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Uniform Attractors for the Kirchhoff Type Suspension Bridge Equation with Nonlinear Damping and Memory Term

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Abstract The goal of this paper is to investigate the long-time dynamics of solutions to a Kirchhoff type suspension bridge equation with nonlinear damping and memory term. For this problem we establish the well-posedness and existence of uniform attractor under some suitable assumptions on the nonlinear term $g(u)$, the nonlinear damping $f(u_t)$ and the external force $h(x,t)$. Specifically, the asymptotic compactness of the semigroup is verified by the energy reconstruction method.

Keywords uniform attractor; Kirchhoff type; suspension bridge equation; nonlinear damping; memory term

MR(2020) Subject Classification 35B40; 35B41

1. Introduction

In this paper, we mainly study the existence of uniform attractors for the Kirchhoff type suspension bridge equation with nonlinear damping and memory term

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u_t+ku^+-M(\|\nabla u\|^2)\Delta u+l(0)\Delta^2 u+\\ \quad\int_0^\infty l'(s)\Delta^2 u(t-s)\mathrm{d}s+\delta f(u_t)+g(u)=h(x,t),\ (x,t)\in\Omega\times\mathbb{R}^+,\\ u(x,t)=\Delta u(x,t)=0,\\ u(x,0)=u_0(x),\\ u_t(x,0)=u_1(x), \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded open domain with a sufficiently smooth boundary $\partial\Omega$, $k>0$ is a spring constant, $u^+=\max\{u,0\}$ is the positive part of u, $-ku^+$ denotes restoring force due to the cables. $l(\cdot)$ admits $l(0)>0,\ l(\infty)>0$, $\delta f(u_t)$ $(0<\delta<1)$ represents the nonlinear damping term, g is the nonlinear term, $h\in L^2(\Omega)$ is an external force.

The function $M(\cdot),\ f(\cdot),\ g(\cdot),\ h$ satisfy the following assumptions:

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(1) The Kirchhoff coefficient $M(\cdot)$ satisfies

$$M(s) = s^{\frac{m}{2}}, \quad \forall\, s \in \mathbb{R}^+,\ m \geq 1; \tag{1.2}$$

(2) The nonlinear damping $f(\cdot) \in C^1(\mathbb{R})$, $f(0) = 0$, $f(\cdot)$ is strictly increasing and

$$\liminf_{|s|\to\infty} f'(s) > 0, \tag{1.3}$$

$$|f(s)| \leq C_1(1 + |s|^p), \quad 1 \leq p < \infty; \tag{1.4}$$

(3) The nonlinear function $g(\cdot) \in C^2(\mathbb{R})$ and satisfies

$$|g(s)| \leqslant C_2(1 + |s|^q), \quad q \geq 1, \tag{1.5}$$

where $C_2 > 0$ and $G(s) = \int_0^s g(\tau)\mathrm{d}\tau$,

$$\limsup_{|s|\to\infty} \frac{g(s)}{s} < \lambda_1, \tag{1.6}$$

where $\lambda_1 > 0$ is the first eigenvalue of A;

(4) The external force h satisfies

$$h(x,t) \in L^\infty(\mathbb{R}; L^2(\Omega)), \tag{1.7}$$

$$\partial_t h \in L^r_b(\mathbb{R}; L^r(\Omega)), \quad r > \frac{2q+2}{2q+1},\ q \geq 1. \tag{1.8}$$

The suspension bridge equations describe the vibration of the suspension bridge pavement in the vertical plane. In 1990, Lazer and McKenna [1] proposed and studied the following suspension bridge model as a new problem in the field of nonlinear analysis

$$u_{tt} + u_t + \partial_{xxxx}u + ku^+ = h(x) + w(x,t).$$

The asymptotic behavior of the suspension bridge model has been studied by many scholars [2,3]. When taking into account the axial tension at the end of suspension bridge pavement, we arrive at the Kirchhoff type suspension bridge equation

$$u_{tt} + \delta u_t + \partial_{xxxx}u + (\alpha - \|\partial_{xx}u\|^2_{L^2(\Omega)})\partial_{xx}u + ku^+ + f(u) = h(x,t).$$

In 2010, the author explored the existence of uniform attractors for non-autonomous suspension bridge equations with strong damping in [4]. Later, Park obtained the global attractor of the autonomous suspension bridge equation with nonlinear damping in [5]. In [6], the author considered global attractors for an extensible beam equation incorporating localized nonlinear damping and linear memory

$$u_{tt} + \Delta^2 u - k(0)\Big(1 + \int_\Omega |\nabla u|^2 \mathrm{d}x\Big)\Delta u - \int_0^\infty k'(s)\Delta u(t-s)\mathrm{d}s + a(x)g(u_t) + f(u) = h(x).$$

Subsequently, he further discussed the long-time behavior of autonomous suspension bridge equation with memory in [7,8]. In [9], the author investigated the global attractor of the extensible suspension bridge equation with linear memory. Recently, Feng, Yang and Qin [10] studied uniform attractors for a non-autonomous extensible plate equation endowed with a strong damping

$$u_{tt} + \Delta^2 u + \alpha u - M(\|\nabla u\|^2)\Delta u - \gamma\Delta u_t + f(u) = \sigma(x,t), \quad x \in \Omega,\ t \geq \tau,\ \tau \in \mathbb{R}^+. \tag{1.9}$$

Inspired by the above literature, we study the existence of uniform attractors for Kirchhoff-type suspension bridge equations with nonlinear damping and memory term. In our opinion, the main difficulties and innovations are presented in the following:

(i) The coefficient of the Kirchhoff term transforms into a nonlinear function $M(\cdot)$. This not only elevates the complexity of estimations when investigating the existence of uniform attractors but also renders practical applications more intricate. Additionally, the nonlinear damping term $\delta f(u_t)$ poses challenges in achieving the uniqueness of the solution and the uniform asymptotic compactness of semigroups. To tackle these problems, we put forward conditions (1) and (2). These conditions are designed to overcome the difficulties brought about by the aforementioned factors, thereby facilitating a more in-depth study of the relevant equation's properties and its potential applications.

(ii) We consider the system with a memory term $\int_0^\infty l'(s)\Delta^2 u(t-s)\mathrm{d}s$, and then, the system can generate a new phase space, which makes the solution space more complex. Furthermore, the problem (1.1) includes not only the memory term $\int_0^\infty l'(s)\Delta^2 u(t-s)\mathrm{d}s$, but also the Kirchhoff term $-M(\|\nabla u\|^2)\Delta u$ and the nonlinear damping term $\delta f(u_t)$, in contrast to previous works that merely focus on either the memory term, the Kirchhoff term, or the nonlinear damping term in isolation, our study is more comprehensive. It takes into account the combined effects of these multiple factors, which enriches the scope of the research.

(iii) In the case $f(u_t) = -\Delta u_t$, $l'(s) = 0$ and $k = 0$, the Eq. (1.1) degenerates into the Eq. (1.9), so we proceed to further extend the results associated with it.

