Journal of Mathematical Research with Applications
Jan., 2026, Vol. 46, No. 1, pp. 134–142
DOI:10.3770/j.issn:2095-2651.2026.01.011
Http://jmre.dlut.edu.cn

Strategic and Regional Investigation of the Exact Controllability of the Vibrating Plate Equation on a Regular Domain

Mouhamadou NGOM[1], **Cheikh SECK**[2,*]

1. *Department of Mathematics, UFR SATIC, Alioune Diop University of Bambey, Member of ER-ANLG, Alioune Diop University of Bambey, Bp 30 Bambey, Senegal;*
2. *Department of Mathematics, FASTEF ex ENS of Dakar, Cheikh Anta Diop University of Dakar, Senegal*

Abstract In this paper, we define for the trace operator, the solution of certain models of vibrating plates standards with initial data in a strategic region spaces of weak regularities. Indeed, we know that the notion of regional controllability is more adapted to systems described by dynamic systems. Regional controllability results in a strategic area were established for vibrating plates by the Hilbertian Uniqueness Method.

Keywords exact controllability; vibrating plates; strategic regional control; Hilbert uniqueness method

MR(2020) Subject Classification 93B07; 93B05

1. Introduction

The mathematical study of plates has taken a meteoric rise with the spectacular development of electronics and computing; and, the need to install detectors and sensors on both static and dynamic devices and systems. The exact controllability of a dynamic system consists of bringing it from one initial state to another by an external action (or criterion), in a given time T. For a reversible linear system, this amounts to bringing it to rest. This method suggests that a vibrating elastic plate can be brought to rest by applying forces to it on the surface or inside. The particularity and contribution of this work in control theory is the use of strategic actuators. In addition, it has been proven that regional controllability (on strategic parts of the $\omega \subset \Omega$ domain) is more suitable and more feasible.

Well-known control devices are those involving materials piezoelectrics for example, which can be used both to detect the vibrations of the plate and to impose control forces on it. Indeed, a dynamical system (S) is said to be exactly regionally controllable (or even exactly ω-controllable) if for every yd (the desired state) given in $L^2(\Omega)$ (of course in $L^2(\omega)$), there exists a control $u \in U$ (realizable or admissible set or domain) such as $y(.,T,u) = yd$.

Received February 11, 2025; Accepted September 28, 2025
* Corresponding author
E-mail address: mouhamadou.ngom@uadb.edu.sn (Mouhamadou NGOM); cheikh5.seck@ucad.edu.sn (Cheikh SECK)

The emphasis in this work was placed on regional controllability, which is more suited to reality. In control theory, we know that local or regional controllability on so-called strategic subdomains of the main domain is more realistic and more in line with reality. Indeed, this type of control is achieved with minimal time and therefore at a lower cost. In this work, we established an exact regional controllability result on a strategic part of the boundary for a two-dimensional elastic plate. Consider Ω a regular open of $\mathbb{R}^2$ and $\omega \subseteq \Omega$ (not arbitrarily chosen but small) of respective boundaries $\Gamma = \partial\Omega, \gamma = \partial\omega$ and $T > 0$. For $x^0 \in \mathbb{R}^2$ a vector with $x^0 = (x_1^0, x_2^0)$, we define the function of multiplier $m(x) = x - x^0$ and a geometric condition partition of the boundaries: $\Sigma_T =]0, T[\times\partial\Omega$ and $\Sigma_{\gamma,T} =]0, T[\times\partial\omega$ in the following way

$$\Gamma_0^+ = \{x \in \Gamma / m(x).\nu(x) > 0\}, \quad \gamma_0^+ = \{x \in \gamma / m(x).\nu(x) > 0\},$$

$$\Gamma_0^- = \{x \in \Gamma / m(x).\nu(x) < 0\}, \ \gamma_0^- = \{x \in \gamma / m(x).\nu(x) < 0\}, \ Q_T =]0, T[\times\omega$$

and

$$\Sigma_0 =]0, T[\times\Gamma_0^+, \quad \Sigma_{0,\omega} =]0, T[\times\gamma,$$

$$\Sigma_0^* =]0, T[\times\Gamma_0^-, \quad \Sigma_{0,\omega}^* =]0, T[\times\gamma_0^-.$$

