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Strong Laws of Large Numbers for Sequences of Blockwise m-Dependent and Sub-Orthogonal Random Variables under Sublinear Expectations

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Abstract In this paper, we establish some strong laws of large numbers, which are for non-independent random variables under the framework of sublinear expectations. One of our main results is for blockwise m-dependent random variables, and another is for sub-orthogonal random variables. Both extend the strong law of large numbers for independent random variables under sublinear expectations to the non-independent case.

Keywords sublinear expectations; strong law of large numbers; blockwise m-dependent; sub-orthogonal random variables

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1. Introduction

One of the most famous results in classical probability theory is Kolmogorov's strong law of large numbers, which means the sample mean converges to the population mean. This fundamental theorem is based on the assumption of independence and identical distribution of random variables. However, due to the complexity of the real world, data can also be dependent and not identically distributed in the observations. Thus, we need to consider the law of large numbers with weaker independence conditions.

A kind of common dependence is m-dependence, which was proposed in Hoeffding and Robbins's paper [1] in 1948 to our knowledge. A sequence $\{X_n\}_{n\geq 1}$ of random variables is called m-dependent if $\{X_1,\ldots,X_r\}$ is independent of $\{X_s,X_{s+1},\ldots\}$ provided $s-r>m$, where m is a fixed nonnegative integer. It is clear that 0-dependence is independence. Moricz [2] introduced the definition of blockwise m-dependence and obtained a strong law of large numbers for the dyadic block $\{X_k: 2^{p-1}<k\leq 2^p\}$, $p\in\mathbb{N}^*$. Later, Gaposhkin [3] and Zhang [4] generalized the dyadic block to arbitrary blocks. In theoretical analysis and practice, it may be much easier to check the orthogonality than independence. So it is reasonable to consider the law of large numbers for orthogonal random variables. Moricz [2] also obtained a strong law of large numbers for blockwise quasi-orthogonal sequences of random variables.

The sublinear expectation was introduced by Peng [5], under whose framework the expectation is not additive anymore. Limit theorems and stochastic analysis can also be established

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under the framework of sublinear expectations. In recent years, there have been many papers about the strong law of large numbers under sublinear expectations. However, the limit theorems for the dependent random variables under sublinear expectations are rare. In the recent work, a strong law of large numbers was established by Gu and Zhang [6] for m-dependent and stationary random variables under sublinear expectations. In this paper, we establish strong laws of large numbers for blockwise m-dependent random variables and sub-orthogonal random variables, respectively.

The rest of this paper is arranged as follows. Some preliminaries for the sublinear expectations are introduced in Section 2. In Section 3, some inequalities and properties needed in our proofs are listed. We present our main theorems in Section 4, and the detailed proofs are presented in Section 5.

2. Preliminaries

In this section, some basic concepts and notations for sublinear expectations and capacities are presented. We use a similar framework of Guo and Li [7], and one can refer to [7, 8] for more details. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{P}$ be a set of probability measures on $(\Omega, \mathcal{F})$. $\mathcal{H}$ denotes a linear space of real-valued functions on $(\Omega, \mathcal{F})$. As the space of random variables, $\mathcal{H}$ satifies that $1_A \in \mathcal{H}$, $\forall\, A \in \mathcal{F}$ and $X_1, \ldots, X_n \in \mathcal{H}$ yields $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for all $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n)$. $C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ denotes the linear space of functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \le C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \text{ for } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$
$$\text{where the constant } C > 0 \text{ and } m \in \mathbb{N} \text{ rely on } \varphi.$$

The space of all Lipschitz functions on $\mathbb{R}^n$ is defined as $C_{\mathrm{Lip}}(\mathbb{R}^n)$. The upper expectation with respect to $\mathcal{P}$ is defined as follows:

$$\widehat{\mathbb{E}}[X] \triangleq \sup_{P \in \mathcal{P}} E_P[X], \quad \forall\, X \in \mathcal{H}.$$

Then $\widehat{\mathbb{E}}$ is a sublinear expectation which satisfies the following properties:

(i) Monotonicity: $\widehat{\mathbb{E}}[X] \le \widehat{\mathbb{E}}[Y]$, if $X \le Y$;

(ii) Constant preserving: $\widehat{\mathbb{E}}[c] = c$, $\forall\, c \in \mathbb{R}$;

(iii) Sub-additivity: for $X, Y \in \mathcal{H}$, $\widehat{\mathbb{E}}[X + Y] \le \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(iv) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$, $\forall\, \lambda \ge 0$.

The triplet $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space. The conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ is defined by

$$\widehat{\mathcal{E}}[X] \triangleq -\widehat{\mathbb{E}}[-X], \quad \forall\, X \in \mathcal{H}.$$

By the sub-additivity of $\widehat{\mathbb{E}}$, it can be checked that for $X, Y \in \mathcal{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite, $\widehat{\mathbb{E}}[X - Y] \ge \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$, $\widehat{\mathcal{E}}[X] \le \widehat{\mathbb{E}}[X]$, $\widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c$ for $c \in \mathbb{R}$, where the last one is called cash translatability. $\widehat{\mathcal{E}}[X]$ is also called the lower expectation of X. We say $\mathbf{X} = (X_1, \ldots, X_n)$ is an n-dimensional random vector in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ if each coordinate of $\mathbf{X}$ is in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$.

Definition 2.1 *Let $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$ be two sublinear expectation spaces. Let $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ be two n-dimensional random vectors in $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$, respectively. $\boldsymbol{X_1}$ is said to be identically distributed with $\boldsymbol{X_2}$, denoted by $\boldsymbol{X_1} \overset{d}{=} \boldsymbol{X_2}$, if*

$$\widehat{\mathbb{E}}_1[\varphi(\boldsymbol{X}_1)] = \widehat{\mathbb{E}}_2[\varphi(\boldsymbol{X}_2)],\ \forall\, \varphi \in C_{\text{Lip}}(\mathbb{R}^n).$$

We say a sequence of random variables $\{X_n\}_{n\geq 1}$ is identically distributed if X_i is identically distributed with X_1 for all $i \geq 1$.

Definition 2.2 *Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sublinear expectation space. Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ and $\boldsymbol{X} = (X_1, \ldots, X_m)$ be two random vectors in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. We say $\boldsymbol{Y}$ is independent of $\boldsymbol{X}$ under $\widehat{\mathbb{E}}$ if*

$$\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]|_{\boldsymbol{x}=\boldsymbol{X}}],$$

for $\varphi \in C_{l,\text{Lip}}(\mathbb{R}^{m+n})$ such that $\hat{\varphi}(\boldsymbol{x}) \triangleq \widehat{\mathbb{E}}[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|] < \infty$ for all $\boldsymbol{x} \in \mathbb{R}^m$, $\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]|_{\boldsymbol{x}=\boldsymbol{X}} \in \mathcal{H}$, $\hat{\varphi}(\boldsymbol{X}) \in \mathcal{H}$ and $\widehat{\mathbb{E}}[\hat{\varphi}(\boldsymbol{X})] < \infty$.

It is important to observe that under the framework of sublinear expectation, Y is independent of X does not mean that X is independent of Y, which is different from the classical linear expectation. One can refer to [5, Example 1.3.15] for more details. We say a sequence of random variables $\{X_n\}_{n\geq 1}$ is independent if X_{i+1} is independent of $(X_1, \ldots, X_i)$ for all $i \geq 1$. It is easy to check that for an independent sequence $\{X_1, \ldots, X_n\}$, if $\widehat{\mathbb{E}}[X_i] < \infty$, $i = 1, \ldots, n$, then $\widehat{\mathbb{E}}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \widehat{\mathbb{E}}[X_i]$.

Next, we introduce the definitions of m-dependence and blockwise m-dependence for sequences of random variables in sublinear expectation spaces.

Definition 2.3 ([6]) *A sequence of random variables $\{X_n; n \geq 1\}$ in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is said to be m-dependent if there exists a nonnegative integer m such that $(X_{n+m+1}, \ldots, X_{n+j})$ is independent of $(X_1, \ldots, X_n)$ for every n and every $j \geq m+1$. In particular, $m = 0$ means $\{X_n; n \geq 1\}$ is an independent sequence.*

Definition 2.4 *Let $\{n_i; i \geq 1\}$ be a given strictly increasing sequence of natural numbers. A sequence of random variables $\{X_n; n \geq 1\}$ in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is said to be blockwise m-dependent with respect to $\{n_i; i \geq 1\}$ if $\{X_{n_i \leq n < n_{i+1}}\}$ is m-dependent for each $i \geq 1$, i.e., either $n_{i+1} - n_i \leq m+1$ or $n_{i+1} - n_i > m+1$ and $\{X_n; s \leq n < n_{i+1}\}$ is independent of $\{X_n; n_i \leq n \leq r\}$ if $s - r > m$.*

For a given probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, we know that two random variables $X, Y \in L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ are orthogonal if $E_{\hat{P}}[XY] = 0$. Under the framework of sublinear expectations, we introduce the concept of sub-orthogonality, which is an extension of orthogonality under classical linear expectation.

