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Global Stability of Traveling Wavefronts for a Belousov-Zhabotinsky Model with Mixed Nonlocal and Degenerate Diffusions

Yuting YANG, Guobao ZHANG*
College of Mathematics and Statistics, Northwest Normal University, Gansu 730070*, P. R. China*

Abstract In this paper, we are concerned with the stability of traveling wavefronts of a Belousov-Zhabotinsky model with mixed nonlocal and degenerate diffusions. Such a system can be used to study the competition among nonlocally diffusive species and degenerately diffusive species. We prove that the traveling wavefronts are exponentially stable, when the initial perturbation around the traveling waves decays exponentially as $x \to -\infty$, but in other locations, the initial data can be arbitrarily large. The adopted methods are the weighted energy with the comparison principle and squeezing technique.

Keywords Belousov-Zhabotinsky model; nonlocal diffusion; stability; comparison principle; weighted energy

MR(2020) Subject Classification 35K57; 35C07; 92D25

1. Introduction

In this paper, we study the following Belousov-Zhabotinskii reaction model with mixed nonlocal and degenerate diffusions [1]

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = J * u(x,t) - u(x,t) + u(x,t)[1 - u(x,t) - rv(x,t)], \\ \frac{\partial v(x,t)}{\partial t} = -bu(x,t)v(x,t) \end{cases} \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \tag{1.2}$$

where $x \in \mathbb{R}$, $t > 0$, the parameters r and b are positive. In this model, $u(x,t)$ is the population density of the invasive species and $v(x,t)$ is the population density of the local species. The invasive species u diffuses non-locally while the native species v is non-diffusive. This model is a nonlocal analog of the one studied in [2], which describes the competition between the precursor and differentiated cells. Since the reaction terms of model (1.1) are the same as that of the classical Belousov-Zhabotinskii model, we refer to it as Belousov-Zhabotinskii reaction model [3–10]. $J * u - u$ represents the nonlocal diffusion [11–14] which is defined by

$$J * u(x,t) - u(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)\,\mathrm{d}y - u(x,t).$$

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* Corresponding author
E-mail address: 17393346688@139.com (Yuting YANG); zhanggb2011@nwnu.edu.cn (Guobao ZHANG)

Throughout this paper, we always make the following assumptions on the kernel function J and the competition coefficients b and r:

(H1) $J(y) \geq 0$, $J(y) = J(-y)$, $\forall y \in \mathbb{R}$, $\int_{\mathbb{R}} J(y)\mathrm{d}y = 1$, and $J'(\cdot) \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} |y|J(y)\mathrm{d}y < +\infty$. Further assume that $\int_{\mathbb{R}} e^{\lambda y}J(y)\mathrm{d}y < +\infty$ for all $\lambda > 0$.

(H2) $r \in (0,1)$, $0 < b < 1 - r$.

The spatial homogeneous system of (1.1) is

$$\begin{cases} u'(t) = u(t)(1 - u(t) - rv(t)), \\ v'(t) = -bu(t)v(t), \end{cases}$$

which has constant equilibria: $(1,0)$ and $(0,\nu)$ with $\nu \in \mathbb{R}$. Since the invasion process is considered, we take $\nu = 1$. It can be seen from [1, Lemma 3.1] that when (H2) holds, $(0,1)$ is an unstable equilibrium and $(1,0)$ is a stable one.

Traveling wave solutions connecting two equilibria $(0,1)$ and $(1,0)$ can provide insight to the outcome in the competition between local species v and nonlocal species u. A traveling wave solution of model (1.1) is a special solution (u,v) with the form $(u(x,t),v(x,t)) = (\phi(\xi),\psi(\xi))$, where $\xi = x + ct$, $c > 0$ is the wave speed and (ϕ,ψ) is called the wave profile. If ϕ and ψ are monotone, we call (ϕ,ψ) a traveling wavefront. Substituting $(\phi(\xi),\psi(\xi))$ into (1.1) leads to the following wave profile system

$$\begin{cases} c\phi'(\xi) = J * \phi(\xi) - \phi(\xi) + \phi(\xi)[1 - \phi(\xi) - r\psi(\xi)], \\ c\psi'(\xi) = -b\phi(\xi)\psi(\xi). \end{cases} \tag{1.3}$$

Furthermore, the asymptotic boundary conditions are imposed

$$\lim_{\xi\to-\infty}(\phi(\xi),\psi(\xi)) = (0,1) \quad \text{and} \quad \lim_{\xi\to+\infty}(\phi(\xi),\psi(\xi)) = (1,0). \tag{1.4}$$

The existence of a traveling wave solution to model (1.1) implies the successful invasion of the nonlocal diffusion species u. In 2019, Huang and Hou [1] investigated traveling wavefronts of a class of nonlocal diffusion system with delay by using the monotone iteration scheme. As an application, they proved the existence and nonexistence of traveling wavefronts of (1.1), i.e., the solutions of (1.3) and (1.4). Moreover, they established the asymptotic behaviors of traveling wavefronts at infinity. More precisely,

Proposition 1.1 *Assume that (H1) and (H2) hold. Then there exists a $c^* > 0$ such that for each $c \geq c^*$, system (1.1) has a unique positive strictly monotone traveling wave solution. When $0 < c < c^*$, there is no monotone traveling wave solution.*

Remark 1.2 We remark that c^* is the minimal wave speed of traveling wavefronts, which can be given by

$$c^* = \min_{\lambda>0}\frac{1}{\lambda}\Big\{\Big(\int_{\mathbb{R}} J(y)e^{\lambda y}\mathrm{d}y - 1\Big) + (1 - r)\Big\}.$$

It is particularly important to mention the work of Pan, Li and Lin [15], where the existence of traveling wavefronts of (1.1) with delay and nonlocal diffusion for v are established.

To the best of our knowledge, there is no result on the stability of the traveling wavefronts of (1.1). Hence, in this paper, we are devoted to solving this problem. It is well known that the stability of traveling wave solutions for various evolution systems has been widely studied, see [3, 6, 16–21] for random diffusion systems, [22–26] for nonlocal diffusion systems. The main methods are the (technical) weighted energy method [20, 23], the sub- and supersolutions method and squeezing technique [27, 28], and the combination of the comparison principle and the weighted energy method [18, 22, 25, 26]. Since the nonlinearity of our model (1.1) has a special structure, by a simple transformation $u_1 = u$ and $u_2 = 1 - v$, the model (1.1) can be reduced into a cooperative model, see (2.1). Thus, the comparison principle works for the transformed model of (1.1). Therefore, in this paper, we take the weighted energy method together with the comparison principle to establish the stability of traveling wavefronts of (1.1). We should point out that the stability issue is only partially solved due to the presence of the nonlocal diffusion term. More precisely, we only prove the stability of traveling wavefronts with relatively large speeds. We will further address the remaining issues in the future works.

The rest of our paper is organized as follows. In Section 2, we introduce some preliminaries and the comparison principle on the solution of Cauchy problems (2.1) and (2.2), and then state our main theorem. In Section 3, we are devoted to proving the stability theorem.