We also introduce the constants

$$R_0 = \max_{x\in\overline{\omega}}\Big(\sum_{k=1}^{2}(x - x_k^0)^2\Big)^{\frac{1}{2}}, \quad R_{0,\omega} = \max_{x\in\overline{\omega}}\Big(\sum_{k=1}^{2}(x - x_k^0)^2\Big)^{\frac{1}{2}},$$

$$T_0 = \frac{R_0}{\lambda_0}, \quad T_{0,\omega} = \frac{R_{0,\omega}}{\lambda_0},$$

where λ_0 is the first eigenvalue of the spectral problem

$$\begin{cases} \Delta^2\zeta = \lambda^2\zeta, \quad \text{in } \ \omega, \\ \zeta \in H_0^2(\omega). \end{cases} \tag{1.1}$$

We denote by $\nu(x)$ the exterior unit normal at all x belonging to γ, ω a non-empty strategic part of the set Ω. Let u be the solution to the following homogeneous equation

$$\begin{cases} u'' + \Delta^2 u = 0, & \text{in } \ Q_T =]0, T[\times\omega, \\ \gamma u = \gamma\dfrac{\partial u}{\partial\nu}, & \text{on } \ \Sigma_\gamma, \\ u(0) = u_0, \ u'(0) = u_1. \end{cases} \tag{1.2}$$

We define for initial data u_0 and u_1 respectively in $H_0^2(\omega)$ and $L^2(\omega)$, the energy associated with (1.2) by

$$E_0 = \frac{1}{2}(\|\Delta u_0\|^2_{L^2(\omega)} + \|u_1\|^2_{L^2(\omega)}). \tag{1.3}$$

In Lions [1], the following direct and inverse inequalities were established and for all $T > T_0$, there exists $C_0 > 0$

$$\int_0^T \int_{\Sigma_{0,\omega}} (\Delta u)^2 \mathrm{d}\sigma\mathrm{d}t \le C_0 E_0, \tag{1.4}$$

$$\int_0^T \int_{\Sigma_{0,\omega}^*} (\Delta u)^2 \mathrm{d}\sigma\mathrm{d}t \ge C_T^1 E_0, \tag{1.5}$$

where

$$C_T^1 = \frac{4}{R_{0,\omega}(T-T_0)},$$

a generic positive constant which may depend on T and but not on Ω or ω. Similarly, define the unbounded operator A of $L^2(\omega)$ by

$$D_A = \{u \in H_0^2(\omega)/\Delta^2 u \in L^2(\omega)\} \text{ and } Au = \Delta^2 u, \text{ for all } u \text{ in } D_A.$$

In Lasiecka-Triggiani [2], for the solution u of (1.2) corresponding to the data $u_0 \in D_{A^{\frac{3}{4}}}$ and $u_1 \in D_{A^{\frac{1}{4}}}$, this inequality has been established

$$\int_{\Sigma_{0,\omega}} |\gamma \frac{\partial \Delta u}{\partial \nu}|^2 \mathrm{d}\sigma \mathrm{d}t \geq C(T-T_0)E_{\frac{3}{4}},$$

where

$$E_{\frac{3}{4}} = \frac{1}{2}\{\|A^{\frac{3}{4}}u_0\|^2_{L^2(\omega)} + \|A^{\frac{1}{4}}u_1\|^2_{L^2(\omega)}\},$$

and $C > 0$.

In [1], Lions established the exact controllability of the vibrating plate equation with control over the normal derivative of the system state. Lasiecka and Triggiani [2] established the exact controllability of the vibrating plate equation for position control using (2.1). Finally Niane [3,4] and Zuazua [5] showed, still for the same types of initial data, the exact controllability in an arbitrarily small time. We generalize the estimate of $\gamma\Delta u$ in $L^2(0,T;L^2(\Sigma_{0,\omega}))$ established in (1.4) to estimates in $L^2(0,T;H^{-\frac{3}{2}\theta}(\Sigma_{0,\omega}))$ where $0 \leq \theta \leq 1$. In the same vein, we make even for $\gamma\frac{\partial \Delta u}{\partial \nu}$ in (1.4) in estimates in $L^2(0,T;H^{-\frac{\theta}{2}}(\Sigma_{0,\omega}))$. These inequalities are obtained by interpolation. Using these new estimates, we establish exact controllability results of the vibrating plate equation for irregular data.

Remark 1.1 (1) The system (1.2) is not hyperbolic. It is a very ill-posed system in Petrovsky's sense [1].

(2) The system is not hyperbolic; however, we could still consider its exact controllability within an arbitrarily small time. That said, establishing such controllability results will require additional developments [5].