Definition 2.5 *A sequence of random variables $\{X_n; n \geq 1\}$ in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is said to be sub-orthogonal if $\widehat{\mathbb{E}}[X_i X_j] \leq 0$ for $i \neq j$.*

Definition 2.6 *A sequence of random variables $\{X_n; n \geq 1\}$ in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is said to be generalized sub-orthogonal if there exists a nonnegative sequence $\{f(j) : j = 0, 1, \ldots\}$ with $\sum_{j=0}^{\infty} f(j) < \infty$*

such that $\widehat{\mathbb{E}}[X_kX_l] \le \sqrt{\widehat{\mathbb{E}}[X_k^2]}\sqrt{\widehat{\mathbb{E}}[X_l^2]} \cdot f(|k-l|)$ *for all* $k, l = 1, 2, \ldots$. *In particular, if* $f(0) = 1$ *and* f *is zero on other values, then* $\{X_n; n \ge 1\}$ *is sub-orthogonal.*

Next, we introduce the capacities corresponding to $\mathcal{P}$ under the framework of Guo and Li [7].

A function $V : \mathcal{F} \to [0, 1]$ is called a capacity if $V(\emptyset) = 0$, $V(\Omega) = 1$ and $V(A) \le V(B)$ if $A \subset B, A, B \in \mathcal{F}$. If

$$V(A \cup B) \le V(A) + V(B), \quad \forall A, B \in \mathcal{F},$$

then we call V is a sub-additive capacity.

A pair of capacities $(\hat{\mathbb{V}}, \hat{\mathcal{V}})$ corresponding to $\mathcal{P}$ is defined as follows:

$$\hat{\mathbb{V}}(A) \triangleq \sup_{P\in\mathcal{P}} P(A),\ \ \hat{\mathcal{V}}(A) \triangleq \inf_{P\in\mathcal{P}} P(A),\ \ \forall A \in \mathcal{F}.$$

$\hat{\mathbb{V}}$ and $\hat{\mathcal{V}}$ are called the upper probability and the lower probability, respectively. They are conjugate to each other, which means

$$\hat{\mathbb{V}}(A) + \hat{\mathcal{V}}(A^c) = 1,\ \ \forall A \in \mathcal{F}.$$

The capacity $\hat{\mathbb{V}}$ has the property that

$$\widehat{\mathbb{E}}[f] \le \hat{\mathbb{V}}(A) \le \widehat{\mathbb{E}}[g] \text{ if } f \le I_A \le g, f, g \in \mathcal{H} \text{ and } A \in \mathcal{F}.$$

The Choquet integral $C_{\hat{\mathbb{V}}}$ is defined by

$$C_{\hat{\mathbb{V}}}[X] \triangleq \int_0^\infty \hat{\mathbb{V}}(X \ge t)\mathrm{d}t + \int_{-\infty}^0 [\hat{\mathbb{V}}(X \ge t) - 1]\mathrm{d}t.$$

If X_1 and X_2 are two random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $X_1 \overset{d}{=} X_2$, then we have

$$\hat{\mathbb{V}}(X_1 \ge x + \epsilon) \le \hat{\mathbb{V}}(X_2 \ge x) \le \hat{\mathbb{V}}(X_1 \ge x - \epsilon),\ \text{ for all } \ \epsilon > 0 \ \text{ and } \ x \in \mathbb{R},$$

and

$$C_{\hat{\mathbb{V}}}[X_1] = C_{\hat{\mathbb{V}}}[X_2].$$

As shown in [5, Theorem 6.1.4], $\hat{\mathbb{V}}$ is countably sub-additive. Let $(\overline{\Omega}, \overline{\mathcal{H}}, \overline{\mathbb{E}})$ be a sublinear expectation space from Peng's original sublinear expectation theory [5]. Another way to construct a countably sub-additive capacity is by the outer extension of $\overline{\mathbb{E}}$ (see [6, 9–11]). Here, we say $\overline{\mathbb{E}}$ is regular if

$$\overline{\mathbb{E}}[X_n] \searrow 0,\ \text{ for } \ \{X_n\}_{n\ge1} \subset \overline{\mathcal{H}} \ \text{ when } \ X_n(\omega) \searrow 0,\ \ \forall \omega \in \Omega.$$

A regular sublinear expectation $\overline{\mathbb{E}}$ can be represented as an upper expectation [5, Theorem 1.2.2]. [8, Example 4.1] illustrates that there exists $\mathcal{P}$ such that the upper expectation $\widehat{\mathbb{E}}$ with respect to $\mathcal{P}$ is irregular.

Throughout the whole paper, we denote $x \wedge y \triangleq \min\{x, y\}, x \vee y \triangleq \max\{x, y\}, x^+ \triangleq x \vee 0, x^- \triangleq (-x) \vee 0$ for real numbers x and y. $[x]$ means the maximum integer not exceeding x. $\mathbb{N}$ represents all natural numbers and $\mathbb{N}^*$ represents all non-zero natural numbers. $C > 0$ is a constant that may change from line to line. For $X \in \mathcal{H}$ and $c > 0$, $X^{(c)} \triangleq (-c) \vee X \wedge c$, $\breve{\mathbb{E}}[X] \triangleq \lim_{c\to\infty} \widehat{\mathbb{E}}[X^{(c)}]$ if the limit exists, and $\breve{\mathcal{E}}[X] \triangleq -\breve{\mathbb{E}}[-X]$. It is clear that $\breve{\mathbb{E}}[X] = \widehat{\mathbb{E}}[X]$ if X is bounded.

3. Some lemmas and inequalities

In this section, we shall present some important lemmas and the crucial inequalities used in our proofs of main results. (i) and (ii) in the next lemma are from [10] and [6], respectively.

Lemma 3.1 *Suppose $X \in \mathcal{H}$.*

(i) ([10])

$$\widehat{\mathbb{E}}[|X|] \le C_{\hat{\mathbb{V}}}[|X|], \tag{3.1}$$

and for any $c \in (0, \infty)$,

$$\widehat{\mathbb{E}}[|X| \wedge c] \le \int_0^c \hat{\mathbb{V}}(|X| > x)\mathrm{d}x.$$

(ii) ([6]) If $C_{\hat{\mathbb{V}}}[|X|] < \infty$, then $\breve{\mathbb{E}}[|X|]$ is well-defined. $\breve{\mathbb{E}}$ is a sublinear expectation on

$$\mathcal{H}_1 \triangleq \{X \in \mathcal{H} | \lim_{c,d\to\infty} \widehat{\mathbb{E}}[(|X| \wedge d - c)^+] = 0\} \supseteq \{X \in \mathcal{H} | \ C_{\hat{\mathbb{V}}}[|X|] < \infty\}.$$

(iii) If $C_{\hat{\mathbb{V}}}[|X|^r] < \infty$ for some $r > 1$, then $C_{\hat{\mathbb{V}}}[|X|] < \infty$.

Proof The proof of (i) and (ii) can be found in [10] and [6], respectively. For (iii), we notice that

$$\begin{aligned} C_{\hat{\mathbb{V}}}[|X|^r] &= \int_0^{+\infty} \hat{\mathbb{V}}(|X|^r \ge t)\mathrm{d}t = \int_0^{+\infty} \hat{\mathbb{V}}(|X| \ge t^{\frac{1}{r}})\mathrm{d}t \\ &= \int_0^{+\infty} rm^{r-1}\hat{\mathbb{V}}(|X| \ge m)\mathrm{d}m \ge \int_1^{+\infty} \hat{\mathbb{V}}(|X| \ge m)\mathrm{d}m. \end{aligned}$$

Hence, if $C_{\hat{\mathbb{V}}}[|X|^r] < \infty$, then $C_{\hat{\mathbb{V}}}[|X|] < \infty$. □

The next lemma is about exponential inequalities and Kolmogorov's maximal inequalities in Gu and Zhang [6]. We use the latter to obtain a maximal inequality for m-dependent random variables under sublinear expectations.