2. Preliminaries and main results

In the following, $C > 0$ denotes a generic constant, while $C_i > 0$ $(i = 0, 1, 2, \ldots)$ represents a special constant. Let I be an interval, especially $I = \mathbb{R}$, $L^2(I)$ the space of the square integrable function on I, and $H^k(I)$ $(k \geq 0)$ the Sobolev space of the L^2-function $f(x)$ defined on I whose derivatives $\frac{\mathrm{d}^i}{\mathrm{d}x^i} f$, $i = 1, \ldots, k$, also belong to $L^2(I)$. $L^2_w(I)$ represents the weighted L^2 - space with the weight $w(x) > 0$ and its norm is defined by

$$\|f\|_{L^2_w} = \Big(\int_I w(x) f^2(x) \mathrm{d}x \Big)^{1/2}.$$

Since H^k_w is the weighted Sobolev space with the norm, we have

$$\|f\|_{H^k_w} = \Big(\sum_{i=0}^{k} \int_I w(x) |\frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)|^2 \mathrm{d}x \Big)^{1/2}.$$

Let $T > 0$ and B space. We denote by $C^0([0, T]; B)$ the space of the B-valued continuous functions on $[0, T]$, and $L^2([0, T]; B)$ as the space of B-valued L^2 -function on $[0, T]$. The corresponding spaces of the B-valued function on $[0, \infty)$ are defined similarly.

By a change of variables $u_1 = u$ and $u_2 = 1 - v$, model (1.1) with the initial condition (1.2) can reduce to the following cooperative model

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = J * u_1(x, t) - u_1(x, t) + u_1(x, t)[1 - r - u_1(x, t) + r u_2(x, t)], \\ \frac{\partial u_2(x,t)}{\partial t} = b u_1(x, t)[1 - u_2(x, t)] \end{cases} \tag{2.1}$$

with the initial condition

$$\begin{cases} u_1(x,0) = u_{10}(x) := u_0(x), & x \in \mathbb{R}, \\ u_2(x,0) = u_{20}(x) := 1 - v_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

Correspondingly, the equilibria $(0,1)$ and $(1,0)$ become, respectively, $(0,0)$ and $(1,1)$, and the wave profile system (1.3) becomes

$$\begin{cases} c\phi'(\xi) = J * \phi(\xi) - \phi(\xi) + \phi(\xi)[1 - r - \phi(\xi) + r\psi(\xi)], \\ c\psi'(\xi) = b\phi(\xi)[1 - \psi(\xi)] \end{cases} \tag{2.3}$$

with the asymptotic boundary conditions

$$\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = \mathbf{0} := (0,0), \quad \lim_{\xi \to +\infty} (\phi(\xi), \psi(\xi)) = \mathbf{K} := (1,1). \tag{2.4}$$

Since the model (2.1) is a monotone system, the following comparison principle can be easily obtained. We omit the proof here. The reader can refer to [29, 30].

Lemma 2.1 (Comparison principle) *Let $(u_1^-(x,t), u_2^-(x,t))$ and $(u_1^+(x,t), u_2^+(x,t))$ be the solution of (2.1) with the initial data $(u_{10}^-(x,0), u_{20}^-(x,0))$ and $(u_{10}^+(x,0), u_{20}^+(x,0))$, respectively. If*

$$\boldsymbol{0} \le (u_{10}^-(x,0), u_{20}^-(x,0)) \le (u_{10}^+(x,0), u_{20}^+(x,0)) \le \boldsymbol{K},$$

for $x \in \mathbb{R}$, then

$$\boldsymbol{0} \le (u_1^-(x,0), u_2^-(x,0)) \le (u_1^+(x,0), u_2^+(x,0)) \le \boldsymbol{K},$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

In order to obtain our stability result, we need a technical assumption.

(H3) $0 < r < \frac{1}{2}$ and $r < b < \frac{3}{4} - \frac{r}{2}$.

Remark 2.2 It is clear that the condition (H3) is stronger than (H2).

Define a function on η as follows:

$$f_1(\eta) = -r - 2b + 2 - \int_{-\infty}^{0} J(y) e^{-\eta y} \mathrm{d}y.$$

According to (H1) and (H3), we can easily obtain

$$f_1(0) = -r - 2b + 2 - \int_{-\infty}^{0} J(y) \mathrm{d}y = -r - 2b + \frac{3}{2} > 0.$$

Then by continuity, there exists a $\eta_0 > 0$ such that $f_1(\eta_0) > 0$.

In addition, we define other two functions on ξ as follows:

$$g_1(\xi) = -r - 2b - 2 + 4\phi(\xi) - \int_{-\infty}^{0} J(y) e^{-\eta_0 y} \mathrm{d}y,$$
$$g_2(\xi) = -b - r + 2b\phi(\xi),$$

where (ϕ, ψ) is a traveling wavefront of (2.1) satisfying (2.3) and (2.4). It is easy to see that

$$\lim_{\xi \to +\infty} g_1(\xi) = f_1(\eta_0) > 0,$$

$$\lim_{\xi\to+\infty} g_2(\xi) = b - r > 0.$$

Then by continuity of $g_1(\xi)$ and $g_2(\xi)$ with respect to ξ, there exists a $\xi_0 > 0$ large enough such that

$$g_1(\xi_0) = -r - 2b - 2 + 4\phi(\xi_0) - \int_{-\infty}^{0} J(y)e^{-\eta_0 y}\mathrm{d}y > 0, \tag{2.5}$$

$$g_2(\xi_0) = -b - r + 2b\phi(\xi_0) > 0. \tag{2.6}$$

Now we define a weight function by

$$w(\xi) = \begin{cases} e^{-\eta_0(\xi-\xi_0)}, & \xi \le \xi_0, \\ 1, & \xi > \xi_0, \end{cases} \tag{2.7}$$

where $\xi_0 > 0$ and $\eta_0 > 0$ are defined above.

Theorem 2.3 (Stability) *Assume that (H1) and (H3) hold. Let $(\phi(\xi), \psi(\xi))$ be any given traveling wavefront of (2.1) with the wave speed $c > \max\{c^*, \tilde{c}\}$, where*

$$\tilde{c} = \frac{1}{\eta_0}\Big(r + 2b + \frac{3}{2} + \int_{\mathbb{R}} J(y)e^{\eta_0 y}\mathrm{d}y\Big).$$

If the initial data satisfies

$$\boldsymbol{0} \le (u_{10}(x), u_{20}(x)) \le \boldsymbol{K}, \quad \forall\, x \in \mathbb{R}$$

and the initial perturbation satisfies

$$u_{10}(x) - \phi(x), u_{20}(x) - \psi(x) \in H^1_w(\mathbb{R}) \subset C(\mathbb{R}),$$

then the nonnegative solution $(u_1(x,t), u_2(x,t))$ of Cauchy problems (2.1) and (2.2) satisfy

$$\boldsymbol{0} \le (u_1(x,t), u_2(x,t)) \le \boldsymbol{K}, \quad (x,t) \in \mathbb{R}\times\mathbb{R}_+,$$

$$u_1(x,t) - \phi(x+ct), \quad u_2(x,t) - \psi(x+ct) \in C([0,+\infty), H^1_w(\mathbb{R})) \cap L^2([0,+\infty), H^1_w(\mathbb{R})),$$

where $w(x)$ is defined by (2.7). Moreover, $(u_1(x,t), u_2(x,t))$ converges to the traveling wavefront $(\phi(x+ct), \psi(x+ct))$ exponentially in time t, i.e.,

$$\sup_{x\in\mathbb{R}} |u_1(x,t) - \phi(x+ct)| \le Ce^{-\mu t},$$

$$\sup_{x\in\mathbb{R}} |u_2(x,t) - \psi(x+ct)| \le Ce^{-\mu t},$$

for all $t > 0$, where C and μ are some positive constants.