(3) The system (1.2) models the vibrations of a plate $\mathcal{U} \subset \omega$ in a very simplified way (see Avalos [6] for the modeling and the exact controllability of some of these models).

2. Regional and strategic control on the normal derivative: preliminaries and main result

To state the main result, we need to introduce an important definition relating to the notions of regional controllability and strategic actuators.

Definition 2.1 *We call an actuator any couple* (ω, g_0) *where*

(1) $\omega \subset \Omega$ *is the geometric support of the action;*

(2) $g_0 \in L^2(\omega)$ *is the spatial distribution of the action in* ω*;*

(3) In the case of a point action exerted at $b \in \Omega$, we will have $\Omega_0 \equiv b$ and $g_0 \equiv \delta b$ (Dirac mass).

We say that an actuator is strategic (resp., ω-strategic) if the system (1.2) it excites is weakly controllable (resp., weakly ω-controllable) [7].

In the sequel, let $0 \leq \theta \leq 1$, $\forall(\varphi_0, \varphi_1) \in D(\omega) \times D(\omega)$ and pose

$$\|(\varphi_0, \varphi_1)\|^2_{F_\theta} = \int_0^T \|\gamma\Delta\varphi\|^2_{H^{-\frac{3\theta}{2}}(\Gamma_0^+)} \mathrm{d}t, \tag{2.1}$$

where φ is the solution of the homogeneous Eq. (1.2) corresponding to the initial data (φ_0, φ_1). We denote by F_θ the complement of $D(\omega) \times D(\omega)$ for the norm $\|\|^2_{F_\theta}$.

Theorem 2.2 (Main theorem) *For all $\theta \in [0, 1]$, the solution φ of the homogeneous equation (1.2), corresponding to the initial data $(\varphi_0, \varphi_1) \in F_\theta$ checks the inequality*

$$\int_0^T \|\gamma\Delta\varphi\|^2_{H^{-\frac{3\theta}{2}}(\Sigma_{0,\omega})} \mathrm{d}t \geq K_\omega\{\|\varphi_0\|^2_{H_0^{2-2\theta}(\omega)} + \|\varphi_1\|^2_{H^{-2\theta}(\omega)}\}, \tag{2.2}$$

where K_ω is a strictly positive constant dependent only on ω.

Proof Let $(\varphi_0, \varphi_1) \in L^2(\omega) \times H^{-2}(\omega)$. We consider the following problem

$$\varphi'' + A\varphi = 0,\ \varphi(0) = \varphi_0,\ \varphi'(0) = \varphi_1.$$

This problem admits a unique solution given by $(\varphi(t), \varphi'(t)) = S_1(t)(\varphi_0, \varphi_1)$ where $\{S_1(t) : t \in \mathbb{R}\}$ is the C^0-unit group on $L^2(\omega) \times H^{-2}(\omega)$ generated by A. Let $\psi = A^{-1}\varphi'$, $\psi_0 = A^{-1}\varphi_1 \in (H_0^2(\omega))$, $\psi_1 = -\varphi_0 \in L^2(\omega)$. Then ψ satisfies

$$\psi'' + A\psi = 0, \quad \psi'(0) = \psi_1.$$

Also this problem has a unique solution given by $(\psi(t), \psi'(t)) = S_2(t)(\psi_0, \psi_1)$ where $\{S_2(t) : t \in \mathbb{R}\}$ is the C_0-unitary group on $H_0^2(\omega) \times L^2(\omega)$ generated by A.

For all $T > 0$, we apply inequality (1.4) to ψ, which gives the estimate

$$(\|\varphi_0\|^2_{L^2(\omega)} + \|\varphi_1\|^2_{H^{-2}(\omega)}) \leq C_T \int_{\Sigma_{0,\omega}} |\gamma\Delta A^{-1}\varphi'|^2 \mathrm{d}\sigma. \tag{2.3}$$

For $\epsilon \geq 0$ sufficiently small (such that $T - T_0 > 2\epsilon$), we define the test function $\varphi_\epsilon \in D(R)$ satisfying

(1) $0 \leq \varphi_\epsilon \leq 1$;

(2) $\mathrm{supp}(\varphi_\epsilon) \subset]0, T[$;

(3) $\varphi_\epsilon|_{[\epsilon, T-\epsilon]} = 1$.