Lemma 3.2 ([6]) *Assume that $\{X_1, \ldots, X_n\}$ is a sequence of independent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Set $S_n = \sum_{i=1}^n X_i$, $B_n^2 = \sum_{i=1}^n \widehat{\mathbb{E}}[X_i^2]$. Then for all $x > 0$, $0 < \delta \le 1$ and $p \ge 2$,*

$$\begin{aligned} &\hat{\mathbb{V}}\Big(\max_{k\le n}(S_k - \widehat{\mathbb{E}}[S_k]) \ge x\Big) \ \Big(\text{resp., } \hat{\mathcal{V}}\Big(\max_{k\le n}(S_k - \widehat{\mathcal{E}}[S_k]) \ge x\Big)\Big) \\ &\le C_p\delta^{-p}x^{-p}\sum_{i=1}^n \widehat{\mathbb{E}}[X_i^p] + \exp\Big\{ - \frac{x^2}{2(1+\delta)B_n^2}\Big\}. \end{aligned}$$

Further, by noting that $xe^{-x} \le e^{-1}$ when $x \ge 0$, we have Kolmogorov's maximal inequalities as follows:

$$\begin{aligned} \hat{\mathbb{V}}\Big(\max_{k\le n}(S_k - \widehat{\mathbb{E}}[S_k]) \ge x\Big) &\le Cx^{-2}\sum_{i=1}^n \widehat{\mathbb{E}}[X_i^2], \\ \hat{\mathcal{V}}\Big(\max_{k\le n}(S_k - \widehat{\mathcal{E}}[S_k]) \ge x\Big) &\le Cx^{-2}\sum_{i=1}^n \widehat{\mathbb{E}}[X_i^2]. \end{aligned} \tag{3.2}$$

By (3.2), we can obtain a maximal inequality for m-dependent random variables, which plays a crucial role in our proof of the strong law of large numbers for blockwise m-dependent random variables.

Lemma 3.3 *Suppose that* $\{X_1, \ldots, X_n\}$ *is a sequence of* m*-dependent random variables in* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *with* $\widehat{\mathbb{E}}[X_i] < \infty$ *for* $i = 1, \ldots, n$. *Then for all* $x > 0$,

$$\hat{\mathbb{V}}\Big(\max_{k\le n}\sum_{i=1}^{k}(X_i - \widehat{\mathbb{E}}[X_i]) \ge x\Big) \le Cx^{-2}\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_i^2],$$

where $C = C_m$ *is a constant only dependent on* m.

Proof In the case of $n < m+1$, we take $Z_i \triangleq X_i - \widehat{\mathbb{E}}[X_i]$. Then

$$\begin{aligned}
\hat{\mathbb{V}}\Big(\max_{k\le n}\sum_{i=1}^{k} Z_i \ge x\Big) &= \hat{\mathbb{V}}\Big(\bigcup_{k=1}^{n}\Big\{\sum_{i=1}^{k} Z_i \ge x\Big\}\Big) \le \sum_{k=1}^{n}\hat{\mathbb{V}}\Big(\sum_{i=1}^{k} Z_i \ge x\Big)\\
&\le \sum_{k=1}^{n}\hat{\mathbb{V}}\Big(\bigcup_{i=1}^{k}\{Z_i \ge \frac{x}{k}\}\Big) \le \sum_{k=1}^{n}\sum_{i=1}^{k}\hat{\mathbb{V}}(Z_i \ge \frac{x}{k})\\
&\le C\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{k^2}{x^2}\widehat{\mathbb{E}}[X_i^2] \quad (\text{by } (3.2))\\
&= C\sum_{i=1}^{n}\sum_{k=i}^{n}\frac{k^2}{x^2}\widehat{\mathbb{E}}[X_i^2] \le C\sum_{k=1}^{n}k^2\cdot x^{-2}\cdot\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_i^2]\\
&\le C\frac{(m+1)(m+2)(2m+3)}{6}\cdot x^{-2}\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_i^2].
\end{aligned}$$

In the last inequality we use the truth that $\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$ and notice that $n < m+1$.

In the case of $n \ge m+1$, we take

$$Z_{ki} \triangleq X_{i(m+1)+k} - \widehat{\mathbb{E}}[X_{i(m+1)+k}], \quad i = 0, \ldots, [(n-k)/(m+1)].$$

Then

$$\begin{aligned}
\hat{\mathbb{V}}\Big(\max_{k\le n}\sum_{i=1}^{k}(X_i - \widehat{\mathbb{E}}[X_i]) \ge x\Big) &\le \hat{\mathbb{V}}\Big(\bigcup_{k=1}^{m+1}\Big\{\max_{0\le j\le[(n-k)/(m+1)]}\sum_{i=0}^{j} Z_{ki} \ge \frac{x}{m+1}\Big\}\Big)\\
&\le \sum_{k=1}^{m+1}\hat{\mathbb{V}}\Big(\max_{0\le j\le[(n-k)/(m+1)]}\sum_{i=0}^{j} Z_{ki} \ge \frac{x}{m+1}\Big)\\
&\le C\sum_{k=1}^{m+1}\frac{(m+1)^2}{x^2}\sum_{i=0}^{[(n-k)/(m+1)]}\widehat{\mathbb{E}}[X_{i(m+1)+k}^2]\\
&= C(m+1)^2\cdot x^{-2}\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_i^2].
\end{aligned}$$

In the third inequality we have used (3.2) because $\{Z_{ki}, i = 0, \ldots, [(n-k)/(m+1)]\}$ is a sequence of independent random variables.

The proof is completed. □

Next, we establish the Rademacher-Mensov inequality under the sublinear expectation, which plays a crucial role in the proof of the strong law of large numbers for sub-orthogonal random variables. One can refer to [12] for the Rademacher-Mensov inequality in classical probability theory.

Lemma 3.4 *Suppose that* $\{X_1, \ldots, X_n\}$ *is a sequence of sub-orthogonal random variables in* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *with* $\widehat{\mathbb{E}}[X_i^2] \leq 1$ *for all* $i \in \{1, \ldots, n\}$. *Let* $\{c_1, \ldots, c_n\}$ *be a sequence of nonnegative real numbers. Then*

$$\widehat{\mathbb{E}}\Big[\max_{1\leq k\leq n}\Big(\sum_{j=1}^{k} c_j X_j\Big)^2\Big] \leq (\log_2 4n)^2 \sum_{j=1}^{n} c_j^2. \tag{3.3}$$

Proof We first prove for the case of $n = 2^v$, $v \in \mathbb{N}^*$. Let

$$\eta_j \triangleq c_1 X_1 + \cdots + c_j X_j, \quad j = 1, \ldots, n,$$
$$\psi_{\alpha\beta} \triangleq c_{\alpha+1} X_{\alpha+1} + \cdots + c_\beta X_\beta,$$

where $\alpha \triangleq \mu \cdot 2^k$; $\beta \triangleq \beta(\alpha) \triangleq (\mu+1)\cdot 2^k$; $k = 0, 1, \ldots, v$; $\mu = 0, 1, \ldots, 2^{v-k} - 1$.

We consider the η_j as the sum of some $\psi_{\alpha\beta}$ and put

$$\eta_j = \sum_i \psi_{\alpha_i\beta_i},$$

where $\beta_1 - \alpha_1 > \beta_2 - \alpha_2 > \cdots$, then the number of the members of the sum is less than v.

By the Cauchy inequality, for $j = 1, \ldots, n$, we have

$$\eta_j^2 = \Big(\sum_i \psi_{\alpha_i\beta_i} \cdot 1\Big)^2 \leq v \cdot \sum_i \psi_{\alpha_i\beta_i}^2 \leq v \cdot \sum_{\alpha\beta} \psi_{\alpha\beta}^2,$$

where $\sum_{\alpha\beta}$ means that α and $\beta = \beta(\alpha)$ run through all their possible values. Therefore,

$$\max_{1\leq j\leq n} \eta_j^2 \leq v \cdot \sum_{\alpha\beta} \psi_{\alpha\beta}^2. \tag{3.4}$$

Taking the sublinear expectation on the both side of (3.4) and by the sub-additivity of $\widehat{\mathbb{E}}$, we have

$$\widehat{\mathbb{E}}\Big[\max_{1\leq j\leq N} \eta_j^2\Big] \leq v \cdot \sum_{\alpha\beta} \widehat{\mathbb{E}}[\psi_{\alpha\beta}^2].$$

By the sub-orthogonality of $\{X_1, \ldots, X_n\}$, we have

$$\sum_{\alpha\beta} \widehat{\mathbb{E}}[\psi_{\alpha\beta}^2] \leq (v+1) \cdot \sum_{j=1}^{n} c_j^2.$$

Finally,

$$\widehat{\mathbb{E}}\Big[\max_{1\leq j\leq N} \eta_j^2\Big] \leq v \cdot (v+1) \cdot \sum_{j=1}^{n} c_j^2 \leq (\log_2 2n)^2 \sum_{j=1}^{n} c_j^2,$$

and (3.3) holds for the case of $n = 2^v$, $v \in \mathbb{N}^*$. As for the case of $2^l < n < 2^{l+1}$ for some $l \in \mathbb{N}^*$, we only need to take $c_{n+1} = 0, \ldots, c_{2^{l+1}} = 0$ in the case of $v = l + 1$, and then the proof is completed. □

The next lemma is the Borel-Cantelli lemma for countably sub-additive capacities in Zhang [9].