3. Proof of stability

In this section, we are devoted to establishing the stability of traveling wavefronts of (2.1). The existence and uniqueness of the solution to the initial value problems (2.1) and (2.2) can be proved via the standard energy method and continuity extension method. That is, if the initial data satisfy

$$(0,0) \le (u_{10}(x), u_{20}(x)) \le (1,1), \quad x \in \mathbb{R},$$

and the initial perturbation satisfies

$$u_{10}(x)-\phi(x)\in H^1_w(\mathbb{R})\subset C(\mathbb{R}),$$
$$u_{20}(x)-\psi(x)\in H^1_w(\mathbb{R})\subset C(\mathbb{R}),$$

then the nonnegative solution of (2.1) and (2.2) exists and satisfies

$$\mathbf{0}\le(u_1(x,t),u_2(x,t))\le\mathbf{K},\quad (x,t)\in\mathbb{R}\times\mathbb{R}_+,$$

$$u_1(x,t)-\phi(x+ct),\quad u_2(x,t)-\psi(x+ct)\in C([0,+\infty,H^1_w(\mathbb{R}))\cap L^2([0,+\infty),H^1_w(\mathbb{R})),$$

where $w(x)$ is defined by (2.7). Here we omit the details. Our main target is to prove the stability of traveling wavefronts.

Define

$$u^-_{10}(x,0)=\min\{u_{10}(x),\phi(x)\},\quad x\in\mathbb{R},$$
$$u^+_{10}(x,0)=\max\{u_{10}(x),\phi(x)\},\quad x\in\mathbb{R},$$
$$u^-_{20}(x,0)=\min\{u_{20}(x),\psi(x)\},\quad x\in\mathbb{R},$$
$$u^+_{20}(x,0)=\max\{u_{20}(x),\psi(x)\},\quad x\in\mathbb{R}.$$

We can easily obtain

$$0\le u^-_{10}(x,0)\le u_{10}(x),\phi(x)\le u^+_{10}(x,0)\le 1, \tag{3.1}$$
$$0\le u^-_{20}(x,0)\le u_{20}(x),\psi(x)\le u^+_{20}(x,0)\le 1, \tag{3.2}$$

for all $x\in\mathbb{R}$. Let $(u^-_1(x,t),u^-_2(x,t))$ and $(u^+_1(x,t),u^+_2(x,t))$ be the nonnegative solution of system (2.1) with initial data $(u^-_{10}(x,0),u^-_{20}(x,0))$ and $(u^+_{10}(x,0),u^+_{20}(x,0))$, respectively. Then it follows from the comparison principle that

$$0\le u^-_1(x,t)\le u_1(x,t),\phi(x+ct)\le u^+_1(x,t)\le 1, \tag{3.3}$$
$$0\le u^-_2(x,t)\le u_2(x,t),\psi(x+ct)\le u^+_2(x,t)\le 1, \tag{3.4}$$

for all $x\in\mathbb{R}$ and $t>0$.

We shall complete the proof of Theorem 2.3 in the following three steps:

Step 1. $(u^+_1(x,t),u^+_2(x,t))$ converges to $(\phi(x+ct),\psi(x+ct))$.

Step 2. $(u^-_1(x,t),u^-_2(x,t))$ converges to $(\phi(x+ct),\psi(x+ct))$.

Step 3. $(u_1(x,t),u_2(x,t))$ converges to $(\phi(x+ct),\psi(x+ct))$.

We only do Step 1, since Step 2 can be done by a similar way. Using a squeezing technique, we can easily get the conclusion of Step 3, which implies Theorem 2.3.

Let

$$U_1(\xi,t)=u^+_1(\xi-ct,t)-\phi(\xi),\quad U_{10}(\xi,0)=u^+_{10}(\xi,0)-\phi(\xi),\quad (\xi,t)\in\mathbb{R}\times\mathbb{R}_+,$$
$$U_2(\xi,t)=u^+_2(\xi-ct,t)-\psi(\xi),\quad U_{20}(\xi,0)=u^+_{20}(\xi,0)-\psi(\xi),\quad (\xi,t)\in\mathbb{R}\times\mathbb{R}_+.$$

It then follows from (3.1)–(3.4) that $(0,0)\le(U_{10}(\xi,0),U_{20}(\xi,0))\le(1,1)$ and $(0,0)\le(U_1(\xi,t),U_2(\xi,t))\le(1,1)$. We are going to show that there exist some positive constants C and μ such

that

$$\sup_{\xi\in\mathbb{R}}|U_1(\xi,t)|\le Ce^{-\mu t}\quad\text{and}\quad\sup_{\xi\in\mathbb{R}}|U_2(\xi,t)|\le Ce^{-\mu t},\quad\forall\, t\ge 0.$$

It is easy to see that $U_1(\xi,t)$ and $U_2(\xi,t)$ satisfy

$$\begin{cases}U_{1t}+cU_{1\xi}-J*U_1+(2\phi+r-rU_2-r\psi)U_1=-U_1^2+r\phi U_2,\\ U_{2t}+cU_{2\xi}+bU_2(U_1+\phi)=-bU_1\psi+bU_1\end{cases}\tag{3.5}$$

with the initial data $U_1(\xi,0)=U_{10}(\xi,0)$ and $U_2(\xi,0)=U_{20}(\xi,0)$ for all $\xi\in\mathbb{R}$.

3.1. Estimates of $U_i(\cdot,t)$ in $L^2_w(\mathbb{R})$

In this subsection, we shall derive the priori estimates for $U_i(\cdot,t)$ in the weighted Sobolev Space $L^2_w(\mathbb{R})$.

Lemma 3.1 *Assume that (H1) and (H3) hold. Let $w(\xi)$ be the weight function given by (2.7). For any $c>\max\{c^*,\tilde{c}\}$, it holds*

$$\begin{aligned}&e^{2\mu t}(\|U_1(t)\|^2_{L^2_w}+\|U_2(t)\|^2_{L^2_w})+\\&\quad\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)(B^1_{\mu,w}(\xi,s)\|U_1^2(\xi,s)\|^2_{L^2_w}+B^2_{\mu,w}(\xi,s)\|U_2^2(\xi,s)\|^2_{L^2_w})\mathrm{d}\xi\mathrm{d}s\\&\le\|U_{10}(0)\|^2_{L^2_w}+\|U_{20}(0)\|^2_{L^2_w},\end{aligned}\tag{3.6}$$

for all $t\ge 0$, where

$$B^1_{\mu,w}(\xi,t):=A^1_w(\xi,t)-2\mu,\quad B^2_{\mu,w}(\xi,t):=A^2_w(\xi,t)-2\mu,$$

where

$$A^1_w(\xi,t):=-c\frac{w'}{w}+2[2\phi(\xi)+r-rU_2-r\psi(\xi)]-r\phi(\xi)-b-1-\int_{\mathbb{R}}J(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y,\tag{3.7}$$

$$A^2_w(\xi,t):=-c\frac{w'}{w}-r\phi(\xi)+2b[\phi(\xi)-\frac{1}{2}+U_1(\xi,t)].\tag{3.8}$$

