $\mathcal{F}:\Omega\to\Omega$.

In the following, we will prove that $\mathcal{F}:\Omega\to\Omega$ is completely continuous.

Let $\{x_n\}\subset\Omega$ be an arbitrary sequence. By applying the Lagrange mean value theorem and (4.3), we obtain

$$\big|[\mathcal{F}(x_n)](t_1)-[\mathcal{F}(x_n)](t_2)\big|\le\rho|t_1-t_2|,\quad t_1,\ t_2\ge T_1,\ n\ge 1.\tag{4.4}$$

For any $t_1,t_2\ge T_1$, without loss of generality, let $t_1\le t_2$, using the similar method to treat (3.5), we have

$$\begin{aligned}\big|[\mathcal{F}(x_n)]'(t_1)-[\mathcal{F}(x_n)]'(t_2)\big| &\le \Big|\varphi^{-1}\Big(\int_{t_1}^\infty f(s,\ x_n)\mathrm{d}s\Big)-\varphi^{-1}\Big(\int_{t_2}^\infty f(s,\ x_n)\mathrm{d}s\Big)\Big|\\ &\le \Big|\int_{t_1}^\infty f(s,\ x_n)\mathrm{d}s-\int_{t_2}^\infty f(s,\ x_n)\mathrm{d}s\Big| \le \int_{t_1}^{t_2}|f(s,\ x_n)|\mathrm{d}s\\ &\le \int_{t_1}^{t_2} a(s)g(|x_n(s)|)\mathrm{d}s \le g\big(|\psi_0|+\rho\big)\int_{t_1}^{t_2}a(s)\mathrm{d}s\\ &\le Ag\big(|\psi_0|+\rho\big)(t_2-t_1),\end{aligned}\tag{4.5}$$

where $A=\max_{s\in[t_1,t_2]}a(s)$. So (4.4) and (4.5) show that $\{\mathcal{F}(x_n)\}$ is equicontinuous in Ω. Clearly, $\{\mathcal{F}(x_n)\}$ is uniformly bounded in Ω. On the other hand, it follows from (4.2) that

$$\big|[\mathcal{F}(x_n)](t)-\psi_0\big|\le g\big(|\psi_0|+\rho\big)\int_t^\infty sa(s)\mathrm{d}s.$$

Then, for any $\varepsilon>0$, there exists $t_\varepsilon>T_1$ such that

$$\big|[\mathcal{F}(x_n)](t)-\psi_0\big|\le\varepsilon,\quad t\ge t_\varepsilon,\ n\ge 1.$$

Similar to (4.3), we obtain

$$\big|[\mathcal{F}(x_n)]'(t)\big|\le\varepsilon,\quad t\ge t_\varepsilon,\ n\ge 1.$$

Then $\{\mathcal{F}(x_n)\}$ is equiconvergent in Ω. Consequently, via the Arzela-Ascoli theorem [22], we obtain that $\{\mathcal{F}(x_n)\}$ is relatively compact in Ω.

Now we just need to prove that $\mathcal{F}:\Omega\to\Omega$ is continuous. From (1.5), for any fixed $\varepsilon>0$, there exists $T_*\ge T_1$ such that

$$g(|\psi_0|+\rho)\int_{T_*}^\infty a(s)\mathrm{d}s<g(|\psi_0|+\rho)\int_{T_*}^\infty sa(s)\mathrm{d}s<\frac{\varepsilon}{3}.$$

It follows from (H1) that $f:[t_0,\ T_*]\times\big[|\psi_0|-\rho,\ |\psi_0|+\rho\big]\to\mathbb{R}$ is uniformly continuous, then there exists a constant $\delta>0$, for any $x,\ y\in\big[|\psi_0|-\rho,\ |\psi_0|+\rho\big]$ with $|x-y|<\delta$, we have

$$\big|f(t,\ x)-f(t,\ y)\big|<\frac{\varepsilon}{3T_*^2},\quad t\in[t_0,\ T_*].$$

Therefore, for any $t \geq T_1$, $x_1, x_2 \in \Omega$ with $\|x_1 - x_2\| < \delta$, using the similar method to treat (3.5), we have

$$\begin{aligned}
\big|[\mathcal{F}(x_1)](t) - [\mathcal{F}(x_2)](t)\big| &= \Big| \int_t^\infty \varphi^{-1}\Big(\int_s^\infty f(\tau,\ x_1)\mathrm{d}\tau\Big)\mathrm{d}s - \int_t^\infty \varphi^{-1}\Big(\int_s^\infty f(\tau,\ x_2)\mathrm{d}\tau\Big)\mathrm{d}s\Big| \\
&\leq \int_t^\infty \int_s^\infty \big|f(\tau,\ x_1) - f(\tau,\ x_2)\big|\mathrm{d}\tau\mathrm{d}s \\
&= \int_t^\infty (s-t)\big|f(s,\ x_1) - f(s,\ x_2)\big|\mathrm{d}s \\
&\leq \int_{T_1}^{T_*} (s-T_1)\big|f(s,\ x_1) - f(s,\ x_2)\big|\mathrm{d}s + \\
&\quad \int_{T_*}^\infty (s-T_*)\big|f(s,\ x_1) - f(s,\ x_2)\big|\mathrm{d}s \\
&= I_1 + I_2.
\end{aligned}$$