In order to facilitate the discussion of problem (1.1), we introduce the following transformation

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s). \tag{1.10}$$

Supposing $\mu(s) = -l'(s)$, $l(\infty) = 1$ and combining (1), then we can transform (1.1) as the following system

$$u_{tt} + 2\Delta^2 u - \Delta u_t + ku^+ - \|\nabla u\|^m \Delta u + \int_0^\infty \mu(s)\Delta^2 \eta^t(x,s)\mathrm{d}s + \delta f(u_t) + g(u) = h(x,t) \tag{1.11}$$

and

$$\eta_t = -\eta_s + u_t, \quad (x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \tag{1.12}$$

with initial boundary value conditions

$$\begin{cases} u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ \eta^t(x,s) = 0, & (x,s) \in \partial\Omega \times \mathbb{R}^+,\ t \geq 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u^t(x,0) = u_1(x), & x \in \Omega, \\ \eta^0(x,s) = \eta_0(x,s), & (x,s) \in \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.13}$$

where

$$\begin{cases} u_0(x) = u_0(x,0), \\ u_1(x) = \partial_t u_0(x,t)|_{t=0}, \\ \eta_0(x,s) = u_0(x,0) - u_0(x,-s). \end{cases} \tag{1.14}$$

(5) With respect to the memory kernel $\mu(\cdot)$, we assume that

$$\mu(s) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \tag{1.15}$$

$$\mu(s) \geq 0, \mu'(s) \leq 0, \tag{1.16}$$

$$\int_0^\infty \mu(s)\mathrm{d}s = \delta_1 > 0, \tag{1.17}$$

$$\mu'(s) + \delta_2 \mu(s) \leq 0. \tag{1.18}$$

The structure of this paper is as follows. In Section 2, we introduce preliminary theories related to this paper. In Section 3, we prove the existence of uniform attractors.

2. Preliminaries

In this section, we present some relevant notations and existence results.

Let $L^p = L^p(\Omega)$, $H^k = H^k(\Omega)$, $H = V_0 = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$, $V_2 = H^2(\Omega) \cap H_0^1(\Omega)$, where $p \geqslant 1$, $k > 0$, H^k is a Sobolev space, H_0^k is the closure of $C_0^\infty(\Omega)$ in H^k.

The inner product and norm of space H are represented by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The dual space of V_2 is represented by V_{-2}, and the dual pairing between dual spaces is denoted by $\langle\cdot,\cdot\rangle$.

We define the operator $A : V_2 \to V_{-2}$,

$$\langle Au, v\rangle = \langle \Delta u, \Delta v\rangle, \quad \forall\, u, v \in V_2,$$

$$D(A) = \{u \in H^2(\Omega) \cap H_0^1(\Omega) | Au \in L^2(\Omega)\},$$

then A is a self-adjoint operator in space H and strictly positive in space V_2. For $s \in \mathbb{R}$, we define the power of operator A as A^s, and the space $V_s = D(A^{\frac{s}{4}})$ is a Hilbert space with the inner product and norm given by

$$(u,v)_s = (A^{\frac{s}{4}}u, A^{\frac{s}{4}}v) = \int_\Omega A^{\frac{s}{4}}u \cdot A^{\frac{s}{4}}v\mathrm{d}x,$$

$$\|u\|_s^2 = (u,u)_s = \|A^{\frac{s}{4}}u\|^2.$$

Apparently,

$$\|u\|_{v_1} = \|u\|_1 = \|A^{\frac{1}{4}}u\| = \|\nabla u\|,$$

$$\|u\|_{v_2} = \|u\|_2 = \|A^{\frac{1}{2}}u\| = \|\Delta u\|.$$

We introduce the following Hilbert space

$$L_\mu^2(\mathbb{R}^+; V_i) = \Big\{\eta : \mathbb{R}^+ \to V_i | \int_0^\infty \mu(s)\|\eta(s)\|_i^2 \mathrm{d}s < \infty\Big\}, \quad i = 1, 2,$$

endowed with the inner product and norm

$$(\eta_1,\eta_2)_{\mu,i}=\int_0^\infty \mu(s)(\eta_1(s),\eta_2(s))_i \mathrm{d}s,$$

$$\|\eta\|^2_{\mu,i}=(\eta,\eta)_{\mu,i}=\int_0^\infty \mu(s)\|\eta(s)\|^2_i \mathrm{d}s,$$

respectively. Furthermore, we define the phase space

$$\mathcal{H}=V_2\times L_2\times L^2_\mu(\mathbb{R}^+;V_2)$$

with corresponding norm

$$\|(u,u_t,\xi)\|^2_{\mathcal{H}}=\|\Delta u\|^2+\|u_t\|^2+\|\eta\|^2_{\mu,2}.$$

By the Poincaré inequality, we have

$$\|\nabla u\|^2 \geqslant \lambda_1^{\frac{1}{2}}\|u\|^2, \quad \forall\, u\in V_1.$$

In addition, we define the translation bounded function space in $L^r_{\mathrm{loc}}(\mathbb{R};L^k(\Omega))$, $r,k\geq 1$ as

$$L^r_b(\mathbb{R};L^k(\Omega))=\Big\{h\in L^r_{\mathrm{loc}}(\mathbb{R};L^k(\Omega)):\sup_{t\in\mathbb{R}}\int_t^{t+1}\Big(\int_\Omega \mid h(x,s)\mid^k \mathrm{d}x\Big)^{\frac{r}{k}}\mathrm{d}s<\infty\Big\},$$

and the translation compact function space in $L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$ as

$$\begin{aligned}L^2_c(\mathbb{R};L^2(\Omega))=&\{h\in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega)):\ \forall [t_1,t_2]\subset\mathbb{R},\\ &\{h(x,s+p):p\in\mathbb{R}\}|_{[t_1,t_2]}\text{ is relatively compact in } L^2(t_1,t_2;L^2(\Omega))\}.\end{aligned}$$

Let X be a Banach space and let a two-parameter family

$$\{U(t,\tau)\}=\{U(t,\tau)\mid t\geq\tau,\tau\in\mathbb{R}\}$$

acting on X:

$$U(t,\tau):X\to X,\quad t\geq\tau\in\mathbb{R}.$$

Definition 2.1 ([4]) *Let Σ be a parameter set. $\{U_\sigma(t,\tau)\mid t\geq\tau,\tau\in\mathbb{R}\}$, $\sigma\in\Sigma$ is said to be a family of processes in Banach space X, if for each $\{U_\sigma(t,\tau)\mid t\geq\tau,\tau\in\mathbb{R}\}$, $\sigma\in\Sigma$ is a process, that is, the two-parameter family of mappings $\{U_\sigma(t,\tau)\}$ from X to X satisfy*

$$U_\sigma(t,s)\ o\ U_\sigma(s,\tau)=U_\sigma(t,\tau),\quad \forall\, t\geq s\geq\tau,\ \tau\in\mathbb{R}, \tag{2.1}$$

$$U_\sigma(\tau,\tau)=Id,\quad \tau\in\mathbb{R}, \tag{2.2}$$

where Id is the identity operator, Σ is called the symbol space and σ is the symbol.