Proof Multiplying differential Eq. (3.5) by $e^{2\mu t}w(\xi)U_1(\xi,t)$ and $e^{2\mu t}w(\xi)U_2(\xi,t)$, respectively, where $\mu>0$ and $w(\xi)$ is defined by (2.7), we can obtain

$$\begin{aligned}&e^{2\mu t}w(\xi)U_1U_{1t}+ce^{2\mu t}w(\xi)U_1U_{1\xi}-e^{2\mu t}w(\xi)U_1J*U_1+\\&\quad e^{2\mu t}w(\xi)U_1^2[2\phi(\xi)+r-rU_2-r\psi(\xi)]\\&=-e^{2\mu t}w(\xi)U_1^3+re^{2\mu t}\phi(\xi)w(\xi)U_1U_2,\end{aligned}\tag{3.9}$$

$$\begin{aligned}&e^{2\mu t}w(\xi)U_2U_{2t}+ce^{2\mu t}w(\xi)U_2U_{2\xi}+be^{2\mu t}w(\xi)U_2^2[\phi(\xi)+U_1(\xi,t)]\\&=-be^{2\mu t}w(\xi)U_2U_1\psi(\xi)+be^{2\mu t}w(\xi)U_2U_1.\end{aligned}\tag{3.10}$$

Since

$$e^{2\mu t}w(\xi)U_1U_{1t}=(\frac{1}{2}e^{2\mu t}w(\xi)U_1^2)_t-\mu e^{2\mu t}w(\xi)U_1^2$$

and

$$ce^{2\mu t}w(\xi)U_1U_{1\xi}=(\frac{c}{2}e^{2\mu t}w(\xi)U_1^2)_\xi-\frac{c}{2}e^{2\mu t}w'(\xi)U_1^2,$$

it follows from (3.9) that

$$\begin{aligned}&(\frac{1}{2}e^{2\mu t}w(\xi)U_1^2)_t+(\frac{c}{2}e^{2\mu t}w(\xi)U_1^2)_\xi-e^{2\mu t}w(\xi)U_1J*U_1+\\&\quad\{-\frac{c}{2}\frac{w'}{w}-\mu+[2\phi(\xi)+r-rU_2-r\psi(\xi)]\}e^{2\mu t}w(\xi)U_1^2\\&=-e^{2\mu t}w(\xi)U_1^3+re^{2\mu t}\phi(\xi)w(\xi)U_1U_2.\end{aligned}\tag{3.11}$$

Similarly, calculating the second equation of (3.10), we can see

$$\begin{aligned}&(\frac{1}{2}e^{2\mu t}w(\xi)U_2^2)_t+(\frac{c}{2}e^{2\mu t}w(\xi)U_2^2)_\xi+\{-\frac{c}{2}\frac{w'}{w}-\mu+b[\phi(\xi)+U_1(\xi,t)]\}e^{2\mu t}w(\xi)U_2^2\\&=-be^{2\mu t}w(\xi)U_1U_2\psi(\xi)+be^{2\mu t}w(\xi)U_2U_1.\end{aligned}\tag{3.12}$$

By Cauchy-Schwarz inequality $2xy\le x^2+y^2$, we have

$$be^{2\mu t}w(\xi)U_2U_1\le\frac{b}{2}e^{2\mu t}w(\xi)U_1^2+\frac{b}{2}e^{2\mu t}w(\xi)U_2^2.$$

Meanwhile, dropping the negative terms $-e^{2\mu t}w(\xi)U_1^3$ and $-be^{2\mu t}w(\xi)U_1U_2\psi(\xi)$ in (3.11) and (3.12), respectively, we can see

$$\begin{aligned}&(\frac{1}{2}e^{2\mu t}w(\xi)U_1^2)_t+(\frac{c}{2}e^{2\mu t}w(\xi)U_1^2)_\xi-e^{2\mu t}w(\xi)U_1J*U_1+\\&\quad\{-\frac{c}{2}\frac{w'}{w}-\mu+[2\phi(\xi)+r-rU_2-r\psi(\xi)]\}e^{2\mu t}w(\xi)U_1^2\\&\le re^{2\mu t}\phi(\xi)w(\xi)U_1U_2,\end{aligned}\tag{3.13}$$

$$\begin{aligned}&(\frac{1}{2}e^{2\mu t}w(\xi)U_2^2)_t+(\frac{c}{2}e^{2\mu t}w(\xi)U_2^2)_\xi+\{-\frac{c}{2}\frac{w'}{w}-\mu+b[\phi(\xi)-\frac{1}{2}+\\&\quad U_1(\xi,t)]\}e^{2\mu t}w(\xi)U_2^2\le\frac{b}{2}e^{2\mu t}w(\xi)U_1^2.\end{aligned}\tag{3.14}$$

Integrating (3.13) and (3.14) over $\mathbb{R}\times[0,t]$ with respect to ξ and t, and noting the vanishing term at far fields

$$\{\frac{c}{2}e^{2\mu t}w(\xi)U_1^2\}|_{\xi=-\infty}^{\infty}=0\ \text{ and }\ \{\frac{c}{2}e^{2\mu t}w(\xi)U_2^2\}|_{\xi=-\infty}^{\infty}=0,$$

since $U_1\in H_w^1$ and $U_2\in H_w^1$, we obtain

$$\begin{aligned}&e^{2\mu t}||U_1(t)||_{L_w^2}^2-2\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)U_1(\xi,s)J*U_1\mathrm{d}\xi\mathrm{d}s+\\&\quad\int_0^t\int_{\mathbb{R}}e^{2\mu s}\{-c\frac{w'}{w}-2\mu+2[2\phi(\xi)+r-rU_2-r\psi(\xi)]\}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\&\le\|U_{10}(0)\|_{L_w^2}^2+2r\int_0^t\int_{\mathbb{R}}\phi(\xi)e^{2\mu s}w(\xi)U_1(\xi,s)U_2(\xi,s)\mathrm{d}\xi\mathrm{d}s,\end{aligned}\tag{3.15}$$

$$\begin{aligned}&e^{2\mu t}\|U_2(t)\|_{L_w^2}^2+\int_0^t\int_{\mathbb{R}}e^{2\mu s}\{-c\frac{w'}{w}-2\mu+2b[\phi(\xi)-\frac{1}{2}+U_1(\xi,s)]\}w(\xi)U_2^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\&\le\|U_{20}(0)\|_{L_w^2}^2+b\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s.\end{aligned}\tag{3.16}$$

Using the Cauchy-Schwarz inequality $2xy \le x^2 + y^2$, we have

$$
\begin{aligned}
&2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)U_1(\xi,s)J*U_1\mathrm{d}\xi\mathrm{d}s\\
&=2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\int_{\mathbb{R}} J(y)U_1(\xi,s)U_1(\xi-y,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\
&\le\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\int_{\mathbb{R}} J(y)(U_1^2(\xi,s)+U_1^2(\xi-y,s))\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\
&=\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s+\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\int_{\mathbb{R}} J(y)U_1^2(\xi-y,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\
&=\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s+\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)U_1^2(\xi,s)\int_{\mathbb{R}} J(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y\mathrm{d}\xi\mathrm{d}s. \qquad (3.17)
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
&2r\int_0^t\int_{\mathbb{R}}\phi(\xi)e^{2\mu s}w(\xi)U_1(\xi,s)U_2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\
&\leqslant r\int_0^t\int_{\mathbb{R}}\phi(\xi)w(\xi)e^{2\mu s}U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s+r\int_0^t\int_{\mathbb{R}}\phi(\xi)w(\xi)e^{2\mu s}U_2^2(\xi,s)\mathrm{d}\xi\mathrm{d}s. \qquad (3.18)
\end{aligned}
$$