Lemma 3.5 ([9]) *Let $\{A_n, n \in \mathbb{N}^*\}$ be in $\mathcal{F}$, and let V be a countably sub-additive capacity on $\mathcal{F}$. If $\sum_{n=1}^{\infty} V(A_n) < \infty$, then*

$$V(A_n, i.o.) = 0,$$

where $\{A_n, i.o.\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$.

Lemma 3.6 *(i) ([13]) Let $\{a_n\}_{n\geq 1}$ be a sequence with $a_n \nearrow \infty$. Then for any $M > 1$ there exists a sequence $\{n_k\}_{k\geq 1}$ with $n_k \nearrow \infty$ such that*

$$Ma_{n_k} \leq a_{n_{k+1}} \leq M^3 a_{n_k+1}.$$

(ii) Let $a_n > 0$, $n \in \mathbb{N}^*$ and $\sum_{n=1}^{\infty} a_n < \infty$. Then there exists $b_n > 0$, $n \in \mathbb{N}^*$ such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\frac{a_n}{b_n} \searrow 0$ as $n \to \infty$.

Proof (i) is [13, Lemma 3.3]. For (ii), we only need to take $b_n \triangleq \sqrt{\sum_{k=n}^{\infty} a_k} - \sqrt{\sum_{k=n+1}^{\infty} a_k}$. $\square$

4. Main results

We often do not distinguish different sets with probability 1 in classical probability theory, but it is worth noting that we need to distinguish different sets with capacity 1. This is because capacities generally do not have the property: For A, B in $\mathcal{F}$, $\hat{\mathbb{V}}(A) = \hat{\mathbb{V}}(B) = 1 \Rightarrow \hat{\mathbb{V}}(A \cap B) = 1$. For instance, let $\Omega = [0, 2]$, and for $A \in \mathcal{B}([0, 2])$, $P_i(A) \triangleq \mu(A \cap [i-1, i])$, $i = 1, 2$, where μ is the Lebesgue measure on $[0, 2]$. $\hat{\mathbb{V}}(A) \triangleq P_1(A) \vee P_2(A), \forall A \in \mathcal{B}([0, 2])$. Then $\hat{\mathbb{V}}([0, 1]) = \hat{\mathbb{V}}([1, 2]) = 1$, but $\hat{\mathbb{V}}([0, 1] \cap [1, 2]) = \hat{\mathbb{V}}(\{1\}) = 0$. Hence, we need to overcome this crux when proving the law of large numbers for capacities.

We first show that the independence condition of [6, Theorem 3.1] can be weakened to m-dependence.

Theorem 4.1 *Assume that $\{X_n\}_{n\geq 1}$ is a sequence of m-dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Suppose $\{a_n\}_{n\geq 1}$ is a sequence such that $1 \leq a_n \nearrow \infty$ and*

$$\sum_{n=1}^{\infty} \frac{\widehat{\mathbb{E}}[X_n^2]}{a_n^2} < \infty.$$

Then

$$\hat{\mathbb{V}}\Big(\limsup_{n\to\infty} \frac{\sum_{i=1}^{n}(X_i - \widehat{\mathbb{E}}[X_i])}{a_n} > 0 \ \text{ or } \ \liminf_{n\to\infty} \frac{\sum_{i=1}^{n}(X_i - \widehat{\mathcal{E}}[X_i])}{a_n} < 0\Big) = 0. \tag{4.1}$$

Remark 4.2 By the sub-additivity of $\hat{\mathbb{V}}$, we know that for $A \in \mathcal{F}$, $\hat{\mathbb{V}}(A) = 0 \Rightarrow \hat{\mathbb{V}}(A^c) = 1$. If $\widehat{\mathbb{E}}[X_i] = \widehat{\mathcal{E}}[X_i] = 0,\ i \geq 1$, then (4.1) means $\hat{\mathbb{V}}(\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{a_n} = 0) = 1$.

The next result shows a strong limit behavior of a sequence of blockwise m-dependent random variables under the sublinear expectation.

Theorem 4.3 *Let $\{n_i\}_{i\geq 1}$ be a given strictly increasing sequence of natural numbers. $\{X_n\}_{n\geq 1}$ is m-dependent with respect to $\{n_i\}_{i\geq 1}$ in the sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Denote*

$$I_k \triangleq \{i | [2^k, 2^{k+1}) \cap [n_i, n_{i+1}) \neq \emptyset\}, \quad v_k \triangleq \# I_k, \ k \in \mathbb{N}^*,$$

$$[l_{ki}, r_{ki}) \triangleq [2^k, 2^{k+1}) \cap [n_i, n_{i+1}), \quad i \in I_k,$$

$$\Phi(n) \triangleq \max_{0\le j\le k} v_j \text{ if } n \in [2^k, 2^{k+1}), \quad S_n \triangleq \sum_{k=1}^{n} X_k.$$

Assume

(i) There is a random variable Z in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ such that $\frac{1}{n}\sum_{k=1}^n \hat{\mathbb{V}}(|X_k| > t) \le C\hat{\mathbb{V}}(|Z| > t)$, $\forall n \in \mathbb{N}^*$, $\forall t > 0$, where C is a numarical constant;

(ii) $C_{\hat{\mathbb{V}}}[|Z|^r] < \infty$, for some $r \in [1, 2)$, then

$$\hat{\mathbb{V}}\Big(\limsup_{n\to\infty} \frac{S_n - \sum_{k=1}^n \breve{\mathbb{E}}[X_k]}{n^{\frac{1}{r}}\Phi(n)} > 0 \text{ or } \liminf_{n\to\infty} \frac{S_n - \sum_{k=1}^n \breve{\mathcal{E}}[X_k]}{n^{\frac{1}{r}}\Phi(n)} < 0\Big) = 0. \tag{4.2}$$

Remark 4.4 By (i) and (ii) we have $\frac{1}{n}\sum_{k=1}^n C_{\hat{\mathbb{V}}}[|X_k|^r] \le C_{\hat{\mathbb{V}}}[|Z|^r]$. If $\breve{\mathbb{E}}[X_k] = \breve{\mathcal{E}}[X_k] = 0$, $k \ge 1$, then (4.2) means $\hat{\mathbb{V}}(\lim_{n\to\infty} \frac{S_n}{n^{\frac{1}{r}}\Phi(n)} = 0) = 1$. Moreover, if $\{n_i\}_{i\ge 1} = \{2^i\}_{i\ge 1}$ and $r = 1$, then (4.2) means $\hat{\mathbb{V}}(\lim_{n\to\infty} \frac{S_n}{n} = 0) = 1$, which is the form of a common strong law of large numbers. Define $h(n) \triangleq v_k$ if $n \in [2^k, 2^{k+1})$. Then, $h(n)$ can be estimated for some specific $\{n_i\}_{i\ge 1}$. For example, $h(n) = O(1)$ if $\{n_i\}_{i\ge 1} = \{[q^i]\}_{i\ge 1}$ with $q > 1$; $h(n) = O(n^{\frac{1}{\alpha}})$ if $\{n_i\}_{i\ge 1} = \{[i^\alpha]\}_{i\ge 1}$ with $\alpha \ge 1$. One can refer to [3] and [4] for more examples.