Let $\{T(s)\}_{s\geq 0}$ be a family of operators acting on the symbol space Σ. Then

$$T(s)\Sigma=\Sigma,\quad \forall\, s\geq 0, \tag{2.3}$$

and the family satisfies the translation identity

$$U_\sigma(t+s,\tau+s)=U_{T(s)\sigma}(t,\tau),\quad \forall\,\sigma\in\Sigma,\ t\geq\tau,\ \tau\in\mathbb{R},\ s\geq 0. \tag{2.4}$$

Definition 2.2 ([10]) *A set $B_0 \in \mathcal{B}(X)$ is said to be a uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set for the family of process $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(X)$, there exists $t_0 = t_0(\tau, B) > \tau$ such that $\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset B_0$ for all $t \geq t_0$.*

A set $\mathcal{A} \subset X$ is said to be uniformly (w.r.t. $\sigma \in \Sigma$) attracting for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if for any fixed $\tau \in \mathbb{R}$ and every $B \in \mathcal{B}(X)$,

$$\lim_{t\to\infty} \Big(\sup_{\sigma\in\Sigma} \operatorname{dist}_X(U_\sigma(t,\tau)B,\ \mathcal{A}\Big) = 0,$$

where $\mathcal{B}(X)$ represents the set of all bounded subsets of space X.

Definition 2.3 ([10]) *Let X be a Banach space and $B \in \mathcal{B}(X)$, Σ be a symbol space. We call a function $\phi(\cdot,\cdot;\cdot,\cdot)$ defined on $(X \times X) \times (\Sigma \times \Sigma)$ a contractive function on $B \times B$, if for any sequence $\{x_n\}_{n=1}^\infty \subset B$ and $\{\sigma_n\} \subset \Sigma$, there are subsequences $\{x_{nk}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ and $\{\sigma_{nk}\}_{k=1}^\infty \subset \{\sigma_n\}_{n=1}^\infty$ such that*

$$\lim_{k\to\infty} \lim_{l\to\infty} \phi(x_{nk}, x_{nl}; \sigma_{nk}, \sigma_{nl}) = 0. \tag{2.5}$$

We denote the set of all contractive functions on $B \times B$ by $\operatorname{contr}(B, \Sigma)$.

Lemma 2.4 ([11]) *Let $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ be a family of processes satisfying the translation identity (2.3) and (2.4) on a Banach space X and having a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set $B_0 \subset X$. Moreover, assume that for any $\varepsilon > 0$ there exist $T = T(\varepsilon)$ and $\Phi_T(\cdot,\cdot) \in \operatorname{contr}(B_0, \Sigma)$ such that*

$$\|U_{\sigma_1}(T,\tau)x - U_{\sigma_2}(T,\tau)y\| \leq \varepsilon + \Phi_T(x, y; \sigma_1, \sigma_2), \quad \forall\, (x,y) \in B_0,\ \forall\, \sigma_1, \sigma_2 \in \Sigma,$$

then $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact in X.

Lemma 2.5 ([11]) *Let $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ be a family of processes satisfying the translation identity (2.3) and (2.4) on a Banach space X. Then $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ has a compactly uniform (w.r.t. $\sigma \in \Sigma$) attractor in X if and only if*

(i) $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ has bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set $B_0 \subset X$;

(ii) $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact in X.

3. Existence of uniform attractors

In the introduction part of this paper, the problem (1.1) has been transformed the equivalent systems (1.11)–(1.14). Following this transformation, our subsequent analysis focuses directly on the system (1.11)–(1.14).

3.1. Well-posedness of solution

In this subsection, we will establish the well-posedness of problems (1.11)–(1.14).

The well-posedness of the solutions to problems (1.11)–(1.14) can be obtained by the Faedo-Galerkin method [5, 7, 12, 13]. Here we give the following theorem directly instead of proving it in detail. Denote $\mathbb{R}_\tau = [\tau, +\infty)$.

Theorem 3.1 *Assume that assumptions (1)–(5) hold and initial value* $(u_0, u_1, \eta_0) \in \mathcal{H}$*, then the problems (1.11)–(1.14) have a unique solution* $\big(u, u_t, \eta\big) \in C([0,T];\mathcal{H})$ *satisfying*

$$u \in L^\infty(0,T;V_2),\ u_t \in L^\infty(0,T;H),\ \eta \in L^\infty(0,T;L^2_\mu(\mathbb{R}^+,V_2)),$$

for any $T > 0$*. The solution* $(u(t), u_t(t), \eta)$ *depends continuously on the initial value* (u_0, u_1, η_0)*.*

Let $y(t) = (u(t), u_t(t), \eta^t(t))$ and $y_\tau = (u^0_\tau, u^1_\tau, \eta^0_\tau)$. We will transform the problems (1.11)–(1.14) into an evolutionary equation. Then the systems (1.11)–(1.14) are equivalent to the following system

$$\begin{cases} \partial_t u_t = -2\Delta^2 u + \Delta u_t - k u^+ - \int_0^\infty \mu(s)\Delta^2 \eta^t(x,s)\mathrm{d}s + \\ \quad \|\nabla u\|^m \Delta u - \delta f(u_t) - g(u) + h(x,t), \quad \forall\, t \geq \tau, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \\ u(x,\tau) = u^0_\tau(x), \\ u_t(x,\tau) = u^1_\tau(x), \\ \eta(x,s) = \eta^0_\tau, \end{cases} \tag{3.1}$$

and its operator form is

$$\partial_t y = A_{\sigma(t)}(y) = y_\tau, \tag{3.2}$$

where $\sigma(t) = h(x,t)$ is the symbol of Eq. (3.2).

We define the symbol space of (3.2), take a fixed symbol

$$\sigma_0(s) = h_0(x,s),\ h_0 \in L^\infty(\mathbb{R};L^2(\Omega)) \cap W^{1,r}_b(\mathbb{R};L^r(\Omega)),\ \text{ for } r > \frac{2q+2}{2q+1},$$

and set

$$\Sigma_0 = \{(x,t) \mapsto h_0(x,t+s) : s \in \mathbb{R}\}, \tag{3.3}$$

$$\Sigma \text{ is the weak closure space of } \Sigma_0 \text{ in } L^\infty(\mathbb{R};L^2(\Omega)) \cap W^{1,r}_b(\mathbb{R};L^r(\Omega)). \tag{3.4}$$

The following statement holds true.

Proposition 3.2 ([14]) *Σ is bounded in $L^\infty(\mathbb{R};L^2(\Omega)) \cap W^{1,r}_b(\mathbb{R};L^r(\Omega))$, and for any $\sigma \in \Sigma$, the following estimate holds*

$$\|\sigma\|_{L^\infty(\mathbb{R};L^2(\Omega))\cap W^{1,r}_b(\mathbb{R};L^r(\Omega))} \leq \|h_0\|_{L^\infty(\mathbb{R};L^2(\Omega))\cap W^{1,r}_b(\mathbb{R};L^r(\Omega))}.$$

Therefore, according to Theorem 3.1, we know that for all $\sigma \in \Sigma$ and the process $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, the problems (1.11)–(1.14) are well-posed, where $U_\sigma(t,\tau)y_\tau = y(t)$. Then $y(t)$ is the solution to the problems (1.11)–(1.14) with respect to $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ and satisfies conditions (2.1) and (2.2). Meanwhile, due to the uniqueness of the solution, it is known that $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ satisfies the translation identity (2.3) and (2.4). In the following, $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is denoted as the family of processes generated by (3.2)–(3.4).

3.2. The existence of uniformly bounded absorbing sets

The following theorem is concerned with the existence of uniformly bounded absorbing sets

of problems (1.11)–(1.14).