Substituting (3.17) and (3.18) into (3.15) yields

$$
\begin{aligned}
&e^{2\mu t}\|U_1(t)\|^2_{L^2_w}+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\{-c\frac{w'}{w}-2\mu+2[2\phi(\xi)+r-rU_2-r\psi(\xi)]\}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s+\\
&\quad\int_0^t\int_{\mathbb{R}} e^{2\mu s}\Big\{-r\phi(\xi)-1-\int_{\mathbb{R}} J(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y\Big\}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\
&\le\|U_{10}(0)\|^2_{L^2_w}+r\int_0^t\int_{\mathbb{R}}\phi(\xi)w(\xi)e^{2\mu s}U_2^2(\xi,s)\mathrm{d}\xi\mathrm{d}s. \qquad (3.19)
\end{aligned}
$$

Adding the two inequalities (3.19) and (3.16), we can obtain

$$
\begin{aligned}
&e^{2\mu t}(\|U_1(t)\|^2_{L^2_w}+\|U_2(t)\|^2_{L^2_w})+\\
&\quad\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)[B^1_{\mu,w}(\xi,s)U_1^2(\xi,s)+B^2_{\mu,w}(\xi,s)U_2^2(\xi,s)]\mathrm{d}\xi\mathrm{d}s\\
&\le\|U_{10}(0)\|^2_{L^2_w}+\|U_{20}(0)\|^2_{L^2_w}.
\end{aligned}
$$

This completes the proof. □

We now show that $B^i_{\mu,w}(\xi,t)$ have the positive lower bound, $i=1,2$.

Lemma 3.2 *Assume that (H1) and (H3) hold. Let $w(\xi)$ be the function given by (2.7). Then there exist positive constants C_2 and C_3 such that*

$$A^1_w(\xi,t)\ge C_2 \text{ and } A^2_w(\xi,t)\ge C_3 \quad \text{for all } (\xi,t)\in\mathbb{R}\times[0,+\infty).$$

Proof We first prove $A^1_w(\xi,t)\ge C_2$ for some positive constant C_2. For any $c>\max\{c^*,\tilde{c}\}$, we have

$$c\eta_0-r-2b-\frac{3}{2}-\int_{\mathbb{R}} J(y)e^{\eta_0 y}\mathrm{d}y>0.$$

Case 1. $\xi\le\xi_0$. In this case, we know $w(\xi)=e^{-\eta_0(\xi-\xi_0)}$, and hence, $\frac{w'}{w}=-\eta_0$. Note that $U_2(\xi,t)=u_2^+(\xi-ct,t)-\psi(\xi)$ and $0\le u_2^+(\xi-ct,t)$, $\phi(\xi)\le 1$ for $\xi\in\mathbb{R}$, $t\ge 0$. It then follows

from (3.7) that

$$\begin{aligned}
A_w^1(\xi,t) &\geq c\eta_0 - r - b - 1 - \int_{-\infty}^{\xi_0-\xi} J(y)e^{-\eta_0 y}\mathrm{d}y - \int_{\xi_0-\xi}^{+\infty} J(y)e^{\eta_0(\xi-\xi_0)}\mathrm{d}y \\
&\geq c\eta_0 - r - b - 1 - \int_{\mathbb{R}} J(y)e^{\eta_0 y}\mathrm{d}y - e^{\eta_0(\xi-\xi_0)}\int_0^{+\infty} J(y)\mathrm{d}y \\
&\geq c\eta_0 - r - b - 1 - \int_{\mathbb{R}} J(y)e^{\eta_0 y}\mathrm{d}y - \frac{1}{2} \\
&\geq c\eta_0 - r - b - \frac{3}{2} - \int_{\mathbb{R}} J(y)e^{\eta_0 y}\mathrm{d}y \\
&> b > 0.
\end{aligned}$$

Case 2. $\xi > \xi_0$. In this case, $w(\xi) = 1$ and $\frac{w'}{w} = 0$. It then follows that

$$\begin{aligned}
A_w^1(\xi,t) &\geq 4\phi(\xi_0) - r - b - 1 - \int_{-\infty}^{\xi_0-\xi} J(y)e^{-\eta_0(y+\xi-\xi_0)}\mathrm{d}y - \int_{\xi_0-\xi}^{+\infty} J(y)\mathrm{d}y \\
&\geq 4\phi(\xi_0) - r - b - 2 - \int_{-\infty}^{\xi_0-\xi} J(y)e^{-\eta_0(y+\xi-\xi_0)}\mathrm{d}y \\
&\geq 4\phi(\xi_0) - r - b - 2 - \int_{-\infty}^{0} J(y)e^{-\eta_0 y}\mathrm{d}y \\
&> b > 0,
\end{aligned}$$

where (2.5) is used. According to the above argument and letting $C_2 = b > 0$, we have $A_w^1(\xi,t) \geq C_2 > 0$.

Next, we prove $A_w^2(\xi,t) \geq C_3$ for some positive constant C_3.

Case 1. $\xi \leq \xi_0$. Note that $w(\xi) = e^{-\eta_0(\xi-\xi_0)}$ and $0 \leq \phi(\xi), U_1(\xi,t) \leq 1$ for $\xi \in \mathbb{R}$, $t \geq 0$. Then by (3.8), we obtain

$$A_w^2(\xi,t) \geq c\eta_0 - r - b > b =: C_{31} > 0.$$

Case 2. $\xi > \xi_0$. According to $w(\xi) = 1$, by (2.6), we have

$$A_w^2(\xi,t) \geq 2b\phi(\xi_0) - b - r := C_{32} > 0.$$

Letting $C_3 = \min\{C_{31}, C_{32}\} > 0$, we have $A_w^2(\xi,t) \geq C_3 > 0$. The proof is completed. □

Lemma 3.3 *Assume that (H1) and (H3) hold. Let $w(\xi)$ be the function given by (2.7). For any $c > \max\{c^*, \tilde{c}\}$, there exists a positive constant C_1 such that*

$$B_{\mu,w}^i(\xi,t) \geq C_1, \quad i = 1, 2,$$

for all $\xi \in \mathbb{R}$, $t \geq 0$ and $0 < \mu < \mu_* = \min\{C_2/2, C_3/2\}$.

The proof is easy, so we omit it here.

Dropping the positive terms of the left side in (3.6), we can obtain the following priori estimate of $U_i(\cdot,t)$ in the space $L_w^2(\mathbb{R})$ for $t \geq 0$.

Lemma 3.4 *Assume that (H1) and (H3) hold. Let $w(\xi)$ be the function given by (2.7). Then*

it holds

$$\|U_i(t)\|^2_{L^2_w} \le \hat{C}_0 e^{-2\mu t}(\|U_{10}(0)\|^2_{L^2_w} + \|U_{20}(0)\|^2_{L^2_w}), \quad i = 1, 2,$$

for all $t \ge 0$ *and* $0 < \mu < \mu_* = \min\{C_2/2, C_3/2\}$, $\hat{C}_0$ *is some positive constant.*

3.2. Estimates of $U_{i\xi}(\cdot, t)$ in $L^2_w(\mathbb{R})$

In this subsection, we further establish the priori estimates for $U_{i\xi}(\cdot, t)$ in the weighted Sobolev Space $L^2_w(\mathbb{R})$.