Remark 4.5 Let $\{X_n\}_{n\ge 1}$ be a sequence of i.i.d. random variables in the sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ and $C_{\hat{\mathbb{V}}}[|X_1|^r] < \infty$ for some $r \in [1, 2)$. We can take $\{n_i\}_{i\ge 1} = \{2^i\}_{i\ge 1}$ so that $\{X_n\}_{n\ge 1}$ is m-dependent with respect to $\{2^i\}_{i\ge 1}$. Taking $Z = X_1$ and by (4.2) we can obtain

$$\hat{\mathbb{V}}\Big(\limsup_{n\to\infty} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{\frac{1}{r}}} > 0 \text{ or } \liminf_{n\to\infty} \frac{S_n - n\breve{\mathcal{E}}[X_1]}{n^{\frac{1}{r}}} < 0\Big) = 0,$$

which is [11, Theorem 3.4, Eq. (3.9)].

The next result is a strong law of large numbers for sub-orthogonal sequences of random variables under the sublinear expectation.

Theorem 4.6 *Let $\{X_n\}_{n\ge 1}$ in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be sub-orthogonal. Set $\sigma_n^2 \triangleq \widehat{\mathbb{E}}[X_n^2] < \infty$, $S_n \triangleq \sum_{k=1}^n X_k$, $n \ge 1$. If*

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2}(\log_2 k)^2 < \infty,$$

then

$$\hat{\mathbb{V}}\Big(\lim_{n\to\infty} \frac{S_n}{n} = 0\Big) = 1. \tag{4.3}$$

Actually, the requirement of sub-orthogonality in Theorem 4.6 can be weakened to generalized sub-orthogonality.

Corollary 4.7 *Let $\{X_n\}_{n\ge 1}$ in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be generalized sub-orthogonal. Other conditions are the same as in Theorem 4.6, then (4.3) still holds.*

5. Proofs

In this section, we give the proofs of the results in Section 4.

Proof of Theorem 4.1 By Lemma 3.6 (ii), there exists a sequence $\epsilon_k \searrow 0$ such that

$$\sum_{n=1}^{\infty} \frac{\widehat{\mathbb{E}}[X_n^2]}{\epsilon_n^2 a_n^2} < \infty.$$

By Lemma 3.6 (i), for all $M > 1$, there exists a sequence $n_k \nearrow \infty$ such that

$$M a_{n_k} \le a_{n_{k+1}} \le M^3 a_{n_k+1}.$$

For $n_k + 1 \le n \le n_{k+1}$,

$$\sum_{i=n_k+1}^{n} (X_i - \widehat{\mathbb{E}}[X_i]) = \sum_{j=1}^{m+1} \sum_{l=0}^{L(n,j)} (X_{n_k+j+l(m+1)} - \widehat{\mathbb{E}}[X_{n_k+j+l(m+1)}]),$$

where $L(n,j) \triangleq [(n - n_k - j)/(m+1)]$.

We know that $\{X_{n_k+j+l(m+1)} | l = 0, \dots, L(n,j)\}$ is a sequence of independent random variables for given $j \in \{1, \dots, m+1\}$ by the m-dependence of $\{X_n; n \ge 1\}$. And it is also clear that

$$\max_{n_k+1 \le n \le n_{k+1}} \sum_{i=n_k+1}^{n} (X_i - \widehat{\mathbb{E}}[X_i]) \le \sum_{j=1}^{m+1} \max_{0 \le L \le L(n_{k+1},j)} \sum_{l=0}^{L} (X_{n_k+j+l(m+1)} - \widehat{\mathbb{E}}[X_{n_k+j+l(m+1)}]).$$

Hence,

$$\begin{aligned} &\Big\{ \max_{n_k+1 \le n \le n_{k+1}} \sum_{i=n_k+1}^{n} (X_i - \widehat{\mathbb{E}}[X_i]) \ge \epsilon_{n_k} a_{n_{k+1}} \Big\} \\ &\subseteq \bigcup_{j=1}^{m+1} \Big\{ \max_{0 \le L \le L(n_{k+1},j)} \sum_{l=0}^{L} (X_{n_k+j+l(m+1)} - \widehat{\mathbb{E}}[X_{n_k+j+l(m+1)}]) \ge \frac{\epsilon_{n_k} a_{n_{k+1}}}{m+1} \Big\}. \end{aligned}$$

By (3.2), we have

$$\begin{aligned} &\hat{\mathbb{V}}\Big(\max_{0 \le L \le L(n_{k+1},j)} \sum_{l=0}^{L} (X_{n_k+j+l(m+1)} - \widehat{\mathbb{E}}[X_{n_k+j+l(m+1)}]) \ge \frac{\epsilon_{n_k} a_{n_{k+1}}}{m+1} \Big) \\ &\le C \frac{(m+1)^2}{\epsilon_{n_k}^2 a_{n_k+1}^2} \sum_{l=0}^{L(n_{k+1},j)} \widehat{\mathbb{E}}[X_{n_k+j+l(m+1)}^2]. \end{aligned}$$

Then by the sub-additivity of $\hat{\mathbb{V}}$, we have

$$\begin{aligned} &\hat{\mathbb{V}}\Big(\max_{n_k+1 \le n \le n_{k+1}} \sum_{i=n_k+1}^{n} (X_i - \widehat{\mathbb{E}}[X_i]) \ge \epsilon_{n_k} a_{n_{k+1}} \Big) \\ &\le \sum_{j=1}^{m+1} \hat{\mathbb{V}}\Big(\max_{0 \le L \le L(n_{k+1},j)} \sum_{l=0}^{L} (X_{n_k+j+l(m+1)} - \widehat{\mathbb{E}}[X_{n_k+j+l(m+1)}]) \ge \frac{\epsilon_{n_k} a_{n_{k+1}}}{m+1} \Big) \\ &\le C \frac{(m+1)^2}{\epsilon_{n_k}^2 a_{n_{k+1}}^2} \sum_{j=1}^{m+1} \sum_{l=0}^{L(n_{k+1},j)} \widehat{\mathbb{E}}[X_{n_k+j+l(m+1)}^2] \\ &= C \frac{(m+1)^2}{\epsilon_{n_k}^2 a_{n_{k+1}}^2} \sum_{j=n_k+1}^{n_{k+1}} \widehat{\mathbb{E}}[X_j^2] \le C(m+1)^2 \sum_{j=n_k+1}^{n_{k+1}} \frac{\widehat{\mathbb{E}}[X_j^2]}{\epsilon_j^2 a_j^2}. \end{aligned}$$

So we still have

$$\sum_{k=1}^{\infty} \hat{\mathbb{V}}\Big(\max_{n_k+1\le n\le n_{k+1}} \sum_{i=n_k+1}^{n} (X_i - \widehat{\mathbb{E}}[X_i]) \ge \epsilon_{n_k} a_{n_{k+1}}\Big) < \infty,$$

and the rest of the proof is the same as the corresponding part in the proof of [6, Theorem 3.1]. □

Proof of Theorem 4.3 Let $\overline{X}_n \triangleq (-n^{\frac{1}{r}}) \vee X_n \wedge (n^{\frac{1}{r}})$, $\overline{Z}_n \triangleq (-n^{\frac{1}{r}}) \vee Z \wedge (n^{\frac{1}{r}})$, $n \in \mathbb{N}^*$. We show the proof in four steps.

Step 1. We show for the truncated random variables $\{\overline{X}_n\}_{n\ge 1}$,

$$\hat{\mathbb{V}}\Big(\frac{1}{2^{k/r}\Phi(2^k)} \max_{2^k\le n<2^{k+1}} \sum_{j=2^k}^{n} (\overline{X}_j - \breve{\mathbb{E}}[\overline{X}_j]) > \epsilon, i.o.\Big) = 0, \quad \forall \epsilon > 0. \tag{5.1}$$

Set $Y_j \triangleq \overline{X}_j - \breve{\mathbb{E}}[\overline{X}_j]$. For any $\epsilon > 0$,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \hat{\mathbb{V}}\Big(\frac{1}{2^{k/r}\Phi(2^k)} \max_{2^k\le n<2^{k+1}} \sum_{j=2^k}^{n} Y_j > \epsilon\Big)\\
&= \sum_{k=1}^{\infty} \hat{\mathbb{V}}\Big(\max_{2^k\le n<2^{k+1}} \sum_{j=2^k}^{n} Y_j > \epsilon\cdot 2^{k/r}\Phi(2^k)\Big)\\
&\le \sum_{k=1}^{\infty} \hat{\mathbb{V}}\Big(\bigcup_{i\in I_k}\Big\{\max_{l_{ki}\le n<r_{ki}} \sum_{j=l_{ki}}^{n} Y_j > \epsilon\cdot\frac{2^{k/r}\Phi(2^k)}{v_k}\Big\}\Big)\\
&\le \sum_{k=1}^{\infty}\sum_{i\in I_k} \hat{\mathbb{V}}\Big(\max_{l_{ki}\le n<r_{ki}} \sum_{j=l_{ki}}^{n} Y_j > \epsilon\cdot\frac{2^{k/r}\Phi(2^k)}{v_k}\Big)\\
&\le \sum_{k=1}^{\infty}\sum_{i\in I_k} C\cdot\frac{v_k^2}{\epsilon^2 2^{2k/r}\Phi^2(2^k)} \sum_{j=l_{ki}}^{r_{ki}-1} \widehat{\mathbb{E}}[\overline{X}_j^2]\\
&\le C\sum_{k=1}^{\infty}\sum_{i\in I_k} \frac{1}{2^{2k/r}} \sum_{j=l_{ki}}^{r_{ki}-1} \widehat{\mathbb{E}}[\overline{X}_j^2]\\
&= C\cdot\sum_{k=1}^{\infty} \frac{1}{2^{2k/r}} \sum_{j=2^k}^{2^{k+1}-1} \widehat{\mathbb{E}}[\overline{X}_j^2].
\end{aligned}$$