Theorem 3.3 *Assume that assumptions (1)–(5) hold and Σ is defined by (3.4), then the process family $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ corresponding to the problems (1.11)–(1.14) has a uniformly (w.r.t. $\sigma\in\Sigma$) bounded absorbing set B_0 in $\mathcal{H}$.*

Proof Taking the inner product with u_t of Eq. (1.11) in $L^2(\Omega)$ and using (1.12), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t)+\|\nabla u_t\|^2+(\eta,\eta_s)_{\mu,2}+\delta(f(u_t),u_t)=0, \tag{3.5}$$

where

$$I(t)=\frac{1}{2}\|u_t\|^2+\|\Delta u\|^2+\frac{1}{m+2}\|\nabla u\|^{m+2}+\frac{k}{2}\|u^+\|^2+\frac{1}{2}\|\eta\|^2_{\mu,2}+\int_\Omega(G(u)-h(t)u)\mathrm{d}x. \tag{3.6}$$

By (1.18), we derive

$$(\eta,\eta_s)_{\mu,2}\geq\frac{\delta_2}{2}\|\eta\|^2_{\mu,2}. \tag{3.7}$$

Combining with (3.5)–(3.7) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t)+\|\nabla u_t\|^2+\frac{\delta_2}{2}\|\eta\|^2_{\mu,2}+\delta(f(u_t),u_t)\leq 0. \tag{3.8}$$

Integrating (3.8) over the interval $[0,t]$, we obtain

$$I(t)+\int_0^t\|\nabla u_s\|^2\mathrm{d}s+\int_0^t\frac{\delta_2}{2}\|\eta\|^2_{\mu,2}\mathrm{d}s+\int_0^t\delta(f(u_s),u_s)\mathrm{d}s\leq I(0). \tag{3.9}$$

From (2), we conclude that $\delta(f(u_t),u_t)>0$, thus

$$\int_0^t\|\nabla u_s\|^2\mathrm{d}s+\int_0^t\frac{\delta_2}{2}\|\eta\|^2_{\mu,2}\mathrm{d}s+\int_0^t\delta(f(u_s),u_s)\mathrm{d}s>0. \tag{3.10}$$

According to (3.9) and (3.10), for any $t>0$, we obtain

$$I(t)\leq I(0). \tag{3.11}$$

Applying (1.6) and Poincaré inequality, there exist constants λ and $k_1>0$ such that

$$|(g(u),u)|<\lambda_1\|u\|^2<\frac{\lambda}{2}\|u\|^2+k_1|\Omega| \tag{3.12}$$

and

$$\int_\Omega G(u)\mathrm{d}x<\frac{\lambda_1}{2}\|u\|^2<\frac{\lambda}{4\lambda_1}\|\Delta u\|^2+k_1|\Omega|, \tag{3.13}$$

where $\lambda_1<\lambda<\frac{3}{2}\lambda_1$, $|\Omega|$ is the measure of Ω. Using Young inequality, there exists a constant $\alpha>0$ such that

$$\Big|\int_\Omega hu\mathrm{d}x\Big|\leq\alpha\|u\|^2+\frac{1}{4\alpha}\|h\|^2. \tag{3.14}$$

Exploiting the previous estimates, there exists a constant λ such that $0<\lambda+4\alpha<3\lambda_1-2k$, then we conclude that

$$I(0)\geq I(t)\geq\frac{1}{2}\|u_t\|^2+\frac{1}{4}\|\Delta u\|^2+\frac{1}{m+2}\|\nabla u\|^{m+2}+\frac{1}{2}\|\eta\|^2_{\mu,2}-k_1|\Omega|-\frac{1}{4\alpha}\|h(t)\|^2. \tag{3.15}$$

Taking the inner product with $v = u_t + \varepsilon u$ of Eq. (1.11) in $L^2(\Omega)$ and using (3.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) + H(t) \leq 0, \tag{3.16}$$

where

$$\begin{aligned} W(t) =& \frac{1}{2}\|v\|^2 - \frac{\varepsilon^2}{2}\|u\|^2 + \|\Delta u\|^2 + \frac{1}{m+2}\|\nabla u\|^{m+2} + \\ & \frac{k}{2}\|u^+\|^2 + \frac{\varepsilon}{2}\|\nabla u\|^2 + \frac{1}{2}\|\eta\|^2_{\mu,2} + \int_\Omega G(u)\mathrm{d}x, \end{aligned} \tag{3.17}$$

$$\begin{aligned} H(t) =& -\varepsilon\|u_t\|^2 + 2\varepsilon\|\Delta u\|^2 + \varepsilon\|\nabla u\|^{m+2} + \|\nabla u_t\|^2 + k\varepsilon\|u^+\|^2 + \\ & \frac{\delta_2}{2}\|\eta\|^2_{\mu,2} + \varepsilon(\eta, u)_{\mu,2} + \varepsilon(g(u), u) - (h, u_t) - \varepsilon(h, u). \end{aligned} \tag{3.18}$$

Now we estimate $W(t)$. Combining with (3.13) and the Poincaré inequality, we take ε small enough such that

$$-\frac{\varepsilon^2}{2\lambda_1} - \frac{\lambda}{4\lambda_1} - \frac{k}{2\lambda_1} + 1 > \frac{1}{2},$$

then

$$W(t) \geq \frac{1}{2}\|v\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{m+2}\|\nabla u\|^{m+2} + \frac{\varepsilon}{2}\|\nabla u\|^2 + \frac{1}{2}\|\eta\|^2_{\mu,2} - k_1|\Omega|. \tag{3.19}$$

Next, we estimate $H(t)$. By (1.17) and the Young inequality, we derive

$$|\ \varepsilon(\eta, u)_{\mu,2}\ | \leq \frac{\delta_1}{4}\|\eta\|^2_{\mu,2} + \varepsilon^2\|\Delta u\|^2, \tag{3.20}$$

$$|\ (h, u_t)\ | \leq \frac{\varepsilon}{2}\|h\|^2 + \frac{2}{\varepsilon}\|u_t\|^2. \tag{3.21}$$

Substituting (3.12), (3.20) and (3.21) into (3.18), then there exists a constant k_2 (dependent on k_1 and ε) such that

$$\begin{aligned} \frac{H(t)}{k_2} \geq& (-\varepsilon - \frac{2}{\varepsilon})\|u_t\|^2 - \frac{\lambda\varepsilon}{2}\|u\|^2 + (2\varepsilon - \varepsilon^2 - \frac{k\varepsilon}{\lambda_1} - \frac{\varepsilon}{2\lambda_1})\|\Delta u\|^2 + \\ & \varepsilon\|\nabla u\|^{m+2} + \|\nabla u_t\|^2 + \frac{\delta_1}{4}\|\eta\|^2_{\mu,2} - \varepsilon\|h(t)\|^2. \end{aligned} \tag{3.22}$$