Lemma 3.5 *Assume that (H1) and (H3) hold. Let* $w(\xi)$ *be defined by (2.7). Then for any* $c > \max\{c^*, \tilde{c}\}$, *it holds*

$$\begin{aligned} &e^{2\mu t}(\|U_{1\xi}(t)\|^2_{L^2_w} + \|U_{2\xi}(t)\|^2_{L^2_w}) + \\ &\quad \int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi)[B^3_{\mu,w}(\xi, s)U^2_{1\xi}(\xi, s) + B^4_{\mu,w}(\xi, s)U^2_{2\xi}(\xi, s)]\mathrm{d}\xi\mathrm{d}s \\ &\le \|U_{1\xi 0}(0)\|^2_{L^2_w} + \|U_{2\xi 0}(0)\|^2_{L^2_w} + 2\int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) Q(\xi, s)\mathrm{d}\xi\mathrm{d}s, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} &Q(\xi, t) = [(r-2)\phi(\xi) + r\psi(\xi)]_\xi U_1 U_{1\xi} - b\phi_\xi(\xi) U_2 U_{2\xi}, \\ &B^3_{\mu,w}(\xi, t) = A^3_w(\xi, t) - 2\mu, \quad B^4_{\mu,w}(\xi, t) = A^4_w(\xi, t) - 2\mu \end{aligned}$$

with

$$\begin{aligned} A^3_w(\xi, t) =& -c\frac{w'}{w} + 2[2\phi(\xi) + r - rU_2 - r\psi(\xi)] - r\phi(\xi) - 1 - b(1 + U_2) - rU_1 - \\ &\int_{\mathbb{R}} J(y)\frac{w(\xi + y)}{w(\xi)}\mathrm{d}y, \\ A^4_w(\xi, t) =& -c\frac{w'}{w} - r[\phi(\xi) + U_1] + 2b[\phi(\xi) + U_1] + b(U_2 - 1). \end{aligned}$$

Proof By differentiating (3.5) with respect to ξ, we have

$$\begin{cases} U_{1t\xi} + cU_{1\xi\xi} - J * U_{1\xi} + [2\phi(\xi) + r - rU_2 - r\psi(\xi)]U_{1\xi} + [2\phi(\xi) - rU_2 - r\psi(\xi)]_\xi U_1 \\ \quad = -2U_1 U_{1\xi} + r\phi_\xi U_2 + r\phi U_{2\xi}, \\ U_{2t\xi} + cU_{2\xi\xi} + bU_{2\xi}(U_1 + \phi) + bU_2(U_1 + \phi)_\xi \\ \quad = -bU_1\psi_\xi + b(1 - \psi)U_{1\xi}. \end{cases} \tag{3.21}$$

Multiplying (3.21) by $e^{2\mu t}w(\xi)U_{1\xi}(\xi, t)$, and $e^{2\mu t}w(\xi)U_{2\xi}(\xi, t)$, respectively, where $\mu > 0$ and $w(\xi)$ is defined by (2.7), we obtain

$$\begin{aligned} &e^{2\mu t}w(\xi)U_{1\xi}U_{1t\xi} + ce^{2\mu t}w(\xi)U_{1\xi}U_{1\xi\xi} - e^{2\mu t}w(\xi)U_{1\xi}J * U_{1\xi} + e^{2\mu t}w(\xi)[2\phi(\xi) + r - \\ &\quad rU_2 - r\psi(\xi)]U^2_{1\xi} + e^{2\mu t}w(\xi)[2\phi(\xi) - rU_2 - r\psi(\xi)]_\xi U_1 U_{1\xi} \\ &= -2e^{2\mu t}w(\xi)U_1 U^2_{1\xi} + re^{2\mu t}w(\xi)\phi_\xi U_2 U_{1\xi} + r\phi e^{2\mu t}w(\xi)U_{2\xi}U_{1\xi}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &e^{2\mu t}w(\xi)U_{2\xi}U_{2t\xi} + ce^{2\mu t}w(\xi)U_{2\xi}U_{2\xi\xi} + be^{2\mu t}w(\xi)U^2_{2\xi}(U_1 + \phi) + be^{2\mu t}w(\xi)U_2 U_{2\xi}(U_1 + \phi)_\xi \\ &= -be^{2\mu t}w(\xi)U_1 U_{2\xi}\psi_\xi + be^{2\mu t}w(\xi)(1 - \psi)U_{1\xi}U_{2\xi}. \end{aligned} \tag{3.23}$$

Notice that

$$e^{2\mu t}w(\xi)U_{1\xi}U_{1t\xi} = (\frac{1}{2}e^{2\mu t}w(\xi)U_{1\xi}^2)_t - \mu e^{2\mu t}w(\xi)U_{1\xi}^2,$$
$$ce^{2\mu t}w(\xi)U_{1\xi}U_{1\xi\xi} = (\frac{c}{2}e^{2\mu t}w(\xi)U_{1\xi}^2)_\xi - \frac{c}{2}e^{2\mu t}w'(\xi)U_{1\xi}^2.$$

Substituting them into (3.22) and dropping the negative term $-2e^{2\mu t}w(\xi)U_1U_{1\xi}^2$ on the right side, we can have

$$\begin{aligned}
&(\frac{1}{2}e^{2\mu t}w(\xi)U_{1\xi}^2)_t + (\frac{c}{2}e^{2\mu t}w(\xi)U_{1\xi}^2)_\xi - e^{2\mu t}w(\xi)U_{1\xi}J*U_{1\xi}+\\
&\quad e^{2\mu t}w(\xi)\{-\frac{c}{2}\frac{w'}{w} - \mu + [2\phi(\xi) + r - rU_2 - r\psi(\xi)]\}U_{1\xi}^2+\\
&\quad e^{2\mu t}w(\xi)[2\phi(\xi) - rU_2 - r\psi(\xi)]_\xi U_1U_{1\xi}\\
&\le re^{2\mu t}w(\xi)\phi_\xi U_2U_{1\xi} + r\phi e^{2\mu t}w(\xi)U_{2\xi}U_{1\xi}. \qquad (3.24)
\end{aligned}$$

Similarly, it follows from (3.23) that

$$\begin{aligned}
&(\frac{1}{2}e^{2\mu t}w(\xi)U_{2\xi}^2)_t + (\frac{c}{2}e^{2\mu t}w(\xi)U_{2\xi}^2)_\xi+\\
&\quad \{-\frac{c}{2}\frac{w'}{w} - \mu + b[U_1 + \phi(\xi)]\}e^{2\mu t}w(\xi)U_{2\xi}^2 + be^{2\mu t}w(\xi)U_2U_{2\xi}(U_1+\phi)_\xi\\
&\le be^{2\mu t}w(\xi)U_{1\xi}U_{2\xi}. \qquad (3.25)
\end{aligned}$$