Multiplying by φ_ϵ and integrating by part (Green formulas), we obtain

$$\begin{aligned}(\|\varphi_0\|^2_{L^2(\omega)} + \|\varphi_1\|^2_{H^{-2}(\omega)}) \leq & C_T \int_{\Sigma_{0,\omega}} [\varphi'_\epsilon(\gamma\Delta A^{-1}\varphi')((\gamma\Delta A^{-1}\varphi)) + \varphi_\epsilon(\gamma\Delta A^{-1}\varphi'')(\gamma\Delta A^{-1}\varphi)]\mathrm{d}\sigma \\ \leq & C_T \int_{\Sigma_{0,\omega}} [\frac{1}{2}\varphi''|\gamma\Delta A^{-1}\varphi|^2 + \varphi_\epsilon(\gamma\Delta\varphi)(\gamma\Delta A^{-1}\varphi)]\mathrm{d}\sigma.\end{aligned}$$

Using estimation (1.4) gives

$$\begin{aligned}\int_{\Sigma_{0,\omega}} \varphi_\epsilon''|\gamma\Delta A^{-1}\varphi|^2 d\sigma \leq & C_T(\|A^{-1}\varphi^2\|^2_{H_0^2(\omega)} + \|A^{-1}\varphi_1\|^2_{L^2(\omega)}) \\ \leq & C_T(\|\varphi_0\|^2_{H^{-2}(\omega)} + \|\varphi_1\|^2_{D'_A}),\end{aligned}$$

where D'_A is the dual of D_A in $L^2(\omega)$. By applying Young's inequality, we have, for any positive $\delta > 0$,

$$\int_{\Sigma_{0,\omega}} \varphi_\epsilon(\gamma\Delta\varphi)(\gamma\Delta A^{-1}\varphi)d\sigma \leq C(\frac{1}{\delta}\|\gamma\Delta\varphi\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma_0))} + \delta\|\gamma\Delta A^{-1}\varphi\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Gamma_0^+))})$$

but $\gamma A^{-1}\varphi = \gamma\frac{\partial}{\partial\nu}A^{-1}\varphi = 0$. So by the regularity theorem applied to the bilaplacian [6, 9]

$$\begin{aligned}\|\gamma\Delta(A^{-1}\varphi)\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Gamma_0^+))} \leq & C_T\|\Delta^2(A^{-1}\varphi)\|^2_{L^2(0,T;L^2(\omega))} \\ \leq & C_T\|\varphi\|^2_{L^2(0,T;L^2(\omega))} \\ \leq & C_T(\|\varphi_0\|^2_{L^2(\omega)} + \|\varphi_1\|^2_{H^{-2}(\omega)}).\end{aligned}$$

By choosing δ small enough, we obtain

$$\|\varphi_0\|^2_{L^2(\omega)} + \|\varphi_1\|^2_{H^{-2}(\omega)} \leq C_T[(\|\varphi_0\|^2_{H^{-2}(\omega)} + \|\varphi_1\|^2_{D'_A}) + \|\gamma\Delta\varphi\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma_{0,\omega}))}].$$

It now remains to show that

$$(\|\varphi_0\|^2_{H^{-2}(\omega)} + \|\varphi_1\|^2_{D'_A}) \leq C_T\|\gamma\Delta\varphi\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma_{0,\omega}))}.$$

Indeed assume that there exists (φ_n) solution of

$$\varphi_n'' + A\varphi_n = 0,\ \varphi_n(0) = \varphi_{0,n},\ \varphi_n' = \varphi_{1,n},$$

where $(\varphi_{0,n}, \varphi_{1,n}) \in L^2(\omega) \times H^{-2}(\omega)$, satisfying

$$(\|\varphi_{o,n}\|^2_{H^{-2}(\omega)} + \|\varphi_{1,n}\|^2_{D'_A}) = 1, \quad \|\gamma\Delta\varphi_n\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma_{0,\omega}))} \to 0.$$