Further, integrating (3.16) over $[0, t]$, we have

$$W(t) + \int_0^t H(s)\mathrm{d}s \leq W(0). \tag{3.23}$$

Combining with (3.19), (3.22) and (3.23), it yields

$$\begin{aligned} & \frac{1}{2}\|v\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{m+2}\|\nabla u\|^{m+2} + \frac{\varepsilon}{2}\|\nabla u\|^2 + \frac{1}{2}\|\eta\|^2_{\mu,2} - \\ & \varepsilon k_2\int_0^t \|u_s\|^2\mathrm{d}s + k_2\int_0^t \|\nabla u_s\|^2\mathrm{d}s - k_2\int_0^t \frac{\lambda\varepsilon}{2}\|u(s)\|^2\mathrm{d}s + \\ & k_2(2\varepsilon - \varepsilon^2 - \frac{\varepsilon}{2\lambda_1} - \frac{k\varepsilon}{\lambda_1})\int_0^t \|\Delta u(s)\|^2\mathrm{d}s - \\ & \varepsilon k_2\int_0^t \|h(s)\|^2_{L^2_b}\mathrm{d}s + k_2\varepsilon\int_0^t \|\nabla u(s)\|^{m+2}\mathrm{d}s + \frac{k_2\delta_1}{4}\int_0^t \|\eta\|^2_{\mu,2}\mathrm{d}s \end{aligned}$$

$$\leq \frac{1}{2}\|v_0\|^2+\frac{1}{2}\|\Delta u_0\|^2+\frac{1}{m+2}\|\nabla u_0\|^{m+2}+\frac{\varepsilon}{2}\|\nabla u_0\|^2+\frac{1}{2}\|\eta_0\|^2_{\mu,2}. \tag{3.24}$$

We note that

$$\frac{1}{m+2}\|\nabla u\|^{m+2}+\frac{\varepsilon}{2}\|\nabla u\|^2+k_2\varepsilon\int_0^t\|\nabla u(s)\|^{m+2}\mathrm{d}s+\frac{k_2\delta_1}{4}\int_0^t\|\eta\|^2_{\mu,2}\mathrm{d}s>0. \tag{3.25}$$

Applying Poincaré inequality and choosing ε small enough such that $\varepsilon^2+(\frac{1+2k}{2\lambda_1}-\frac{3}{2})\varepsilon\leq 0$ and $0<\varepsilon<\lambda_1^{\frac{1}{2}}$, then we deduce that

$$k_2(2\varepsilon-\varepsilon^2-\frac{\varepsilon}{2\lambda_1}-\frac{k\varepsilon}{\lambda_1})\int_0^t\|\Delta u(s)\|^2\mathrm{d}s-k_2\varepsilon\int_0^t\|u_s\|^2\mathrm{d}s+$$
$$k_2\int_0^t\|\nabla u_s\|^2\mathrm{d}s-\frac{k_2\varepsilon\lambda_1}{2}\int_0^t\|u(s)\|^2\mathrm{d}s\geq 0. \tag{3.26}$$

Therefore, from (3.25) and (3.26), it is easy to get that

$$\frac{1}{2}\|v\|^2+\frac{1}{2}\|\Delta u\|^2+\frac{1}{2}\|\eta\|^2_{\mu,2}\leq\frac{1}{2}\|v_0\|^2+\frac{1}{2}\|\Delta u_0\|^2+\frac{1}{m+2}\|\nabla u_0\|^{m+2}+\frac{\varepsilon}{2}\|\nabla u_0\|^2+$$
$$\frac{1}{2}\|\eta_0\|^2_{\mu,2}+k_2\varepsilon\int_0^t\|h(s)\|^2_{L_b^2}\mathrm{d}s. \tag{3.27}$$

By (3.15), there exists $M_0>0$ such that

$$\frac{1}{2}\|v_0\|^2+\frac{1}{2}\|\Delta u_0\|^2+\frac{1}{m+2}\|\nabla u_0\|^{m+2}+\frac{\varepsilon}{2}\|\nabla u_0\|^2+\frac{1}{2}\|\eta_0\|^2_{\mu,2}\leq M_0.$$

Therefore, there exists $k_3>\max\{2,\frac{2\varepsilon^2}{\lambda_1}+1\}$ such that

$$\|u_t\|^2+\|\Delta u\|^2+\|\eta\|^2_{\mu,2}\leq k_3(\|v\|^2+\|\Delta u\|^2+\|\eta\|^2_{\mu,2})+\tilde{k}\int_0^t\|h(s)\|^2_{L_b^2}\mathrm{d}s\leq R_1, \tag{3.28}$$

where $\tilde{k}$ depends on k_2, k_3, ε.

By the definition of Σ, for all $\sigma\in\Sigma$, we have

$$\|\sigma\|^2_{L_b^2}\leq\|h_0\|^2_{L_b^2}.$$

Then there is

$$\|h\|^2_{L_b^2}\leq\|h_0\|^2_{L_b^2},\quad\text{for all } h\in\Sigma.$$

Thus, the uniform (w.r.t. $\sigma\in\Sigma$) absorbing set B_0 in $\mathcal{H}$ is

$$B_0=\{y=(u,u_t,\eta^t)|\|y\|^2_{\mathcal{H}}\leq R_1\}.$$

That is, for any bounded subset B in $\mathcal{H}$, there exists $t_0=t_0(\tau,B)\geq\tau$ such that

$$\bigcup_{\sigma\in\Sigma}U_\sigma(t,\tau)B\subset B_0,\quad\forall\, t\geq t_0.$$

This completes the proof. □

3.3. Uniformly asymptotically compact

In order to obtain the uniformly asymptotic compactness of system, we need the following results.

Proposition 3.4 ([14]) *Let $h \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}; L^r(\Omega))$, $r > \frac{2q+2}{2q+1}$. Then there exists a constant $K > 0$ such that*

$$\sup_{t\in R} \|h(x, t+s)\|_{L^2(\Omega)} \le K, \text{ for all } s \in R.$$

Proposition 3.5 ([14]) *Let $s_i \in \mathbb{R}$ $(i = 1, 2, \ldots)$, $h \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}; L^r(\Omega))$ $(r > \frac{2q+2}{2q+1})$, $\{u_n(t)|t \ge 0,\ n = 1, 2, \ldots\}$ be bounded in $H^2(\Omega) \cap H^1(\Omega)$, and for any $T_1 > 0$, $\{u_{n_t}(t)|n = 1, 2, \ldots\}$ is bounded $L^\infty(0, T_1; L^2(\Omega))$. Then for any $T > 0$, there exist subsequences $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and $\{s_{n_k}\}_{k=1}^\infty$ of $\{s_n\}_{n=1}^\infty$, respectively, such that*

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T\int_s^T\int_\Omega (h(x, \tau + s_{n_k}) - (h(x, \tau + s_{n_l}))(u_{n_k} - u_{n_l})_t(\tau)\mathrm{d}x\mathrm{d}s\mathrm{d}\tau = 0.$$

Theorem 3.6 *Assume that assumptions (1)–(5) hold and Σ is defined by (3.4), then the process family $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ corresponding to the problems (1.11)–(1.14) is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact in $\mathcal{H}$.*