Integrating (3.24) and (3.25) with respect to (ξ, t) over $\mathbb{R}\times[0,t]$, respectively, and in view of

$$\begin{aligned}
&2r\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\phi(\xi)U_{1\xi}U_{2\xi}\mathrm{d}\xi\mathrm{d}s\\
&\quad\le r\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\phi(\xi)U_{1\xi}^2\mathrm{d}\xi\mathrm{d}s + r\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\phi(\xi)U_{2\xi}^2\mathrm{d}\xi\mathrm{d}s,
\end{aligned}$$

we obtain the following inequality

$$\begin{aligned}
&e^{2\mu t}\|U_{1\xi}(t)\|_{L_w^2}^2 - 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\phi(\xi)U_{1\xi}\int_{\mathbb{R}} J(y)U_{1\xi}(\xi-y,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s+\\
&\quad\int_0^t\int_{\mathbb{R}} e^{2\mu s}\{-c\frac{w'}{w} - 2\mu - rU_1 - r\phi(\xi) + 2[2\phi(\xi) + r - rU_2 - r\psi(\xi)]\}w(\xi)U_{1\xi}^2\mathrm{d}\xi\mathrm{d}s\\
&\le \|U_{1\xi0}(0)\|_{L_w^2}^2 + r\int_0^t\int_{\mathbb{R}}[U_1+\phi(\xi)]e^{2\mu s}w(\xi)U_{2\xi}^2\mathrm{d}\xi\mathrm{d}s+\\
&\quad 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}[-2\phi(\xi) + r\psi(\xi)]_\xi w(\xi)U_1U_{1\xi}\mathrm{d}\xi\mathrm{d}s+\\
&\quad 2r\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\phi_\xi(\xi)U_1U_{1\xi}\mathrm{d}\xi\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
&e^{2\mu t}\|U_{2\xi}(t)\|_{L_w^2}^2 + \int_0^t\int_{\mathbb{R}} e^{2\mu s}\{-c\frac{w'}{w} - 2\mu + bU_2 - b + 2b[U_1 - \phi(\xi)]\}w(\xi)U_{2\xi}^2\mathrm{d}\xi\mathrm{d}s\\
&\le \|U_{2\xi0}(0)\|_{L_w^2}^2 + b\int_0^t\int_{\mathbb{R}}[1+U_1]e^{2\mu s}w(\xi)U_{1\xi}^2\mathrm{d}\xi\mathrm{d}s - 2b\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\phi_\xi(\xi)U_2U_{2\xi}\mathrm{d}\xi\mathrm{d}s.
\end{aligned}$$

According to the Cauchy-Schwarz inequality, it holds

$$e^{2\mu t}\|U_{1\xi}(t)\|_{L_w^2}^2 +$$

$$\int_0^t\int_{\mathbb{R}} e^{2\mu s}\{-c\frac{w'}{w}-2\mu-rU_1-r\phi(\xi)+2[2\phi(\xi)+r-rU_2-r\psi(\xi)]\}w(\xi)U_{1\xi}^2\mathrm{d}\xi\mathrm{d}s-$$
$$\int_0^t\int_{\mathbb{R}} e^{2\mu s}\Big\{\int_{\mathbb{R}} J(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y-1\Big\}w(\xi)U_{1\xi}^2\mathrm{d}\xi\mathrm{d}s$$
$$\le \|U_{1\xi 0}(0)\|_{L_w^2}^2+r\int_0^t\int_{\mathbb{R}}[U_1+\phi(\xi)]e^{2\mu s}w(\xi)U_{2\xi}^2\mathrm{d}\xi\mathrm{d}s+$$
$$2\int_0^t\int_{\mathbb{R}} e^{2\mu s}[-2\phi(\xi)+r\psi(\xi)]_\xi w(\xi)U_1U_{1\xi}\mathrm{d}\xi\mathrm{d}s+2r\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\phi_\xi(\xi)U_1U_{1\xi}\mathrm{d}\xi\mathrm{d}s.$$

Thus, adding the two inequalities, we have

$$e^{2\mu t}(\|U_{1\xi}(t)\|_{L_w^2}^2+\|U_{2\xi}(t)\|_{L_w^2}^2)+$$
$$\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)[B_{\mu,w}^3(\xi,s)U_{1\xi}^2(\xi,s)+B_{\mu,w}^4(\xi,s)U_{2\xi}^2(\xi,s)]\mathrm{d}\xi\mathrm{d}s$$
$$\le \|U_{1\xi 0}(0)\|_{L_w^2}^2+\|U_{2\xi 0}(0)\|_{L_w^2}^2+2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)Q(\xi,s)\mathrm{d}\xi\mathrm{d}s.$$

The proof is completed. □

Next, we will first prove that $B_{\mu,w}^i(\xi,t)$ have the positive lower bound, $i=3,4$. It is easy to see that

$$A_w^3(\xi,t)=A_w^1(\xi,t)-rU_1-bU_2,\quad A_w^4(\xi,t)=A_w^2(\xi,t)-rU_1+bU_2.$$

Lemma 3.6 *Assume that (H1) and (H3) hold. Let $w(\xi)$ be the function given by (2.7). For any $c>\max\{c^*,\tilde{c}\}$, there exist positive constants C_4 and C_5 such that*

$$A_w^3(\xi,t)\ge C_4 \text{ and } A_w^4(\xi,t)\ge C_5 \text{ for all } (\xi,t)\in\mathbb{R}\times[0,+\infty).$$

Proof We first prove $A_w^3(\xi,t)\ge C_4$, for some positive constant C_4.

Case 1. $\xi\le\xi_0$. Since $U_2(\xi,t)=u_2^+(x,t)-\psi(\xi)$ and $U_1(\xi,t)=u_1^+(x,t)-\phi(\xi)$, according to (2.7), we have

$$\begin{aligned}A_w^3(\xi,t)&\ge c\eta_0-r-2b-1-\int_{-\infty}^{\xi_0-\xi}J(y)e^{-\eta_0 y}\mathrm{d}y-\int_{\xi_0-\xi}^{+\infty}J(y)e^{\eta_0(\xi-\xi_0)}\mathrm{d}y\\&\ge c\eta_0-r-2b-1-\int_{\mathbb{R}}J(y)e^{\eta_0 y}\mathrm{d}y-e^{\eta_0(\xi-\xi_0)}\int_0^{+\infty}J(y)\mathrm{d}y\\&\ge c\eta_0-r-2b-\frac{3}{2}-\int_{\mathbb{R}}J(y)e^{\eta_0 y}\mathrm{d}y\\&:=C_{41}>0.\end{aligned}$$

Case 2. $\xi>\xi_0$. In this case, $w(\xi)=1$ and $\frac{w'}{w}=0$. Thus we have

$$\begin{aligned}A_w^3(\xi,t)&\ge 4\phi(\xi_0)-r-2b-1-\int_{-\infty}^{\xi_0-\xi}J(y)e^{-\eta_0(y+\xi-\xi_0)}\mathrm{d}y-\int_{\xi_0-\xi}^{+\infty}J(y)\mathrm{d}y\\&\ge 4\phi(\xi_0)-r-2b-2-\int_{-\infty}^{\xi_0-\xi}J(y)e^{-\eta_0(y+\xi-\xi_0)}\mathrm{d}y\\&\ge 4\phi(\xi_0)-r-2b-2-\int_{-\infty}^{0}J(y)e^{-\eta_0 y}\mathrm{d}y\end{aligned}$$

$$:= C_{42} > 0.$$

Letting $C_4 := \min\{C_{41}, C_{42}\}$, we have $A_w^3(\xi,t) \geq C_4$.

Next, we prove $A_w^4(\xi,t) \geq C_5 > 0$.