Note that

$$I_1 \leq \frac{\varepsilon}{3T_*^2}\int_{T_1}^{T_*}(s-T_1)\mathrm{d}s = \frac{\varepsilon}{3T_*^2}\frac{(T_*-T_1)^2}{2} < \frac{\varepsilon}{6}$$

and

$$\begin{aligned}
I_2 &\leq \int_{T_*}^\infty s\big|f(s,\ x_1) - f(s,\ x_2)\big|\mathrm{d}s \leq \int_{T_*}^\infty s\big(|f(s,\ x_1)| + |f(s,\ x_2)|\big)\mathrm{d}s \\
&\leq \int_{T_*}^\infty sa(s)\big(g(|x_1(s)|) + g(|x_2(s)|)\big)\mathrm{d}s \leq 2g(|\psi_0| + \rho)\int_{T_*}^\infty sa(s)\mathrm{d}s < \frac{2\varepsilon}{3}.
\end{aligned}$$

Then

$$\big|[\mathcal{F}(x_1)](t) - [\mathcal{F}(x_2)](t)\big| < \varepsilon. \tag{4.6}$$

Moreover, using the similar method to treat (3.5), we have

$$\begin{aligned}
\big|[\mathcal{F}(x_1)]'(t) - [\mathcal{F}(x_2)]'(t)\big| &= \Big|\varphi^{-1}\Big(\int_t^\infty f\big(s,\ x_1\big)\mathrm{d}s\Big) - \varphi^{-1}\Big(\int_t^\infty f\big(s,\ x_2\big)\mathrm{d}s\Big)\Big| \\
&\leq \int_t^\infty \big|f(s,\ x_1) - f(s,\ x_2)\big|\mathrm{d}s \\
&\leq \int_{T_1}^{T_*} \big|f(s,\ x_1) - f(s,\ x_2)\big|\mathrm{d}s + \int_{T_*}^\infty \big|f(s,\ x_1) - f(s,\ x_2)\big|\mathrm{d}s \\
&\leq \frac{\varepsilon}{3T_*^2}(T_* - T_1) + 2g\big(|\psi_0| + \rho\big)\int_{T_*}^\infty a(s)\mathrm{d}s \\
&\leq \frac{\varepsilon}{3T_*} + \frac{2\varepsilon}{3} < \varepsilon.
\end{aligned} \tag{4.7}$$

Thus, it follows from (4.6) and (4.7) that $\mathcal{F} : \Omega \to \Omega$ is a continuous operator.

Consequently, it follows from the Schauder fixed-point theorem that $\mathcal{F}$ has a fixed-point $x \in \Omega$. So (2.1) has at least one solution $x : \big[T_1,\ \infty\big) \to \mathbb{R}$ with $\lim_{t\to\infty} x(t) = \psi_0$. □

Proof of Theorem 1.3 It follows from Theorem 1.2 that (2.1) has at least one solution $x : [T_1,\ \infty) \to \mathbb{R}$ with $\lim_{t\to\infty} x(t) = \psi_0$. Now we only need to prove the uniqueness of the

solution of (2.1). Suppose that x_1 and x_2 are two solutions of (2.1) with

$$\lim_{t\to\infty} x_1(t) = \psi_0 = \lim_{t\to\infty} x_2(t). \tag{4.8}$$

According to (H5), (2.1) and (3.5), we have

$$\begin{aligned}
|x_1(t) - x_2(t)| &= \Big| \int_t^\infty \varphi^{-1}\Big(\int_s^\infty f(\tau,\ x_1)\mathrm{d}\tau\Big)\mathrm{d}s - \int_t^\infty \varphi^{-1}\Big(\int_s^\infty f(\tau,\ x_2)\mathrm{d}\tau\Big)\mathrm{d}s\Big| \\
&\le \int_t^\infty \int_s^\infty \big|f(\tau,\ x_1) - f(\tau,\ x_2)\big|\mathrm{d}\tau\mathrm{d}s \\
&\le \int_t^\infty (s-t)\big|f(s,\ x_1) - f(s,\ x_2)\big|\mathrm{d}s \\
&\le \int_t^\infty sa(s)G(|x_1 - x_2|)\mathrm{d}s.
\end{aligned}$$