Then with (2.1), $(\varphi_{o,n}(t), \varphi_{1,n}(t))$ is bounded in $L^2(\omega) \times H^{-2}(\omega)$. Even if it means extracting a subsequence, we have $(\varphi_{o,n}, \varphi_{1,n}) \rightharpoonup (\varphi_o, \varphi_1)$ weakly in $L^2(\omega) \times H^{-2}(\omega)$ like the injection of $L^2(\omega) \times H^{-2}(\omega)$ in $H^{-2}(\omega) \times D'_A$ being compact, $(\varphi_{o,n}, \varphi_{1,n}) \to (\varphi, \varphi')$ strongly in $H^{-2}(\omega) \times D'_A$. So

$$\|\varphi_0\|^2_{H^{-2}(\omega)} + \|\varphi_1\|^2_{D'_A} = 1. \tag{2.4}$$

Let φ be the solution corresponding to φ_0 and φ_1. We have

$$\|\gamma\Delta\varphi\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma_{0,\omega}))} = 0,$$

and φ is solution of

$$\begin{gathered}\varphi'' + \Delta^2\varphi = 0, \\ \varphi = \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Sigma, \quad \Delta\varphi = 0 \text{ on } \Sigma_0.\end{gathered}$$

Then by the classical uniqueness theorem $\varphi \equiv 0$ on Σ which contradicts (2.3). Thus there exists a constant C_T^1 such that

$$\int_0^T \|\gamma\Delta\varphi\|^2_{H^{-\frac{3}{2}}(\Gamma_{0,\omega})} dt \geq C_T^1\{\|\varphi_0\|^2_{L^2(\omega)} + \|\varphi_1\|^2_{H^{-2}(\omega)}\}. \tag{2.5}$$

The inequalities (2.2) and (2.5) are given by interpolation, for all $\theta \in [0,1]$ and initial data $(\varphi_0, \varphi_1) \in (H_0^{2-2\theta} \times H^{-2\theta}(\omega))$. We have

$$\int_0^T \|\gamma\Delta\varphi\|^2_{H^{-\frac{3}{2}}(\Gamma_{0,\omega})} \mathrm{d}t \geq K\{\|\varphi_0\|^2_{H_0^{2-2\theta}} + \|\varphi_1\|^2_{H^{-2\theta}}\}. \quad \square$$

3. Regional controllability result strategic border

We denote by Δ_Γ the Laplace Beltrami operator [8]. A consequence of equality (2.4) is the following exact controllability theorem:

Theorem 3.1 *There exists a time T_0 such that if $T > T_0$, then for all $\theta \in [0,1]$ and any pair of initial data $(y_0, y_1) \in (H_0^{2\theta} \times H^{2\theta-2}(\omega))$, there exists $v \in L^2(0,T; H^{-\frac{3}{2}\theta}(\Gamma_{0,\omega}))$ such that, if y is the solution of*

$$\begin{cases} y'' + \Delta^2 y = 0, & \text{in }]0,T[\times\omega, \\ \gamma y = 0, & \text{on } \gamma_0^+(x^0), \\ \gamma\frac{\partial y}{\partial\nu} = (-\Delta_{\Gamma^+})^{-\frac{3}{2}\theta} v, & \text{on } \Sigma_0, \\ y(0) = y_0;\ y'(0) = y_1, \end{cases}$$

then, at time T, $y(T) = y'(T) = 0$.

Proof We apply the HUM method from Lions [1]. We start from $\varphi_o \in H_0^{2-2\theta}(\omega), \varphi_1 \in H^{-2\theta}(\omega)$. Let $\varphi \in C(0,T; H_0^{2-2\theta}(\omega)) \cap (0,T; H^{-2\theta)}(\omega))$ be the only solution to the problem

$$\begin{cases} \varphi'' + \Delta^2\varphi = 0, & \text{in } Q_T, \\ \varphi(0) = \varphi_o, \\ \varphi'(0) = \varphi_1, \\ \gamma\varphi = \gamma\frac{\partial\varphi}{\partial\nu} = 0, & \text{in } \Sigma_0. \end{cases}$$

Let ξ be the unique solution by transposition of the equation

$$\begin{cases} \xi'' + \Delta^2\xi = 0, & \text{in }]0,T[\times\omega, \\ \xi(T) = 0, & \text{in } \omega, \\ \xi'(t) = 0, & \text{in } \omega, \\ \gamma\xi = 0, & \text{on }]0,T[\times\partial\omega, \\ \gamma\frac{\partial\xi}{\partial\nu} = z, & \text{on }]0,T[\times\partial\omega, \end{cases}$$