Case 1. $\xi \leq \xi_0$. Since $U_1(\xi,t) = u_1^+(x,t) - \phi(\xi)$, it follows from $w(\xi) = e^{-\eta_0(\xi-\xi_0)}$ and the monotonicity of $w(\xi)$ that

$$A_w^4(\xi,t) \geq c\eta_0 - b - r =: C_{51} > 0.$$

Case 2. $\xi > \xi_0$. According to $w(\xi) = 1$ and $\frac{w'}{w} = 0$, we see

$$A_w^4(\xi,t) \geq 2b\phi(\xi_0) - b - r =: C_{52} > 0.$$

Thus, we can prove $A_w^4(\xi,t) \geq C_5$, where $C_5 := \min\{C_{51}, C_{52}\}$. This completes the proof. □

Similar to Lemma 3.3, we obtain the following result.

Lemma 3.7 *Assume that (H1) and (H3) hold. Let $w(\xi)$ be the function given by (2.7). For any $c > \max\{c^*, \tilde{c}\}$, there exists a positive constant C_1 such that*

$$B_{\mu,w}^i(\xi,t) \geq C_1, \quad i = 3, 4,$$

for all $\xi \in \mathbb{R}$, $t \geq 0$ and $0 < \mu < \mu^* = \min\{C_4/2, C_5/2\}$.

Furthermore, we estimate the last term $\int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) Q(\xi,s) \mathrm{d}\xi \mathrm{d}s$. Note that

$$\mathbf{0} \leq (U_1(\xi,t), U_2(\xi,t)) \leq \mathbf{K},$$

and $\phi'(\xi)$, $\psi'(\xi)$ are bounded for $\xi \in \mathbb{R}$. Thus, there exists a positive constant C_0 such that

$$|(r-2)\phi'(\xi) + r\psi'(\xi)| \leq C_0, \quad |b\phi'(\xi)| \leq C_0.$$

According to Lemma 3.1, we easily obtain the following inequality

$$\int_0^t (\|U_1(s)\|_{L_w^2}^2 + \|U_2(s)\|_{L_w^2}^2) e^{2\mu s} \mathrm{d}s \leq C(\|U_{10}(0)\|_{L_w^2}^2 + \|U_{20}(0)\|_{L_w^2}^2).$$

Using Young-inequality $2xy \leq \varepsilon x^2 + \frac{1}{\varepsilon} y^2$, it holds

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) Q(\xi,s) \mathrm{d}\xi \mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}} \{[(r-2)\phi(\xi) + r\psi(\xi)]_\xi U_1 U_{1\xi} - b\phi_\xi(\xi) U_2 U_{2\xi}\} e^{2\mu s} w(\xi) \mathrm{d}\xi \mathrm{d}s \\
&\leq C_0 \int_0^t \int_{\mathbb{R}} [\frac{1}{\varepsilon}(U_1^2(\xi,s) + U_2^2(\xi,s)) + \varepsilon(U_{1\xi}^2(\xi,s) + U_{2\xi}^2(\xi,s))] e^{2\mu s} w(\xi) \mathrm{d}\xi \mathrm{d}s \\
&\leq \frac{C_0}{\varepsilon} \int_0^t (\|U_1(s)\|_{L_w^2}^2 + \|U_2(s)\|_{L_w^2}^2) e^{2\mu s} \mathrm{d}s + C_0 \varepsilon \int_0^t (\|U_{1\xi}(s)\|_{L_w^2}^2 + \|U_{2\xi}(s)\|_{L_w^2}^2) e^{2\mu s} \mathrm{d}s \\
&\leq \frac{C_0 C}{\varepsilon} (\|U_{10}(0)\|_{L_w^2}^2 + \|U_{20}(0)\|_{L_w^2}^2) + C_0 \varepsilon \int_0^t (\|U_{1\xi}(s)\|_{L_w^2}^2 + \|U_{2\xi}(s)\|_{L_w^2}^2) e^{2\mu s} \mathrm{d}s.
\end{aligned}$$

Choosing $\varepsilon > 0$ such that $C_0\varepsilon < \min\{C_4, C_5\}$ and combining with (3.20), we have

$$e^{2\mu t}(\|U_{1\xi}(t)\|_{L_w^2}^2 + \|U_{2\xi}(t)\|_{L_w^2}^2) +$$

$$\int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi)[B^3_{\mu,w}(\xi,s)U^2_{1\xi}(\xi,s) + B^4_{\mu,w}(\xi,s)U^2_{2\xi}(\xi,s)]\mathrm{d}\xi\mathrm{d}s$$
$$\leq C_6(\|U_{1\xi 0}(0)\|^2_{L^2_w} + \|U_{2\xi 0}(0)\|^2_{L^2_w} + \|U_{10}(0)\|^2_{L^2_w} + \|U_{20}(0)\|^2_{L^2_w}), \tag{3.26}$$

where C_6 is a positive constant.

Dropping the positive terms of the left side in (3.26), we obtain the following priori estimate of $U_{i\xi}(\cdot,t)$ in the space $L^2_w(\mathbb{R})$ for $t \geq 0$.

Lemma 3.8 *Assume that (H1) and (H3) hold. Let $w(\xi)$ be the function given by (2.7). Then*

$$\|U_{i\xi}(t)\|^2_{L^2_w} \leq \tilde{C}_0 e^{-2\mu t}(\|U_{10}(0)\|^2_{H^1_w} + \|U_{20}(0)\|^2_{H^1_w}), \quad i = 1, 2,$$

for all $t \geq 0$ and $0 < \mu < \mu^* = \min\{C_4/2, C_5/2\}$, $\tilde{C}_0$ is some positive constant.

3.3. Proof of Step 1

Combining Lemmas 3.4 and 3.8, we immediately obtain a prior estimate of $U_i(\cdot,t)$ in the weighted Sobolev Space $H^1_w(\mathbb{R})$ for all $t > 0$, i.e.,

$$\|U_i(t)\|^2_{H^1_w} \leq Ce^{-2\mu t}(\|U_{10}(0)\|^2_{H^1_w} + \|U_{20}(0)\|^2_{H^1_w}), \quad i = 1, 2, \tag{3.27}$$

for all $t > 0$, where C and μ are some positive constants.

By the standard Sobolev embedding inequality $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, and the embedding inequality $H^1_w(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$ since $w \geq 1$, we can obtain

$$\sup_{\xi\in\mathbb{R}} |U_i(\xi,t)| \leq C\|U_i(t)\|^2_{H^1} \leq C\|U_i(t)\|^2_{H^1_w}, \quad i = 1, 2, \tag{3.28}$$

for all $t > 0$. By (3.27) and (3.28), we can directly derive the following result.

Lemma 3.9 *Assume that (H1) and (H3) hold. Let $w(\xi)$ be defined by (2.7). Then for any $c > \max\{c^*, \tilde{c}\}$, it holds*

$$\sup_{x\in\mathbb{R}} |u^+_i(x,t) - \phi(x+ct)| = \sup_{\xi\in\mathbb{R}} |U_i(\xi,t)| \leq Ce^{-\mu t}, \quad i = 1, 2,$$

for all $t > 0$, where C and μ are some positive constants.

By Lemmas 3.9, we obtain that $(u^+_1(x,t), u^+_2(x,t))$ converges to $(\phi(x+ct), \psi(x+ct))$ as $t \to +\infty$. Hence, the step 1 is done.

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