Proof Let $(u(t), u_t(t), \eta^t(t))$ and $(v(t), v_t(t), \zeta^t(t))$ be two weak solutions of problems (1.11)–(1.14) corresponding to the symbols $\sigma_1(t) = h_1(x, t)$ and $\sigma_2(t) = h_2(x, t)$, and they depend on the initial value $(u_0, u_1, \eta_0(\cdot))$, $(v_0, v_1, \zeta_0(\cdot)) \in B_0$, respectively. B_0 is the uniformly (w.r.t. $\sigma \in \Sigma$) bounded absorbing set obtained in Theorem 3.3. Let $w(t) = u(t) - v(t)$, $\xi^t(\cdot) = \eta^t(\cdot) - \zeta^t(\cdot)$. Then $w(t), \xi^t(\cdot)$ satisfy the following equation

$$\begin{aligned} &w_{tt} + 2\Delta^2 w - \|\nabla u\|^m \Delta u + \|\nabla v\|^m \Delta v - \Delta w_t + \int_0^\infty \mu(s)\Delta^2 \xi^t(x, s)\mathrm{d}s + \\ &k(u^+ - v^+) + \delta f(u_t) - \delta f(v_t) + g(u) - g(v) = h_1(t) - h_2(t), \end{aligned} \tag{3.29}$$

$$\xi_t^t = -\xi_s^t + w_t, \tag{3.30}$$

where $(w(0), w_t(0), \xi^0(\cdot)) = (u_0, u_1, \eta_0(\cdot)) - (v_0, v_1, \zeta_0(\cdot))$. We define

$$E_w(t) = \frac{1}{2}\|\Delta w(t)\|^2 + \frac{1}{2}\|w_t(t)\|^2 + \frac{1}{2}\|\xi^t\|_{\mu,2}^2. \tag{3.31}$$

Taking the inner product in $L^2(\Omega)$ of (3.29) with w and integrating over $[0, T]$, we obtain

$$\begin{aligned} 2\int_0^T \|\Delta w\|^2 \mathrm{d}\tau \le &-\int_\Omega w_t(T)w(T)\mathrm{d}x + \int_\Omega w_t(0)w(0)\mathrm{d}x + \int_0^T \|w_t\|^2\mathrm{d}\tau + \varepsilon_0\int_0^T \|\xi^t\|_{\mu,2}^2\mathrm{d}\tau + \\ &Lk(\|w(T)\|^2 - \|w(0)\|^2) - \int_0^T \|\nabla u\|^m\|\nabla w\|^2\mathrm{d}\tau - \int_0^T\int_\Omega (\|\nabla u\|^m - \|\nabla v\|^m)\nabla v\nabla w\mathrm{d}x\mathrm{d}\tau + \\ &\delta_1\int_0^T \|\Delta w\|^2\mathrm{d}\tau - \frac{1}{2}\|\nabla w(T)\|^2 + \frac{1}{2}\|\nabla w(0)\|^2 - \int_0^T\int_\Omega (\delta f(u_t) - \delta f(v_t))w\mathrm{d}x\mathrm{d}\tau - \\ &\int_0^T\int_\Omega (g(u) - g(v))w\mathrm{d}x\mathrm{d}\tau - \int_0^T\int_\Omega (h_1 - h_2)w\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{3.32}$$

The above estimates utilize the inequality $|u_1^+ - u_2^+| \le L|u_1 - u_2| = L|w|$ and

$$\Big|\int_0^T (\xi^t, w)_{\mu,2}\mathrm{d}\tau\Big| \le \varepsilon_0\int_0^T \|\xi^t\|_{\mu,2}^2\mathrm{d}\tau + \delta_1\int_0^T \|\Delta w\|^2\mathrm{d}\tau,$$

where $L>0$, $\varepsilon_0>0$.

Next, taking the inner product in L^2 of (3.29) with w_t and combining with (3.7) and (3.31), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_w(t)+\frac{\delta_2}{2}\|\xi^t\|^2_{\mu,2}+\|\nabla w_t\|^2+\int_\Omega(\|\nabla u\|^m-\|\nabla v\|^m)\nabla v\nabla w_t\mathrm{d}x+\int_\Omega\|\nabla u\|^m\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nabla w)^2\mathrm{d}x+$$
$$\int_\Omega(g(u)-g(v))w_t\mathrm{d}x-\frac{kL}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2+\int_\Omega(\delta f(u_t)-\delta f(v_t))w_t\mathrm{d}x\le\int_\Omega(h_1-h_2)w_t\mathrm{d}x. \tag{3.33}$$

Integrating (3.33) over $[s,T]$ and then integrating the result over $[0,T]$, it follows that

$$TE_w(T)+\frac{\delta_2}{2}\int_0^T\int_s^T\|\xi^t\|^2_{\mu,2}\mathrm{d}s\mathrm{d}\tau+\int_0^T\int_s^T\|\nabla w_t\|^2\mathrm{d}s\mathrm{d}\tau-\frac{kL}{2}\int_0^T\|w(T)\|^2\mathrm{d}\tau+$$
$$\int_0^T\int_s^T\int_\Omega(\|\nabla u\|^m-\|\nabla v\|^m)\nabla v\nabla w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau+\frac{1}{2}\int_0^T\int_\Omega\|\nabla u(T)\|^m(\nabla w(T))^2\mathrm{d}x\mathrm{d}\tau-$$
$$\frac{m}{2}\int_0^T\int_\Omega\int_s^T\|\nabla u\|^{m-1}\nabla u_t(\nabla w)^2\mathrm{d}s\mathrm{d}x\mathrm{d}\tau-\frac{1}{2}\int_0^T\int_\Omega\|\nabla u(s)\|^m(\nabla w(s))^2\mathrm{d}x\mathrm{d}\tau+$$
$$\int_0^T\int_s^T\int_\Omega(\delta f(u_t)-\delta f(v_t))w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau+\int_0^T\int_s^T\int_\Omega(g(u)-g(v))w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau-$$
$$\int_0^T\int_s^T\int_\Omega(h_1-h_2)w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau+\frac{kL}{2}\int_0^T\|w(s)\|^2\mathrm{d}\tau\le\int_0^T E_w(s)\mathrm{d}\tau. \tag{3.34}$$

Note that

$$\frac{\delta_2}{2}\int_0^T\int_s^T\|\xi^t\|^2_{\mu,2}\mathrm{d}s\mathrm{d}\tau+\int_0^T\int_s^T\|\nabla w_t\|^2\mathrm{d}s\mathrm{d}\tau+\frac{1}{2}\int_0^T\int_\Omega\|\nabla u(T)\|^m(\nabla w(T))^2\mathrm{d}x\mathrm{d}\tau\ge 0. \tag{3.35}$$

From (1.3) and (1.4), there exist $C_{\delta,M}>0$ and $C_\delta>0$ (dependent on δ) such that

$$\int_0^T\int_s^T\int_\Omega(\delta f(u_t)-\delta f(v_t))w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau\ge C_{\delta,M}\int_0^T\int_s^T\int_\Omega w_t^2\mathrm{d}x\mathrm{d}s\mathrm{d}\tau\ge 0, \tag{3.36}$$

$$\int_0^T\int_\Omega(\delta f(u_t)-\delta f(v_t))w_t\mathrm{d}x\mathrm{d}\tau\ge C_\delta+\int_0^T\|w_t\|^2\mathrm{d}\tau. \tag{3.37}$$