The second inequality follows from the sub-additivity of $\hat{\mathbb{V}}$ and the third inequality follows from Lemma 3.3. The fourth inequality is due to $v_k^2/\Phi^2(2^k) \le 1$. By (3.1), we have

$$\begin{aligned}
\sum_{j=2^k}^{2^{k+1}-1} \widehat{\mathbb{E}}[\overline{X}_j^2] &\le \sum_{j=2^k}^{2^{k+1}-1} C_{\hat{\mathbb{V}}}[\overline{X}_j^2] = \sum_{j=2^k}^{2^{k+1}-1} \int_0^{+\infty} \hat{\mathbb{V}}(|\overline{X}_j|^2 \ge t)\mathrm{d}t\\
&\le \sum_{j=1}^{2^{k+1}} \int_0^{(2^{\frac{k+1}{r}})^2} \hat{\mathbb{V}}(|\overline{X}_j|^2 \ge t)\mathrm{d}t = \sum_{j=1}^{2^{k+1}} \int_0^{(2^{\frac{k+1}{r}})^2} \hat{\mathbb{V}}(|\overline{X}_j| > \sqrt{t})\mathrm{d}t\\
&\le \int_0^{(2^{\frac{k+1}{r}})^2} 2^{k+1}\hat{\mathbb{V}}(|\overline{Z}_{2^{k+1}}| > \sqrt{t})\mathrm{d}t = 2^{k+1}C_{\hat{\mathbb{V}}}[|\overline{Z}_{2^{k+1}}|^2].
\end{aligned}$$

Then

$$\sum_{k=1}^{\infty}\frac{1}{2^{2k/r}}\sum_{j=2^k}^{2^{k+1}-1}\widehat{\mathbb{E}}[\overline{X}_j^2]\le C\sum_{k=0}^{\infty}\frac{1}{2^{2k/r}}2^kC_{\hat{\mathbb{V}}}[|\overline{Z}_{2^k}|^2]=C\sum_{k=0}^{\infty}2^k\int_0^{+\infty}\hat{\mathbb{V}}(|\frac{\overline{Z}_{2^k}}{2^{k/r}}|^2\ge t)\mathrm{d}t$$
$$=C\sum_{k=0}^{\infty}2^k\int_0^{1}\hat{\mathbb{V}}(|\frac{\overline{Z}_{2^k}}{2^{k/r}}|^2\ge t)\mathrm{d}t\le C\sum_{k=0}^{\infty}2^k\int_0^{1}t\cdot\hat{\mathbb{V}}(|\frac{\overline{Z}_{2^k}}{2^{k/r}}|\ge t)\mathrm{d}t$$
$$=C\int_0^1 t\cdot\sum_{k=0}^{\infty}2^k\hat{\mathbb{V}}(|Z|\ge 2^{k/r}t)\mathrm{d}t=C\int_0^1 t\cdot\sum_{k=0}^{\infty}2^k\hat{\mathbb{V}}(|\frac{Z}{t}|^r\ge 2^k)\mathrm{d}t$$
$$\le C\int_0^1 t\cdot C_{\hat{\mathbb{V}}}[|\frac{Z}{t}|^r]\mathrm{d}t=C\cdot C_{\hat{\mathbb{V}}}[|Z|^r]\cdot\int_0^1\frac{1}{t^{r-1}}\mathrm{d}t<\infty.$$

Therefore,

$$\sum_{k=1}^{\infty}\hat{\mathbb{V}}\Big(\frac{1}{2^{k/r}\Phi(2^k)}\max_{2^k\le n<2^{k+1}}\sum_{j=2^k}^{n}Y_j>\epsilon\Big)<\infty,\quad\forall\epsilon>0.\tag{5.2}$$

By Lemma 3.5, (5.2) yields (5.1).

Step 2. We show

$$\hat{\mathbb{V}}(X_k\ne\overline{X}_k,i.o.)=0.\tag{5.3}$$

By Lemma 3.5, we only need to prove $\sum_{k=1}^{\infty}\hat{\mathbb{V}}(X_k\ne\overline{X}_k)<\infty$.

$$\sum_{k=1}^{\infty}\hat{\mathbb{V}}(X_k\ne\overline{X}_k)=\sum_{k=1}^{\infty}\hat{\mathbb{V}}(|X_k|>k^{\frac{1}{r}})\le\sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\hat{\mathbb{V}}(|X_n|>2^{k/r})$$
$$\le C\sum_{k=0}^{\infty}2^k\hat{\mathbb{V}}(|Z|>2^{k/r})\le C\cdot C_{\hat{\mathbb{V}}}[|Z|^r]<\infty.$$

Hence, (5.3) holds.

Step 3. We show

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}(\breve{\mathbb{E}}[X_k]-\breve{\mathbb{E}}[\overline{X}_k])}{n^{\frac{1}{r}}}=0.\tag{5.4}$$

For $1\le r<2$,

$$\Big|\sum_{k=1}^{n}\frac{(\breve{\mathbb{E}}[X_k]-\breve{\mathbb{E}}[\overline{X}_k])}{k^{\frac{1}{r}}}\Big|\le\sum_{k=1}^{n}\frac{\breve{\mathbb{E}}[(|X_k|-k^{\frac{1}{r}})^+]}{k^{\frac{1}{r}}}$$
$$\le\sum_{k=1}^{\infty}\frac{C_{\hat{\mathbb{V}}}[(|X_k|-k^{\frac{1}{r}})^+]}{k^{\frac{1}{r}}}\le\sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\frac{C_{\hat{\mathbb{V}}}[(|X_n|-2^{\frac{k}{r}})^+]}{2^{\frac{k}{r}}}$$
$$=\sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\frac{\int_{2^{k/r}}^{+\infty}\hat{\mathbb{V}}(|X_n|>t)\mathrm{d}t}{2^{\frac{k}{r}}}$$
$$\le C\sum_{k=0}^{\infty}2^k\cdot\frac{\int_{2^{k/r}}^{+\infty}\hat{\mathbb{V}}(|Z|>t)\mathrm{d}t}{2^{\frac{k}{r}}}$$
$$=C\sum_{k=0}^{\infty}2^k\cdot\int_1^{+\infty}\hat{\mathbb{V}}(|Z|>2^{k/r}m)\mathrm{d}m$$

$$
\begin{aligned}
&= C\int_1^{+\infty}\sum_{k=0}^{\infty}2^k\cdot\hat{\mathbb{V}}(|\frac{Z}{m}|^r>2^k)\mathrm{d}m\\
&\le C\int_1^{+\infty}C_{\hat{\mathbb{V}}}(|\frac{Z}{m}|^r)\mathrm{d}m\\
&= C\cdot C_{\hat{\mathbb{V}}}[|Z|^r]\cdot\int_1^{+\infty}\frac{1}{m^r}\mathrm{d}m<\infty.
\end{aligned}
$$

Therefore, according to the Kronecker's lemma, (5.4) holds.

Step 4. We show

$$\hat{\mathbb{V}}\Big(\limsup_{n\to\infty}\frac{S_n-\sum_{k=1}^n\breve{\mathbb{E}}[X_k]}{n^{1/r}\Phi(n)}>\epsilon\Big)=0,\quad\forall\,\epsilon>0. \tag{5.5}$$

Let $2^k\le n<2^{k+1}$, $\epsilon>0$. For every $\omega\in\{\frac{1}{2^{k/r}\Phi(2^k)}\max_{2^k\le n<2^{k+1}}\sum_{j=2^k}^n(\overline{X}_j-\breve{\mathbb{E}}[\overline{X}_j])>\epsilon, i.o.\}^c$, there exists a positive integer $k_0(\omega)$ such that for every $k\ge 2^{k_0(\omega)}$,

$$\frac{1}{2^{k/r}\Phi(2^k)}\max_{2^k\le n<2^{k+1}}\sum_{j=2^k}^n Y_j\le\epsilon, \tag{5.6}$$

where $Y_j\triangleq\overline{X}_j-\breve{\mathbb{E}}[\overline{X}_j]$.