Set $\phi(t) = |x_1 - x_2|$. Then

$$\phi(t) \le \int_t^\infty sa(s)G(\phi(s))\mathrm{d}s. \tag{4.9}$$

We only need to prove that $\phi(t) \equiv 0$ for $t \ge T_1$. Now we proceed as in [21], repeating the arguments for completeness. On the contrary, suppose that

$$\Phi(t) = \max_{s\ge t} \phi(s) > 0, \quad \text{for any } t \ge T_1,$$

then $\phi(t) \le \Phi(t)$ and it follows from (4.8) that

$$\lim_{t\to\infty} \Phi(t) = \lim_{t\to\infty} \phi(t) = 0,$$

which implies there exists $t^* \ge T_1$ such that $\Phi(t) = \phi(t^*)$ for any $t \ge t^*$. Therefore,

$$\Phi(t) = \phi(t^*) \le \int_{t^*}^\infty sa(s)G(\phi(s))\mathrm{d}s \le \int_t^\infty sa(s)G(\phi(s))\mathrm{d}s.$$

Set

$$\Psi(t) = \int_t^\infty sa(s)G(\phi(s))\mathrm{d}s,$$

then from (4.9), we get $\phi(t) \le \Psi(t)$ and $\lim_{t\to\infty} \Psi(t) = 0$. Further,

$$\Psi'(t) = -ta(t)G(\phi(t)) \ge -ta(t)G(\Psi(t)).$$

Therefore,

$$\frac{\Psi'(t)}{G(\Psi(t))} \ge -ta(t).$$

Integrate the above equation to obtain

$$\int_{t^*}^\infty \frac{\Psi'(t)}{G(\Psi(t))}\mathrm{d}t \ge -\int_{t^*}^\infty ta(t)\mathrm{d}t. \tag{4.10}$$

Set

$$r^* = \Psi(t^*) = \int_{t^*}^\infty sa(s)G(\phi(s))\mathrm{d}s < \infty.$$

It deduces from (4.10) that

$$\int_0^{r^*} \frac{1}{G(r)}\mathrm{d}r \le \int_{t^*}^\infty ta(t)\mathrm{d}t < \infty,$$

which contradicts (1.7). Therefore, $\phi(t) \equiv 0$ for all $t \geq t_0$. □

Remark 4.1 Let $f(t, x) = e^{-2t}x(t)$, $\psi_0 = 0$. Clearly, (H1) holds. According to (1.2), $x(t)$ is bounded. Then

$$\lim_{t\to\infty} f(t,x)e^t = \lim_{t\to\infty} e^{-t}x(t) = 0.$$

Hence (H2) holds. Let $a(t) = e^{-2t}$. Then

$$\int_{t_0}^{\infty} sa(s)\mathrm{d}s = \int_{t_0}^{\infty} se^{-2s}\mathrm{d}s = \frac{2t_0+1}{4e^{2t_0}} < \infty$$

and

$$|e^{-2t}x(t) - e^{-2t}y(t)| \leq e^{-2t}|x(t) - y(t)|, \quad \text{for any } x,\ y \in \mathbb{R}.$$

Thus (H3) holds. Let $g(|x|) = x$. Then (H4) holds. Set $r = |x(t) - y(t)|$, $G(r) = r^2$, we can see $G(0) = 0$, $G(r) > 0$ and satisfies

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} \frac{1}{G(r)}\mathrm{d}r = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} \frac{1}{r^2}\mathrm{d}r = \infty.$$

Then (H5) is valid. Consequently, there exists a unique stable bounded solution of (1.1) and (1.2). □

Remark 4.2 Set

$$f(t, x) = \frac{e^{-t}}{t}x\ln(1 + |x|).$$

Obviously, (H1) holds. According to (1.2), x is bounded. Thus

$$\sup_{x\in\mathbb{R}}\Big|\frac{x\ln(1+|x|)}{t}\Big| \to 0, \quad t \to \infty.$$

This means that (H2) is established. Let $a(t) = \frac{e^{-t}}{t}$, $g(|x|) = |x|\ln(1 + |x|)$. We can have

$$\int_{t_0}^{\infty} e^{-s}\mathrm{d}s = e^{-t_0} < \infty,$$

and (H4) holds. According to Lagrange mean value theorem, there exists a z between x and y such that

$$|x\ln(1+|x|) - y\ln(1+|y|)| \leq \Big[\ln(1+|z|) + \frac{|z|}{1+|z|}\Big]|x-y| \leq |x-y|\ln(1+|x-y|) + |x-y|.$$

Choose $G(r) = r\ln(1 + r) + r$, clearly,

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} \frac{1}{r\ln(1+r)+r}\mathrm{d}r = \infty.$$

According to Theorems 1.2 and 1.3, (1.1) and (1.2) has at least one unique bounded solution. □

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