Combining with (3.35)–(3.37), it is easy to get that

$$TE_w(T)\le\int_0^T E_w(s)\mathrm{d}\tau+\frac{kL}{2}\int_0^T\|w(T)\|^2\mathrm{d}\tau-\frac{kL}{2}\int_0^T\|w(s)\|^2\mathrm{d}\tau+$$
$$\frac{1}{2}\int_0^T\int_\Omega\|\nabla u(s)\|^m(\nabla w(s))^2\mathrm{d}x\mathrm{d}\tau+\frac{m}{2}\int_0^T\int_\Omega\int_s^T\|\nabla u\|^{m-1}\nabla u_t(\nabla w)^2\mathrm{d}s\mathrm{d}x\mathrm{d}\tau-$$
$$\int_0^T\int_s^T\int_\Omega(\|\nabla u\|^m-\|\nabla v\|^m)\nabla v\nabla w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau-\int_0^T\int_s^T\int_\Omega(g(u)-g(v))w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau+$$
$$\int_0^T\int_s^T\int_\Omega(h_1-h_2)w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau. \tag{3.38}$$

Further, integrating (3.33) over $[0,T]$ and combining with (3.37), it yields

$$E_w(T)+\int_0^T\|w_t\|^2\mathrm{d}\tau+\frac{\delta_2}{2}\int_0^T\|\xi^t\|^2_{\mu,2}\mathrm{d}\tau+\int_0^T\|\nabla w_t\|^2\mathrm{d}\tau$$

$$\leq -\int_0^T\int_\Omega(\|\nabla u\|^m-\|\nabla v\|^m)\nabla v\nabla w_t\mathrm{d}x\mathrm{d}\tau-\frac{1}{2}\int_\Omega\|\nabla u(T)\|^m(\nabla w(T))^2\mathrm{d}x+\frac{kL}{2}\|w(T)\|^2-$$
$$\frac{kL}{2}\|w(0)\|^2+\frac{m}{2}\int_\Omega\int_0^T\|\nabla u\|^{m-1}\nabla u_t(\nabla w)^2\mathrm{d}\tau\mathrm{d}x+\frac{1}{2}\int_\Omega\|\nabla u(0)\|^m(\nabla w(0))^2\mathrm{d}x-$$
$$\int_0^T\int_\Omega(g(u)-g(v))w_t\mathrm{d}x\mathrm{d}\tau+\int_0^T\int_\Omega(h_1-h_2)w_t\mathrm{d}x\mathrm{d}\tau-C_\delta+E_w(0). \tag{3.39}$$

Applying $E_w(T)+\int_0^T\|\nabla w_t\|^2\mathrm{d}\tau\geq 0$, (3.31)–(3.32) and (3.39), we conclude that

$$\int_0^T E_w(t)\mathrm{d}\tau\leq(2-\delta_1)\int_0^T\|\Delta w\|^2\mathrm{d}\tau+\frac{1}{2}\int_0^T\|w_t\|^2\mathrm{d}\tau+(1-\varepsilon_0)\int_0^T\|\xi^t_{\mu,2}\|^2\mathrm{d}\tau$$
$$\leq -C_{2,\alpha}\int_0^T\int_\Omega(\|\nabla u\|^m-\|\nabla v\|^m)\nabla v\nabla w_t\mathrm{d}x\mathrm{d}\tau-\frac{C_{2,\alpha}}{2}\int_\Omega\|\nabla u(T)\|^m(\nabla w(T))^2\mathrm{d}x+$$
$$\frac{kLC_{2,\alpha}}{2}\|w(T)\|^2-\frac{kLC_{2,\alpha}}{2}\|w(0)\|^2+\frac{mC_{2,\alpha}}{2}\int_\Omega\int_0^T\|\nabla u\|^{m-1}\nabla u_t(\nabla w)^2\mathrm{d}\tau\mathrm{d}x+$$
$$\frac{C_{2,\alpha}}{2}\int_\Omega\|\nabla u(0)\|^m(\nabla w(0))^2\mathrm{d}x-C_{2,\alpha}C_\delta-C_{2,\alpha}\int_0^T\int_\Omega(g(u)-g(v))w_t\mathrm{d}x\mathrm{d}\tau+$$
$$C_{2,\alpha}E_w(0)+C_{2,\alpha}\int_0^T\int_\Omega(h_1-h_2)w_t\mathrm{d}x\mathrm{d}\tau-\int_0^T\|\nabla u\|^m\|\nabla w\|^2\mathrm{d}\tau-\int_\Omega w_t(T)w(T)\mathrm{d}x+$$
$$\int_\Omega w_t(0)w(0)\mathrm{d}x-\int_0^T\int_\Omega(\|\nabla u\|^m-\|\nabla v\|^m)\nabla v\nabla w\mathrm{d}x\mathrm{d}\tau+kL\|w(T)\|^2-kL\|w(0)\|^2-$$
$$\frac{1}{2}\|\nabla w(T)\|^2+\frac{1}{2}\|\nabla w(0)\|^2-\int_0^T\int_\Omega(\delta f(u_t)-\delta f(v_t))w\mathrm{d}x\mathrm{d}\tau-$$
$$\int_0^T\int_\Omega(g(u)-g(v))w\mathrm{d}x\mathrm{d}\tau+\int_0^T\int_\Omega(h_1-h_2)w\mathrm{d}x\mathrm{d}\tau, \tag{3.40}$$

where $C_{2,\alpha}>0$ is a constant with respect to δ_2. Exploiting (1.3), (1.4) and Hölder inequality, we have

$$\Big|\int_0^T\int_\Omega(\delta f(u_t)-\delta f(v_t))w\mathrm{d}x\mathrm{d}\tau\Big|\leq 2C_{\delta,p}C_pC_{\alpha_1}\Big(\int_0^T\int_\Omega|w|^{p+1}\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{p+1}}. \tag{3.41}$$

It is evident that $0<s<t<T$, then substituting (3.41) into (3.40) and combining with (3.38), we obtain

$$E_w(T)\leq\frac{C_E}{T}+\frac{1}{T}\Phi_T((u_0,u_1,\eta_0(\cdot)),(v_0,v_1,\zeta_0(\cdot))),$$

where

$$C_E=\frac{kLC_{2,\alpha}}{2}\|w(T)\|^2-\frac{kLC_{2,\alpha}}{2}\|w(0)\|^2-\frac{C_{2,\alpha}}{2}\int_\Omega\|\nabla u(T)\|^m(\nabla w(T))^2\mathrm{d}x+$$
$$\frac{C_{2,\alpha}}{2}\int_\Omega\|\nabla u(0)\|^m(\nabla w(0))^2\mathrm{d}x-C_{2,\alpha}C_\delta+C_{2,\alpha}E_w(0)-\int_\Omega w_t(T)w(T)\mathrm{d}x+$$
$$\int_\Omega w_t(0)w(0)\mathrm{d}x-\frac{1}{2}\|\nabla w(T)\|^2+\frac{1}{2}\|\nabla w(0)\|^2+\frac{kL}{2}\int_0^T\|w(T)\|^2\mathrm{d}\tau+$$
$$kL\|w(T)\|^2-kL\|w(0)\|^2, \tag{3.42}$$

$$\Phi_T((u_0,u_1,\eta_0(\cdot)),(v_0,v_1,\zeta_0(\cdot)))=-C_{2,\alpha}\int_0^T\int_\Omega(\|\nabla u\|^m-\|\nabla v\|^m)\nabla v\nabla w_t\mathrm{d}x\mathrm{d}\tau-$$