On $\{\frac{1}{2^{k/r}\Phi(2^k)}\max_{2^k\le n<2^{k+1}}\sum_{j=2^k}^n(\overline{X}_j-\breve{\mathbb{E}}[\overline{X}_j])>\epsilon, i.o.\}^c$, we have

$$
\begin{aligned}
\frac{\sum_{j=1}^n Y_j}{n^{1/r}\Phi(n)}&\le\frac{\sum_{j=1}^{2^k-1}Y_j}{n^{1/r}\Phi(n)}+\frac{\max_{2^k\le i<2^{k+1}}\sum_{j=2^k}^i Y_j}{n^{1/r}\Phi(n)}\\
&=\frac{\sum_{j=1}^{2^{k_0}-1}Y_j}{n^{1/r}\Phi(n)}+\frac{\sum_{m=k_0}^{k-1}\sum_{j=2^m}^{2^{m+1}-1}Y_j}{n^{1/r}\Phi(n)}+\frac{2^{k/r}\Phi(2^k)}{n^{1/r}\Phi(n)}\cdot\frac{\max_{2^k\le i<2^{k+1}}\sum_{j=2^k}^i Y_j}{2^{k/r}\Phi(2^k)}.
\end{aligned} \tag{5.7}
$$

The first term in (5.7) converges to zero when $n\to\infty$. By $\frac{2^{k/r}\Phi(2^k)}{n^{1/r}\Phi(n)}\le 1$ and (5.6), we have $\limsup_{n\to\infty}\frac{2^{k/r}\Phi(2^k)}{n^{1/r}\Phi(n)}\cdot\frac{\max_{2^k\le i<2^{k+1}}\sum_{j=2^k}^i Y_j}{2^{k/r}\Phi(2^k)}\le\epsilon$ for the third term in (5.7). As for the middle term,

$$
\begin{aligned}
\frac{\sum_{m=k_0}^{k-1}\sum_{j=2^m}^{2^{m+1}-1}Y_j}{n^{1/r}\Phi(n)}&\le\sum_{m=k_0}^{k-1}\frac{2^{m/r}\Phi(2^m)}{n^{1/r}\Phi(n)}\cdot\frac{\max_{2^m\le n<2^{m+1}}\sum_{j=2^m}^n Y_j}{2^{m/r}\Phi(2^m)}\\
&\le\epsilon\cdot\sum_{m=k_0}^{k-1}\frac{2^{m/r}\Phi(2^m)}{n^{1/r}\Phi(n)}\le\epsilon\cdot\frac{\sum_{m=k_0}^{k-1}2^{m/r}}{n^{1/r}}\\
&\le\epsilon\cdot\frac{\frac{1}{\ln(2^{1/r})}\cdot 2^{k/r}}{n^{1/r}}\le 3\epsilon,
\end{aligned}
$$

where the fourth inequality follows from the truth that for $\alpha>1$, $\sum_{m=0}^{k-1}\alpha^m\le\frac{\alpha^k}{\ln\alpha}$. Hence, we have $\limsup_{n\to\infty}\frac{\sum_{k=1}^n(\overline{X}_k-\breve{\mathbb{E}}[\overline{X}_k])}{n^{1/r}\Phi(n)}\le 4\epsilon$ on $\{\frac{1}{2^{k/r}\Phi(2^k)}\max_{2^k\le n<2^{k+1}}\sum_{j=2^k}^n Y_j>\epsilon, i.o.\}^c$, which implies

$$\hat{\mathbb{V}}\Big(\limsup_{n\to\infty}\frac{\sum_{k=1}^n(\overline{X}_k-\breve{\mathbb{E}}[\overline{X}_k])}{n^{1/r}\Phi(n)}>4\epsilon\Big)\le\hat{\mathbb{V}}\Big(\frac{1}{2^{k/r}\Phi(2^k)}\max_{2^k\le n<2^{k+1}}\sum_{j=2^k}^n Y_j>\epsilon, i.o.\Big)=0,\quad\forall\,\epsilon>0. \tag{5.8}$$

It is clear that

$$\frac{S_n-\sum_{k=1}^n \breve{\mathbb{E}}[X_k]}{n^{1/r}\Phi(n)}=\frac{\sum_{k=1}^n (X_k-\overline{X}_k)}{n^{1/r}\Phi(n)}+\frac{\sum_{k=1}^n (\overline{X}_k-\breve{\mathbb{E}}[\overline{X}_k])}{n^{1/r}\Phi(n)}+\frac{\sum_{k=1}^n (\breve{\mathbb{E}}[\overline{X}_k]-\breve{\mathbb{E}}[X_k])}{n^{1/r}\Phi(n)},$$

which implies

$$\Big\{\limsup_{n\to\infty}\frac{S_n-\sum_{k=1}^n \breve{\mathbb{E}}[X_k]}{n^{1/r}\Phi(n)}>\epsilon\Big\}\subseteq\{X_k\neq\overline{X}_k, i.o.\}$$
$$\bigcup\Big\{\limsup_{n\to\infty}\frac{\sum_{k=1}^n (\overline{X}_k-\breve{\mathbb{E}}[\overline{X}_k])}{n^{1/r}\Phi(n)}>\epsilon\Big\},\quad \forall\epsilon>0.$$

Therefore, by the sub-additivity of $\hat{\mathbb{V}}$, (5.3) and (5.8), we have (5.5).

So far, we have obtained (5.1), (5.3)–(5.5).

We notice that

$$\hat{\mathbb{V}}\Big(\limsup_{n\to\infty}\frac{S_n-\sum_{k=1}^n \breve{\mathbb{E}}[X_k]}{n^{1/r}\Phi(n)}>0\Big)=\hat{\mathbb{V}}\Big(\bigcup_{m=1}^{\infty}\Big\{\limsup_{n\to\infty}\frac{S_n-\sum_{k=1}^n \breve{\mathbb{E}}[X_k]}{n^{1/r}\Phi(n)}>\frac{1}{m}\Big\}\Big).$$

Hence, by the countable sub-additivity of $\hat{\mathbb{V}}$ and (5.5) we have

$$\hat{\mathbb{V}}\Big(\limsup_{n\to\infty}\frac{S_n-\sum_{k=1}^n \breve{\mathbb{E}}[X_k]}{n^{1/r}\Phi(n)}>0\Big)=0.$$

Taking X_k by $-X_k$, we have

$$\hat{\mathbb{V}}\Big(\liminf_{n\to\infty}\frac{S_n-\sum_{k=1}^n \breve{\mathcal{E}}[X_k]}{n^{1/r}\Phi(n)}<0\Big)=0.$$

Finally, by the sub-additivity of $\hat{\mathbb{V}}$, (4.2) is proved. □

Next, we show the proof of Theorem 4.6.

Proof of Theorem 4.6 By Lemma 3.6 (ii), there exists a sequence $\epsilon_k\searrow 0$ such that

$$\sum_{k=1}^{\infty}\frac{\sigma_k^2}{\epsilon_k^2 k^2}(\log_2 k)^2<\infty,$$

then

$$\hat{\mathbb{V}}\Big(\max_{2^k+1\le n\le 2^{k+1}}|S_n-S_{2^k}|\ge\epsilon_{2^k}\cdot 2^{k+1}\Big)\le\frac{\widehat{\mathbb{E}}[(\max_{2^k+1\le n\le 2^{k+1}}|S_n-S_{2^k}|)^2]}{\epsilon_{2^k}^2(2^{k+1})^2}$$
$$\le(\log_2 2^{k+2})^2\frac{\sum_{j=2^k+1}^{2^{k+1}}\widehat{\mathbb{E}}[X_j^2]}{\epsilon_{2^k}^2(2^{k+1})^2}\le(\log_2 2^{k+2})^2\sum_{j=2^k+1}^{2^{k+1}}\frac{\widehat{\mathbb{E}}[X_j^2]}{\epsilon_j^2 j^2}$$
$$\le\sum_{j=2^k+1}^{2^{k+1}}\frac{\widehat{\mathbb{E}}[X_j^2]}{\epsilon_j^2 j^2}(2+\log_2 j)^2,$$

where we use the Rademacher-Mensov inequality (3.3) in the second inequality. Hence,

$$\sum_{k=1}^{\infty}\hat{\mathbb{V}}\Big(\max_{2^k+1\le n\le 2^{k+1}}|S_n-S_{2^k}|\ge\epsilon_{2^k}\cdot 2^{k+1}\Big)\le\sum_{j=1}^{\infty}\frac{\sigma_j^2}{\epsilon_j^2 j^2}(2+\log_2 j)^2<\infty,$$

and by Lemma 3.5 we have $\hat{\mathbb{V}}(\max_{2^k+1\le n\le 2^{k+1}}|S_n-S_{2^k}|\ge\epsilon_{2^k}\cdot 2^{k+1}, i.o.)=0$.