where $z \in L^2(0,T; H_0^{\frac{3}{2}\theta})$. By setting $u = \gamma\Delta\varphi$ and taking $z = (-\Delta_\Gamma)^{-\frac{3}{2}\theta} v$, we have $\xi \in C(0, H_0^{2\theta}(\omega)) \cap C^1(0,T; H^{2\theta-2}(\omega))$ and $\xi(0) \in H^{2\theta}(\omega), \xi'(0) \in H^{2\theta-2}(\omega)$. By virtue of Theorem 3.1, we have $F_\theta \subset H_0^{2-2\theta}(\omega) \times H^{-2\theta}(\omega)$ and so $H^{2\theta-2}(\omega) \times H_0^{2\theta}(\omega) \subset F'_\theta$. We define the following operator

$$\begin{aligned} \Lambda:\quad & F_\theta \to F'_\theta \\ & (\varphi_o, \varphi_1) \mapsto (\xi'(0); -\xi(0)), \end{aligned}$$

and, we verify that

$$\langle\Lambda(\varphi_o,\varphi_1), (\varphi_o,\varphi_1)\rangle_{F_\theta, F'_\theta} = \int_{\gamma_0^+} \langle(-\Delta_\Gamma)^{-\frac{3}{2}\theta} v, v\rangle_{H_0^{\frac{3}{2}\theta}(\Gamma_{0,\omega}), H^{-\frac{3}{2}\theta}(\Gamma_{0,\omega})} \mathrm{d}\sigma\mathrm{d}t$$

$$=\|v\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma_{0,\omega}))},$$

and that Λ is an isomorphism. So for all $T > T_0$ and for all $(y_0, y_1) \in F'_\theta$, there exists a $(\varphi_o, \varphi_1) \in F_\theta$ such that $\Lambda(\varphi_o, \varphi_1) = (y_1, -y_0)$.

Let φ and ξ be the functions defined previously with $\xi_0 = y_0$ and $\xi_1 = y_1$. Thanks to the uniqueness of the solution, we have $y = \xi$. So $y(T) = y'(T) = 0$, which proves the theorem. □

4. Specific case of strategic position control

Let $0 \leq \theta \leq 1$. For all $(\varphi_0, \varphi_1) \in D(\omega) \times D(\omega)$, we pose

$$\|(\varphi_0, \varphi_1)\|^2_{G_\theta} = \int_0^T \|\gamma \Delta\varphi\|^2_{H^{-\frac{\theta}{2}}(\Gamma_0)} \mathrm{d}t,$$

where φ is the solution of the homogeneous Eq. (1.1) corresponding to the initial state (φ_0, φ_1). We denote by G_θ the complement of $D(\omega) \times D(\omega)$ for the norm $\|.\|^2_{G_\theta}$.

Proposition 4.1 *For all $\theta \in [0,1]$ and the initial state $(\varphi_0, \varphi_1) \in G_\theta$ we have, for all $T > 0$ and $\alpha_\omega > 0$, the following inequality (better known in control theory as the inverse inequality):*

$$\int_0^T \|\gamma \frac{\partial \Delta\varphi}{\partial \nu}\|^2_{H^{-\frac{\theta}{2}}(\Gamma_{0,\omega})} \mathrm{d}t \geq \alpha_\omega (T - T_0)\{\|A^{\frac{3}{4}-\frac{\theta}{2}\varphi_o}\|^2_{L^2(\omega)} + \|A^{\frac{1}{4}-\frac{\theta}{2}\varphi_1}\|^2_{L^2(\omega)}\}.$$

Proof The proof is identical to that of Theorem 2.2. Let the initial state be $(\varphi_0, \varphi_1) \in D_{A^{\frac{1}{4}}} \times D_{A^{\frac{-1}{4}}}$ and the problem

$$\varphi'' + A\varphi = 0,\ \varphi(0) = \varphi_0,\ \varphi'(0) = \varphi_1.$$

This problem has a unique solution given by $(\varphi(t), \varphi'(t)) = S_1(t)(\varphi_0, \varphi_1)$, where $\{S_1(t) : t \in \mathbb{R}\}$ is the C_0-unitary group on $D_{A^{\frac{1}{4}}} \times D_{A^{\frac{-1}{4}}}$ generated by A.