$$\int_0^T\int_s^T\int_\Omega(\|\nabla u\|^m-\|\nabla v\|^m)\nabla v\nabla w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau-\int_0^T\int_\Omega(g(u)-g(v))w\mathrm{d}x\mathrm{d}\tau-$$
$$\int_0^T\|\nabla u\|^m\|\nabla w\|^2\mathrm{d}\tau-\int_0^T\int_\Omega(\|\nabla u\|^m-\|\nabla v\|^m)\nabla v\nabla w\mathrm{d}x\mathrm{d}\tau+$$
$$\frac{m}{2}\int_0^T\int_\Omega\int_s^T\|\nabla u\|^{m-1}\nabla u_t(\nabla w)^2\mathrm{d}s\mathrm{d}x\mathrm{d}\tau+\frac{mC_{2,\alpha}}{2}\int_\Omega\int_0^T\|\nabla u\|^{m-1}\nabla u_t(\nabla w)^2\mathrm{d}\tau\mathrm{d}x-$$
$$C_{2,\alpha}\int_0^T\int_\Omega(g(u)-g(v))w_t\mathrm{d}x\mathrm{d}\tau-\int_0^T\int_s^T\int_\Omega(g(u)-g(v))w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau+$$
$$\int_0^T\int_s^T\int_\Omega(h_1-h_2)w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau+\frac{1}{2}\int_0^T\int_\Omega\|\nabla u(s)\|^m(\nabla w(s))^2\mathrm{d}x\mathrm{d}\tau+$$
$$C_{2,\alpha}\int_0^T\int_\Omega(h_1-h_2)w_t\mathrm{d}x\mathrm{d}\tau+2C_{\delta,p}C_pC_{\alpha_1}\Big(\int_0^T\int_\Omega|w|^{p+1}\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{P+1}}+$$
$$\int_0^T\int_\Omega(h_1-h_2)w\mathrm{d}x\mathrm{d}\tau-\frac{kL}{2}\int_0^T\|w(s)\|^2\mathrm{d}\tau. \tag{3.43}$$

Since the process family $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ has a uniformly bounded absorbing set, for any $\nu>0$, choosing T large enough such that

$$\frac{C_E}{T}\le\nu,\quad T\to\infty. \tag{3.44}$$

According to Lemma 2.4, it now suffices to prove that Φ_T is a contraction function, that is, $\Phi_T(\cdot,\cdot)\in\mathrm{contr}(B_0,\Sigma)$. Let the sequence $(u_n,\partial_t u_n,\eta_n)$ be the solutions to the problems (1.11)–(1.14) corresponding to the initial data $(u_0^n,u_1^n,\eta_0^n)\in B_0$. Since B_0 is a bounded positively invariant set obtained in Theorem 3.3, by Alaoglu's theorem, without loss of generality, we assume that

$$u_n\to u\text{ weak star in }L^\infty(0,T;V_2), \tag{3.45}$$
$$\partial_t u_n\to\partial_t u\text{ weak star in }L^\infty(0,T;L^2), \tag{3.46}$$
$$u_n\to u\text{ strongly in }L^2(0,T;L^2), \tag{3.47}$$
$$u_n\to u\text{ strongly in }L^{p+1}(0,T;L^{p+1}), \tag{3.48}$$

where $V_2\hookrightarrow\hookrightarrow L^{p+1}$. In the following we deal with each term of (3.43).

Firstly, by (3.45)–(3.48), it follows that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega\|\nabla u_n\|^{m-1}(\nabla\partial_t u_n)\cdot(\nabla u_n-\nabla u_m)^2\mathrm{d}x\mathrm{d}\tau=0,$$
$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega(\|\nabla u_n\|^m-\|\nabla u_m\|^m)\cdot\nabla u_m(\nabla\partial_t u_n-\nabla\partial_t u_m)\mathrm{d}x\mathrm{d}\tau=0,$$
$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega\|\nabla u_n\|^m\cdot(\nabla u_n-\nabla u_m)^2\mathrm{d}x\mathrm{d}\tau=0,$$
$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega(\|\nabla u_n\|^m-\|\nabla u_m\|^m)\cdot\nabla u_m(\nabla u_n-\nabla u_m)\mathrm{d}x\mathrm{d}\tau=0.$$

Secondly, from (3.47) and (3.48), we deduce that

$$\lim_{n\to\infty}\lim_{m\to\infty}\Big(\int_0^T\int_\Omega|u_m-u_n|^{p+1}\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{p+1}}=0.$$

Combining with (1.5), (1.6) and (3.46), it yields

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (g(u_n)-g(u_m))\cdot(\partial_t u_n-\partial_t u_m)\mathrm{d}x\mathrm{d}\tau=0,$$
$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (g(u_n)-g(u_m))(u_n-u_m)\mathrm{d}x\mathrm{d}\tau=0.$$

From Proposition 3.5 and (3.48), we derive that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (h_n-h_m)\cdot(\partial_t u_n-\partial_t u_m)\mathrm{d}x\mathrm{d}\tau=0,$$
$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (h_n-h_m)(u_n-u_m)\mathrm{d}x\mathrm{d}\tau=0,$$
$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_s^T\int_\Omega (h_n-h_m)\cdot(\partial_t u_n-\partial_t u_m)\mathrm{d}x\mathrm{d}\tau=0.$$

Finally, for each fixed t, using the Lebesgue control convergence theorem, we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_s^T\int_\Omega (g(u_n)-g(u_m))\cdot(\partial_t u_n-\partial_t u_m)\mathrm{d}x\mathrm{d}s\mathrm{d}\tau$$
$$=\int_0^T\Big(\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_\Omega (g(u_n)-g(u_m))\cdot(\partial_t u_n-\partial_t u_m)\mathrm{d}x\mathrm{d}s\Big)\mathrm{d}\tau=0.$$

Analogously,

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_s^T\int_\Omega (\|\nabla u\|^m-\|\nabla v\|^m)\cdot\nabla v\nabla w_t\mathrm{d}x\mathrm{d}s\mathrm{d}\tau$$
$$=\int_0^T\Big(\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_\Omega (\|\nabla u\|^m-\|\nabla v\|^m)\cdot\nabla v\nabla w_t\mathrm{d}x\mathrm{d}s\Big)\mathrm{d}\tau=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_s^T\int_\Omega \|\nabla u\|^{m-1}\cdot\nabla u_t(\nabla w)^2\mathrm{d}x\mathrm{d}s\mathrm{d}\tau$$
$$=\int_0^T\Big(\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_\Omega \|\nabla u\|^{m-1}\cdot\nabla u_t(\nabla w)^2\mathrm{d}x\mathrm{d}s\Big)\mathrm{d}\tau=0.$$

In summary, $\Phi_T(\cdot,\cdot)\in\mathrm{contr}(B_0,\Sigma)$, which completes the proof. □

3.4. Existence of uniform attractors

In this subsection, we prove the existence of uniform attractors of problems (1.11)–(1.14).

Theorem 3.7 *Assume that assumptions (1)–(5) hold and Σ is defined by (3.4), then the process family $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ corresponding to the problems (1.11)–(1.14) possesses a compact uniform (w.r.t. $\sigma\in\Sigma$) attractor $\mathcal{A}_\Sigma$.*

Proof Theorems 3.3 and 3.6 imply the existence of a uniform attractor. □

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