For every $\omega \in \{\max_{2^k+1\le n\le 2^{k+1}} |S_n - S_{2^k}| \ge \epsilon_{2^k}\cdot 2^{k+1}, i.o.\}^c$, there exists a positive integer $K_0(\omega)$ such that for every $k \ge K_0(\omega)$, $\max_{2^k+1\le n\le 2^{k+1}} |S_n - S_{2^k}| < \epsilon_{2^k}\cdot 2^{k+1}$. Let $n > 2^{K_0(\omega)+1}$, $2^k+1 \le n \le 2^{k+1}$, $K_0(\omega) \le k_0 < k$. On $\{\max_{2^k+1\le n\le 2^{k+1}} |S_n - S_{2^k}| \ge \epsilon_{2^k}\cdot 2^{k+1}, i.o.\}^c$, we have

$$\begin{aligned}
|S_n| &\le |S_n - S_{2^k}| + \sum_{i=k_0+1}^{k} |S_{2^i} - S_{2^{i-1}}| + |S_{2^{k_0}}| \\
&\le \epsilon_{2^k}\cdot 2^{k+1} + \sum_{i=k_0+1}^{k} \epsilon_{2^{i-1}} 2^i + |S_{2^{k_0}}| \\
&\le \epsilon_{2^{k_0}}\cdot 2^{k+1}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-k_0}}\Big) + |S_{2^{k_0}}| \\
&\le \epsilon_{2^{k_0}}\cdot 2^{k+1}\cdot 2 + |S_{2^{k_0}}|.
\end{aligned}$$

Then

$$\frac{|S_n|}{n} \le \epsilon_{2^{k_0}}\cdot \frac{2^{k+1}}{n}\cdot 2 + \frac{|S_{2^{k_0}}|}{n} \le \epsilon_{2^{k_0}}\cdot \frac{2^{k+1}}{2^k}\cdot 2 + \frac{|S_{2^{k_0}}|}{n} = 4\epsilon_{2^{k_0}} + \frac{|S_{2^{k_0}}|}{n}.$$

Hence

$$\limsup_{n\to\infty} \frac{|S_n|}{n} \le 4\epsilon_{2^{k_0}}.$$

Then letting $k_0 \to \infty$, we have $\lim_{n\to\infty} \frac{S_n}{n} = 0$, which means

$$\Big\{\max_{2^k+1\le n\le 2^{k+1}} |S_n - S_{2^k}| \ge \epsilon_{2^k}\cdot 2^{k+1}, i.o.\Big\}^c \subseteq \Big\{\lim_{n\to\infty} \frac{S_n}{n} = 0\Big\}.$$

Therefore,

$$1 = \hat{\mathbb{V}}\Big(\Big\{\max_{2^k+1\le n\le 2^{k+1}} |S_n - S_{2^k}| \ge \epsilon_{2^k}\cdot 2^{k+1}, i.o.\Big\}^c\Big) \le \hat{\mathbb{V}}\Big(\lim_{n\to\infty} \frac{S_n}{n} = 0\Big).$$

(4.3) is proved. □

Proof of Corollary 4.7 We just need to show that Rademacher-Mensov type inequality is still valid for the generalized sub-orthogonal sequence of random variables $\{X_n\}_{n\ge1}$, and the rest of the proof is the same as that of Theorem 4.6. By the definition of generalized sub-orthogonality (see Definition 2.6) we know that there exists a nonnegative sequence $\{f(j): j = 0, 1, \ldots\}$ with $\sum_{j=0}^{\infty} f(j) < \infty$ such that $\widehat{\mathbb{E}}[X_k X_l] \le \sigma_k\sigma_l f(|k-l|)$ for all $k, l = 1, 2, \ldots$. For $n \ge 2$,

$$\begin{aligned}
\widehat{\mathbb{E}}\Big[\Big(\sum_{k=1}^{n} X_k\Big)^2\Big] &= \widehat{\mathbb{E}}\Big[\sum_{k=1}^{n} X_k^2 + 2\sum_{j=1}^{n-1}\sum_{k=1}^{n-j} X_k X_{k+j}\Big] \\
&\le \sum_{k=1}^{n} \widehat{\mathbb{E}}[X_k^2] + 2\sum_{j=1}^{n-1}\sum_{k=1}^{n-j} \widehat{\mathbb{E}}[X_k X_{k+j}] \\
&\le \sum_{k=1}^{n} \sigma_k^2 + 2\sum_{j=1}^{n-1}\sum_{k=1}^{n-j} \sigma_k\sigma_{k+j} f(j) \\
&\le \sum_{k=1}^{n} \sigma_k^2 + \sum_{j=1}^{n-1} f(j) \sum_{k=1}^{n-j} (\sigma_k^2 + \sigma_{k+j}^2)
\end{aligned}$$

$$\leq \Big(1+2\sum_{j=1}^{\infty} f(j)\Big)\sum_{k=1}^{n}\sigma_k^2.$$

Then manipulating the proof of Lemma 3.4 again, we have

$$\widehat{\mathbb{E}}\Big[\max_{1\leq k\leq n}\Big(\sum_{j=1}^{k}X_j\Big)^2\Big] \leq \Big(1+2\sum_{j=1}^{\infty} f(j)\Big)(\log_2 4n)^2\sum_{j=1}^{n}\widehat{\mathbb{E}}[X_j^2].$$

The proof is completed. □

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References

[1] W. HOEFFDING, H. ROBBINS. *The central limit theorem for dependent random variables.* Duke Math. J., 1948, **15**(3): 773–780.

[2] F. MORICZ. *Strong limit theorems for blockwise m-dependent and blockwise quasiorthogonal sequences of random variables.* Proc. Amer. Math. Soc., 1987, **101**(4): 709–715.

[3] V. F. GAPOSHKIN. *On the strong law of large numbers for blockwise independent and blockwise orthogonal random variables.* Theory Probab. Appl., 1994, **39**(4): 677–684.

[4] Bo ZHANG. *Marcinkiewicz-Zygmund law for sequences of blockwise m-dependent random variables.* Statist. Probab. Lett., 1998, **38**(1): 83–88.

[5] Shige PENG. *Nonlinear Expectations and Stochastic Calculus under Uncertainty.* Springer, Berlin, 2019.

[6] Wangyun GU, Lixin ZHANG. *Strong law of large numbers for m-dependent and stationary random variables under sub-linear expectations.* Sci. Sin. Math., 2026, **56**(1): 1–20. (in Chinese)

[7] Xiaofan GUO, Xingpeng LI. *On the laws of large numbers for pseudo-independent random variables under sublinear expectation.* Statist. Probab. Lett., 2021, **172**: Paper No. 109042, 8 pp.

[8] Xiaofan GUO, Shan LI, Xinpeng LI. *On the laws of the iterated logarithm with mean-uncertainty under sublinear expectations.* Probab. Uncertain. Quant. Risk., 2022, **7**(1): 1–12.

[9] Lixin ZHANG. *Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications.* Sci. China Math., 2016, **59**(4): 751–768.

[10] Lixin ZHANG. *Exponential inequalities under the sub-linear expectations with applications to laws of the iterated logarithm.* Sci. China Math., 2016, **59**(12): 2503–2526.

[11] Lixin ZHANG. *The sufficient and necessary conditions of the strong law of large numbers under sub-linear expectations.* Acta Math. Sin. (Engl. Ser.), 2023, **39**(12): 2283–2315.

[12] P. REVESZ. *The Laws of Large Numbers.* Academic Press, New York, 1968.

[13] R. WITTMANN. *A general law of iterated logarithm.* Z. Wahrsch. Verw. Gebiete., 1985, **68**(4): 521–543.