Let us assume that $\psi = A^{-1}\varphi'$, $\psi_0 = A^{-1}\varphi_1 \in H^2_0(\omega)$, $\psi_1 = -\varphi_0$. Then ψ satisfies

$$\psi'' + A\psi = 0,\ \psi(0) = \psi_0 \in D_{A^{\frac{3}{4}}},\ \varphi'(0) = \varphi_1 \in D_{A^{\frac{1}{4}}}.$$

This problem has a unique solution given by $(\psi(t), \psi'(t)) = S_2(t)(\psi_0, \psi_1)$ where $\{S_2(t) : t \in \mathbb{R}\}$ is a C_0 unitary semi-group on $D_{A^{\frac{1}{4}}} \times D_{A^{\frac{-1}{4}}}$. For $T > T_0$, using the estimation (1.4), we have

$$\|A^{\frac{1}{4}}\varphi_0\|^2_{L^2(\omega)} + \|A^{\frac{-1}{4}}\varphi_1\|^2_{L^2(\omega)} \leq C_T \int_{\Sigma_0^+} |\gamma \frac{\partial}{\partial \nu} \Delta A^{-1}\varphi'|^2 \mathrm{d}\sigma.$$

Multiplying by φ_ϵ and integrating by part, we obtain

$$(\|A^{\frac{1}{4}}\varphi_0\|^2_{L^2(\omega)} + \|A^{\frac{-1}{4}}\varphi_1\|^2_{H^{-2}(\omega)}) \leq C_T \int_{\Sigma_0^+} [(\frac{1}{2})\varphi''|\gamma \frac{\partial}{\partial \nu}\Delta A^{-1}\varphi|^2 + \varphi_\epsilon(\gamma \frac{\partial}{\partial \nu}\Delta\varphi)(\gamma \frac{\partial}{\partial \nu}\Delta A^{-1}\varphi)] \mathrm{d}\sigma.$$

Using estimation (1.3), we obtain

$$\int_{\Sigma_0^+} \varphi''_\epsilon |\gamma \frac{\partial}{\partial \nu}\Delta A^{-1}\varphi|^2 \mathrm{d}\sigma \leq C_T(\|A^{\frac{-1}{4}}\varphi_0\|^2_{L^2(\omega)} + \|\varphi_1\|^2_{D'_{A^{\frac{3}{4}}}}),$$

where $D'_{A^{\frac{3}{4}}}$ is the dual of $D_{A^{\frac{3}{4}}}$ in $L^2(\omega)$. By applying Young's inequality and using the following estimate

$$\|\gamma \frac{\partial}{\partial \nu}\Delta\varphi\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma_0))} \leq C\|\Delta A^{-1}\varphi\|^2_{L^2(0,T;H^2(\Gamma_{0,\omega}))}$$

$$\leq \|\varphi\|^2_{L^2(0,T;L^2(\omega))}$$
$$\leq C_T(\|A^{\frac{1}{4}}\varphi_0\|^2_{L^2(\omega)} + \|A^{-\frac{1}{4}}\varphi_1\|^2_{L^2(\omega)}),$$

we have

$$(\|A^{\frac{1}{4}}\varphi_0\|^2_{L^2(\omega)} + \|A^{-\frac{1}{4}}\varphi_1\|^2_{L^2(\omega)}) \leq C_T[(\|A^{-\frac{1}{4}}\varphi_0\|^2_{L^2(\omega)} + \|\varphi_1\|^2_{D'_{A^{\frac{3}{4}}}}) + \|\gamma\frac{\partial}{\partial\nu}\Delta\varphi\|^2_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_{0,\omega}))}].$$

By compactness reasoning ($D_{A^{\frac{1}{4}}} \times D_{A^{-\frac{1}{4}}} \subset D_{A^{-\frac{1}{4}}} \times D'_{A^{\frac{3}{4}}}$) with injection compact, we have for initial state $\varphi_0 \in D_{A^{\frac{1}{4}}}$ and $\varphi_1 \in D_{A^{-\frac{1}{4}}}$. The solution φ of problem (1.1) corresponding to these data verifies

$$\int_0^T \|\frac{\partial\Delta\varphi}{\partial\nu}\|^2_{H^{-\frac{1}{2}}(\Gamma_0)} \mathrm{d}t \geq C_T^2\{\|A^{\frac{1}{4}}\varphi_0\|^2 + \|A^{-\frac{1}{4}}\varphi_1\|^2\}.$$

And we obtain by interpolation, for all $\theta \in [0{,}1]$ and initial state $(\varphi_o, \varphi_1) \in (D_{A^{\frac{3}{4}-\frac{\theta}{2}}} \times D_{A^{\frac{1}{4}-\frac{\theta}{2}}})$, the inequality

$$\int_0^T \|\frac{\partial\Delta\varphi}{\partial\nu}\|^2_{H^{-\frac{\theta}{2}}(\Gamma_{0,\omega})} \mathrm{d}t \geq C_T^3\{\|A^{\frac{3}{4}-\frac{\theta}{2}}\varphi_0\|^2 + \|A^{\frac{1}{4}-\frac{\theta}{2}}\varphi_1\|^2\},$$

which leads to the following Theorem:

Theorem 4.2 *Let $T > T_0$, $y_0 \in D_{A^{-\frac{1}{4}+\frac{\theta}{2}}}$, $y_1 \in D_{A^{-\frac{3}{4}+\frac{\theta}{2}}}$ and $\theta \in [0,1]$. There exists a border control function $v \in L^2(0,T;H^{-\frac{\theta}{2}}(\Gamma_{0,\omega}))$ such that, if y is the unique solution of the equation*

$$\begin{cases} y'' + \Delta^2 y = 0, & \text{in }]0,T[\times\omega, \\ y(0) = y_0;\ y'(0) = y_1, & \\ \gamma y = (-\Delta_\Gamma)^{-\frac{\theta}{2}} v, & \text{on } \Sigma_0^+, \\ \gamma\frac{\partial}{\partial\nu} y = 0, & \text{on } \Sigma_0^*, \end{cases}$$

then $y(T) = y'(T) = 0$.

Proof We also apply the Hilbertian Uniqueness Method, see Lions [1]. □

5. Conclusion and perspectives

One of our objectives in this work was to establish a strong result of exact controllability of the vibrating plate equation on a strategic area of the domain. The interest in regional controllability lies in the fact that this notion is more adapted to systems described by partial differential equations. Among these reasons:

- It is a concept closer to concrete concerns (for example maintains of a given temperature in a region of placed or vibrating plates, ...);

- A regional transfer is lower cost compared to controlling the entire interior of the domain or the entire border;

- Ultimately, regional control is more adapted and precise (with regard to the objective to be achieved).

In the near future, we intend to study the minimum time for the exact (or approximate) controllability of the equation of placed plates and the minimum cost to achieve it. This work has already been done [9–11] for the case of a parabolic equation of the heat type via a strategic zone profile.

Acknowledgements The authors thank in advance the anonymous referees for their time, valuable comments, suggestions and remarks.

Conflict of Interest The authors declare no conflict of interest.

References

[1] J. L. LIONS. *Exact Controllability of Perturbation and Stabilization of Distributed Systems (I)*. Masson, Paris, 1988.

[2] R. TRIAGGIANI, I. LASIECKA. *Exact Controllability of the Euler-Bernouilli Equation with L_2 Control in the Dirichlet Boundary Condition*. Rendiconti Accademia Nazionale dei Lincei, Roma, Italy; Classe Sci. Fis. Matem, LXXX, 1988.

[3] M. T. NIANE. *Exact controllability of the vibrating plate equation in a polygon*. C. R. Acad. Sci. Paris Sér. I Math., 1988, **307**(10): 517–521.

[4] M. T. NIANE. *Exact Controllability of the Wave Equation for Initial State in Fractional Power Sobolev Spaces*. State Thesis, Cheikh Anta Diop University of Dakar, 1990.

[5] E. ZUAZUA. *Exact Controllability of some Plate Models in an Arbitrarily Small Time*. Lions Appendix, 1988.

[6] G. AVALOS, I. LASIECKA. *Boundary controllability of thermoelastic plates via the free boundary conditions*. SIAM J. Control Optim., 2000, **38**(2): 337–383.

[7] A. El JAI. *Analyse régionale des systémes distribués*. ESAIM Control Optim. Calc. Var., 2002, **8**: 663–692. (in Frence)

[8] M. T. NIANE, A. SENE. *On the exact controllability of the vibrating plate equation*. Rev. Mat. Complut., 2002, **15**(2): 619–629.

[9] C. SECK, M. L. L. ANE, A. SENE. *Boundary exact controllability of the heat equation in 1D by strategic actuators and a linear surjective compact operator*. Appl. Math., 2020, **11**: 991–999.

[10] C. SECK, M. L. L. ANE. *A best cost of null controllability for the 1D heat equation by strategic zone profile*. J. Math., 2021, Art. ID 4643405, 6 pp.

[11] C. SECK, M. L. L. ANE, B. DIOUF. *A new approach to find the best cost of null controllability for the 1D heat equation by a strategic zone profile*. Univers. J. Math. Math. Sci., 2022, **17**: 47–65.